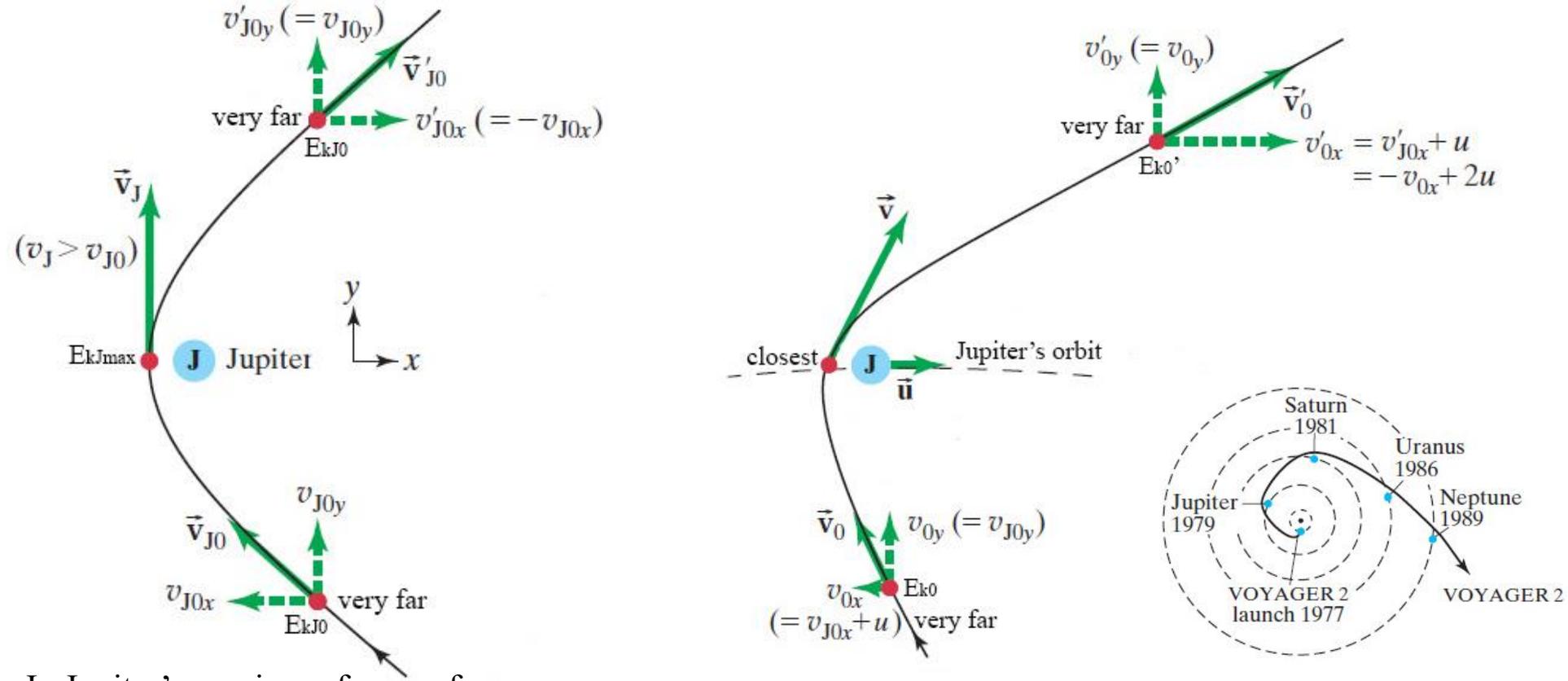


# Gravity assist (slingshot, swing-by)



In Jupiter's moving reference frame  
When  $r \rightarrow \infty$  then  $E_P \rightarrow 0$

In Sun's frame of reference:

Using Galilean transformation:  $\vec{v} = \vec{v}_J + \vec{u}$

$$v'_{0x} = v'_{J0x} + u = -v_{J0x} + u = -(v_{0x} - u) + u = -v_{0x} + 2u$$

Increase in kinetic energy:

$$\begin{aligned} E'_{k0} - E_{k0} &= \frac{1}{2} m [(-v_{0x} + 2u)^2 + v_{0y}^2] - \frac{1}{2} m (v_{0x}^2 + v_{0y}^2) = \\ &= \frac{1}{2} m (v_{0x}^2 - 4v_{0x}u + 4u^2 + v_{0y}^2 - v_{0x}^2 - v_{0y}^2) = 2mu(u - v_{0x}) > 0 \quad (v_{0x} < 0) \end{aligned}$$

# Waves

Waves are created when the vibration of a part of an elastic medium propagates through the medium, with neighboring points also taking over the vibration.

E.g. guitar string (1D), water surface (2D), sound or light (3D)

The speed of propagation is the **phase velocity** of the wave ( $c$ ). This determines the time delay between the vibrations of two distant points.

Consider a plane wave propagating in the  $x$  direction (or a wave propagating on a 1D string).

The vibration at  $x = 0$  is the usual harmonic function:  $y(t) = A\sin(\omega t)$

In comparison, at location  $x$ , the vibration is delayed by  $x/c$  time:



$$\begin{aligned}y(x, t) &= A\sin\left[\omega\left(t - \frac{x}{c}\right)\right] = A\sin\left(\omega t - \frac{\omega x}{c}\right) \\ &= A\sin\left(\omega t - \frac{2\pi x}{Tc}\right) = A\sin\left(\omega t - \frac{2\pi x}{\lambda}\right) = A\sin(\omega t - kx)\end{aligned}$$

$T$ : **period** of vibration     $\omega$ : **angular frequency** of the vibration  $\omega = 2\pi f = 2\pi/T$

$\lambda$ : **wavelength** (distance traveled during one period)  $\lambda = Tc$

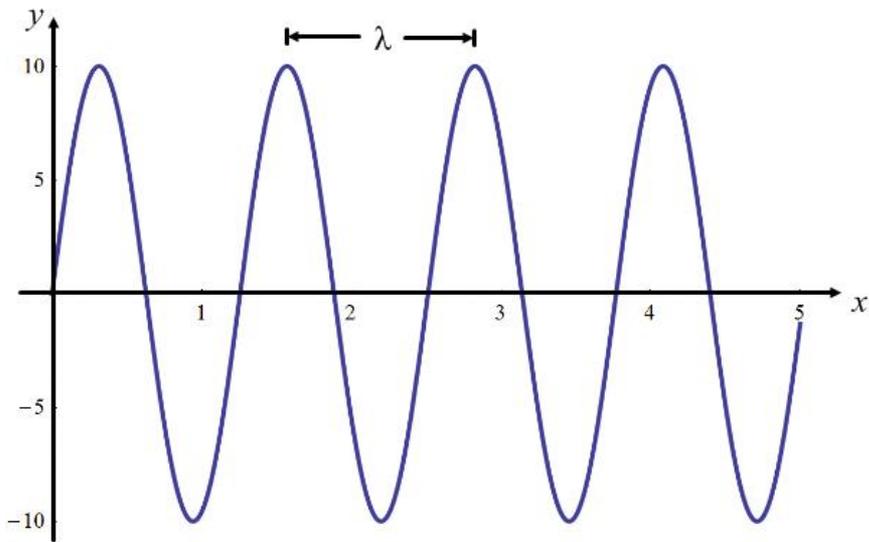
$k$ : **angular wave number**  $k = 2\pi/\lambda$

Since:  $T = 1/f$  therefore  $c = \lambda f$

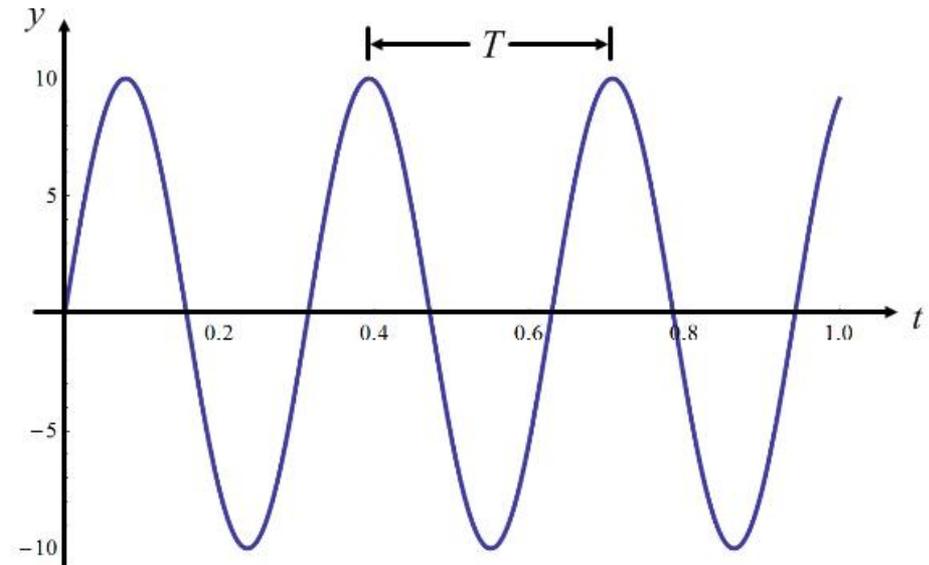
# Waves: spatial and temporal dependence

For a wave, both the spatial and temporal dependences are periodic functions:

$$y(x, t) = A \sin(\omega t - kx)$$

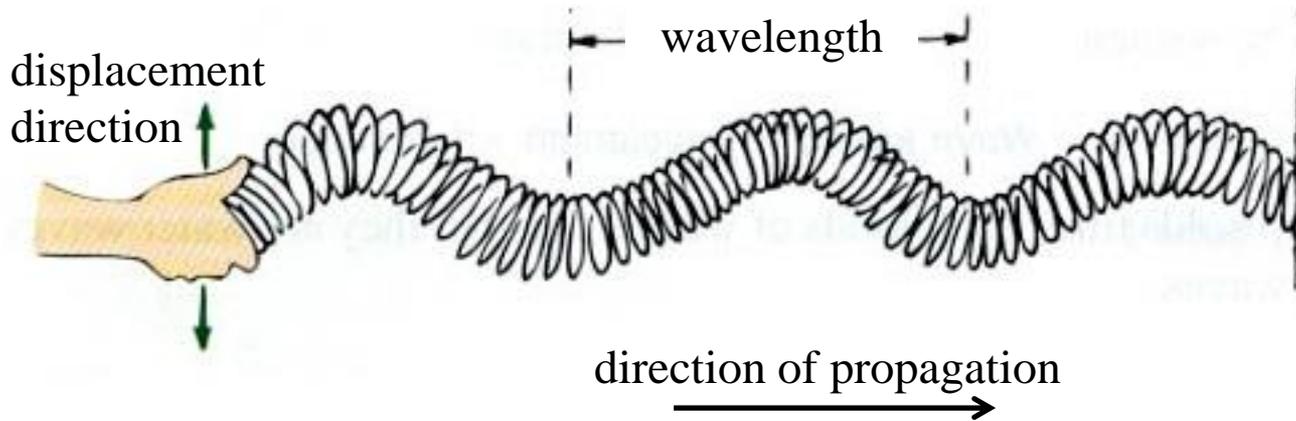


Spatial periodicity is the wavelength  
(snapshot at a given time)

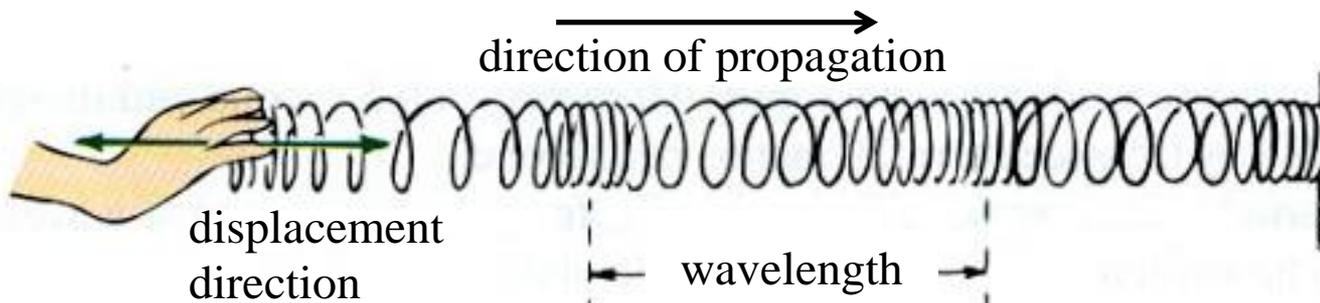


Temporal periodicity is the period  
(time dependence at a given location)

# Transversal and longitudinal waves

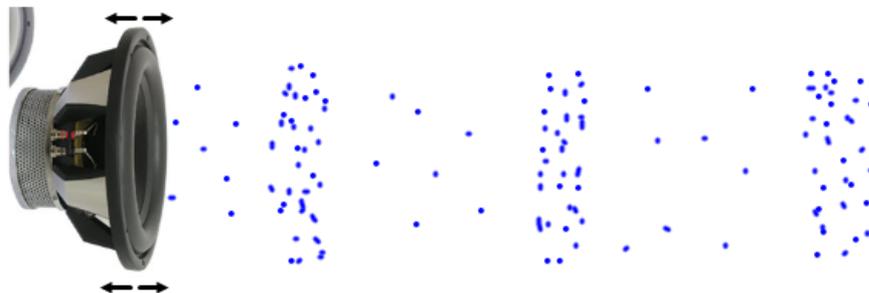


**transversal wave:**  
the displacement is perpendicular to the direction of propagation



**longitudinal wave:**  
the displacement is parallel with the direction of propagation

sound is also a longitudinal wave  
(20Hz – 20kHz)



# Standing waves

Reaching the boundary of the medium, the wave is reflected. When the incoming and reflected waves meet, their displacements add up (they **interfere**). In some cases, a **standing wave** may be created. Consider a wave traveling in the  $x$  direction and a reflected wave traveling in the  $-x$  direction:

$$y(x, t) = y_1(x, t) + y_2(x, t)$$

$$y(x, t) = A\sin(\omega t - kx) + A\sin(\omega t + kx)$$

$$y(x, t) = A\sin(\omega t)\cos(kx) - A\cos(\omega t)\sin(kx) + A\sin(\omega t)\cos(kx) + A\cos(\omega t)\sin(kx)$$

$$y(x, t) = 2A\sin(\omega t)\cos(kx) = 2A\cos(kx)\sin(\omega t) = 2A(x)\sin(\omega t)$$

The amplitude becomes position dependent: **nodes** and **antinodes**.

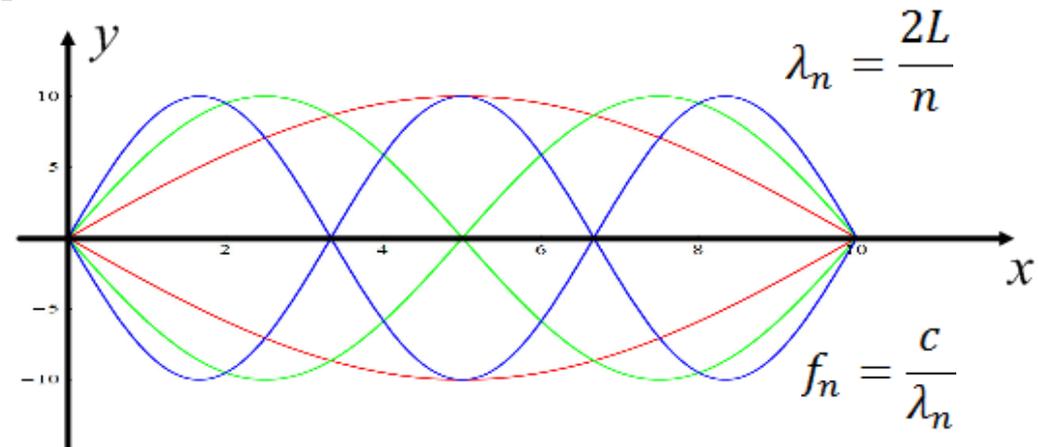
However, the phase no longer depends on time (position and time dependence are decoupled).

The wavelength of the standing wave can

only be the wavelengths allowed by the given geometry: **boundary conditions**

E.g. node at the end of fixed-end string, antinode at the end of open-end whistle.

Therefore, the frequency cannot be arbitrary either: fundamental frequency and harmonics.



# Doppler effect

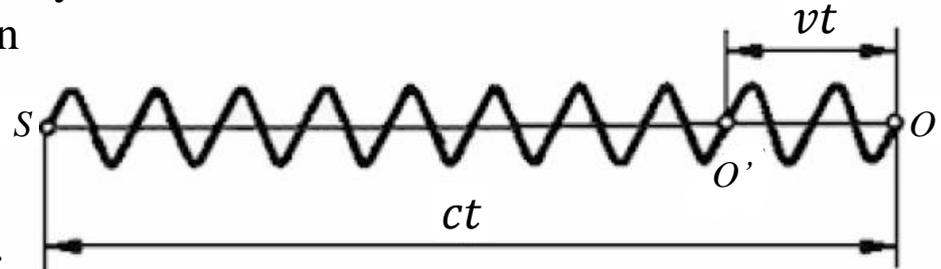
If the source of the wave and the observer are moving relative to each other, the emitted frequency will not match the detected frequency.

E.g. sound of approaching and receding siren

If **observer  $O$  approaches** with speed  $v$ :

Detects not only a wave train of length  $ct$ , but length  $ct + vt$ .

Number of vibrations detected per unit time:



$$f' = \frac{ct \pm vt}{\lambda t} = f \left( 1 \pm \frac{v}{c} \right) \quad (-v/c \text{ if receding})$$

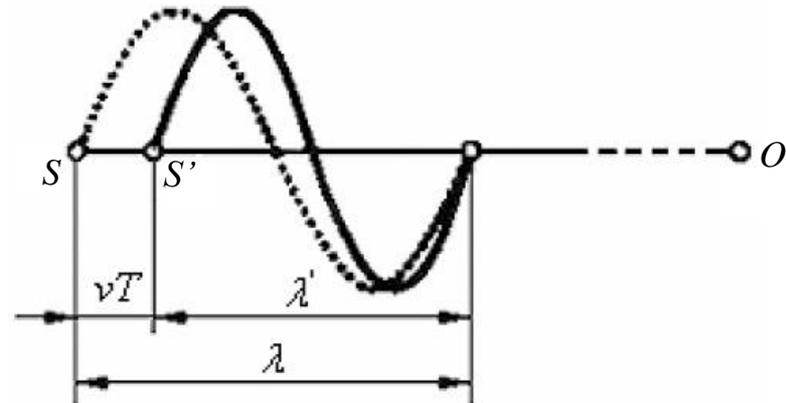
If **source  $S$  approaches** with speed  $v$ :

The wavelength appears to shorten with  $vT$ :

$$\lambda' = \lambda - vT$$

$$f' = \frac{c}{\lambda'} = \frac{c}{\lambda \mp vT} = \frac{c}{\lambda} \frac{1}{1 \mp \frac{vT}{\lambda}} = f \frac{1}{1 \mp \frac{v}{c}}$$

( $+v/c$  if receding)



If both are moving **relative to the medium**:  $f' = f \frac{1 \pm \frac{v_O}{c}}{1 \mp \frac{v_S}{c}}$

# Momentum-force law for mass point systems

For a system of points, let's write the **momentum-force law** for one of the points ( $i$ ):

The net external force acting on it:  $\vec{F}_i$

The force exerted by point  $j$ :  $\vec{F}_{ji}$

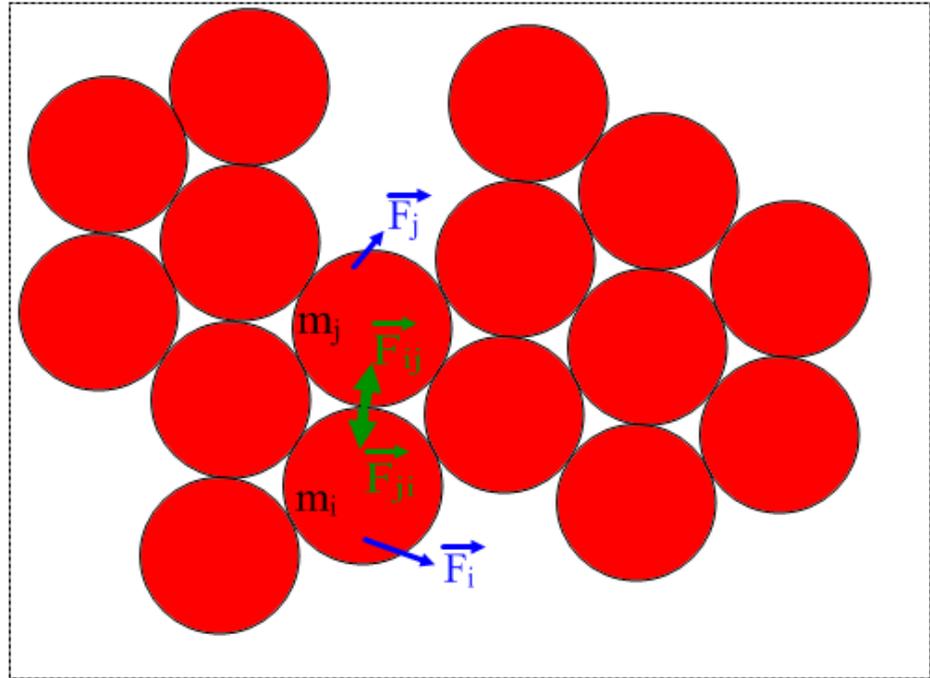
Using the basic equation of dynamics for mass point  $i$ :

$$\frac{d}{dt}\vec{p}_i = m_i\vec{a}_i = \vec{F}_i + \sum_{j=1}^N \vec{F}_{ji}$$

Summarized for all the points:

$$\sum_{i=1}^N \frac{d}{dt}\vec{p}_i = \sum_{i=1}^N \vec{F}_i + \sum_{i=1}^N \sum_{j=1}^N \vec{F}_{ji}$$

Internal forces cancel (Newton's 3rd axiom):  $\vec{F}_{ji} = -\vec{F}_{ij}$



Momentum-force law for mass point systems:

$$\frac{d}{dt}\vec{p} = \sum_{i=1}^N \vec{F}_i$$

If the net external force acting on a point system is zero, then the **momentum is conserved**.

# Collisions

If the colliding bodies form a closed system (no external forces act on them), then **momentum conservation** is always satisfied during the collision.

The total momentum of the members of the system is the same before and after collision:

$$\vec{p}_{A1} + \vec{p}_{B1} + \vec{p}_{C1} + \dots = \vec{p}_{A2} + \vec{p}_{B2} + \vec{p}_{C2} + \dots$$

$$m_A \vec{v}_{A1} + m_B \vec{v}_{B1} + m_C \vec{v}_{C1} + \dots = m_A \vec{v}_{A2} + m_B \vec{v}_{B2} + m_C \vec{v}_{C2} + \dots$$

This usually means 3 independent equations for the  $x$ ,  $y$ , and  $z$  components of momentum.

This is necessary if the collision occurs in space and is not central

(e.g. two balls smashed together whose centers don't line up, or it can also be used in the case of an exploding firework).

The collision of billiard balls in the plane of the table gives two equations.

The central collision of two bodies moving along a straight line gives only one:

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

If the two bodies move in opposite directions, then one velocity is negative.

Extreme cases:

In perfectly inelastic collision, the two bodies stick together:  $v_{A2x} = v_{B2x}$  and  $m = m_A + m_B$

In a perfectly elastic collision, kinetic energy is also conserved:

$$\frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2 = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2$$

# Coefficient of restitution

Real collisions are neither perfectly inelastic nor perfectly elastic.

The **coefficient of restitution** shows how elastic the collision is.

This number is the ratio of the velocity of separation and the velocity of approach.

The coefficient of restitution:

$$k = \frac{v_{B2x} - v_{A2x}}{v_{A1x} - v_{B1x}}$$

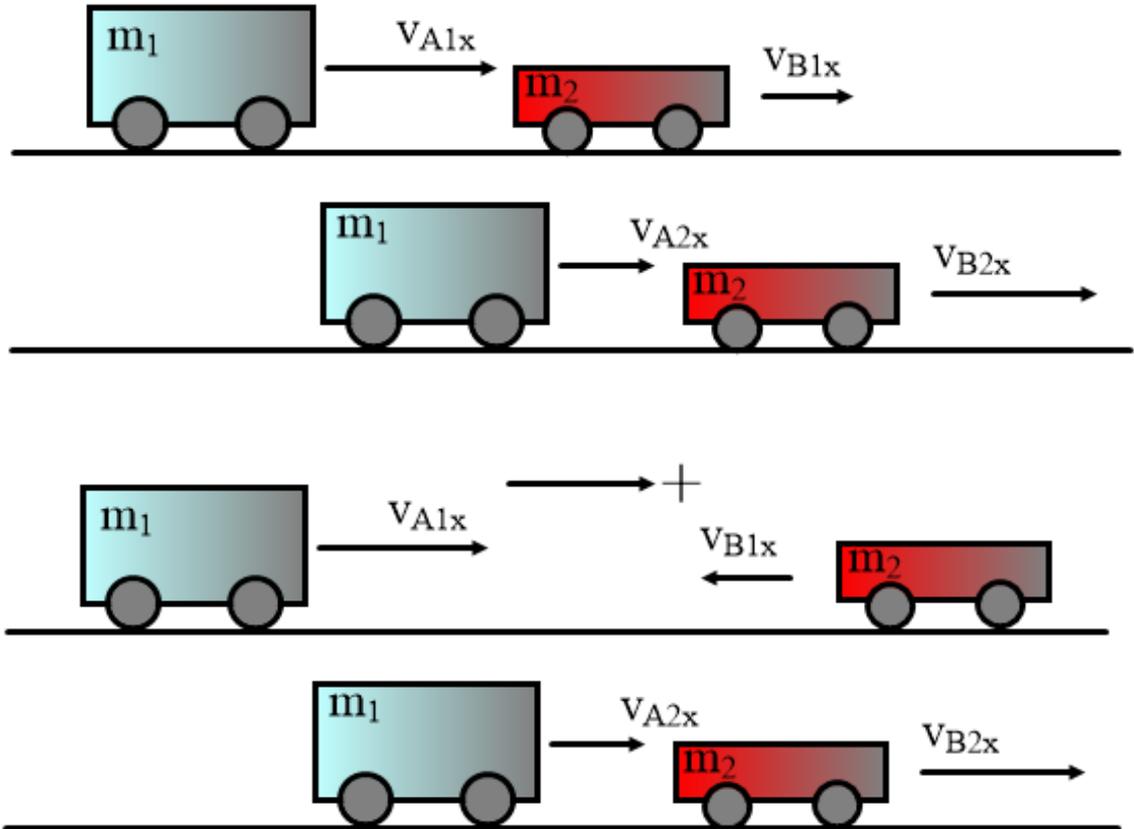
Extreme cases:

perfectly inelastic:  $k = 0$

perfectly elastic:  $k = 1$

in general:  $0 \leq k \leq 1$

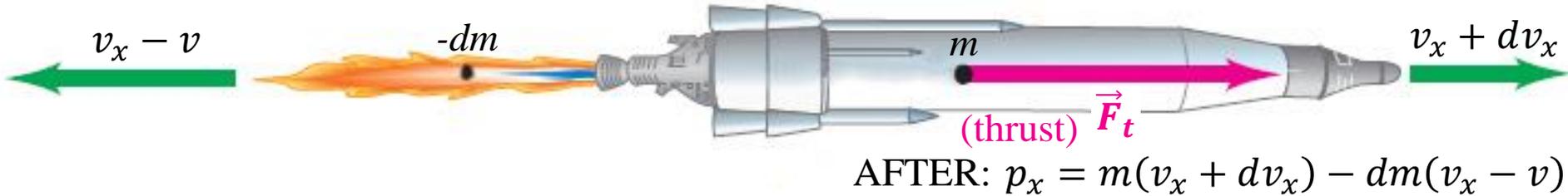
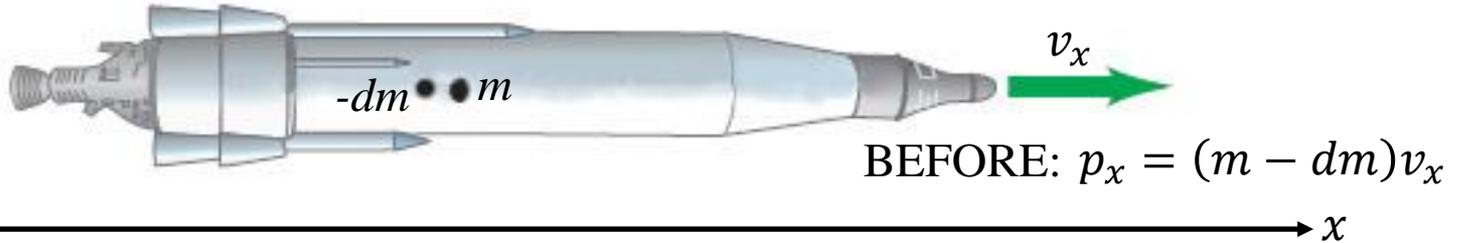
This is easier to use instead of conservation of kinetic energy, because this is not quadratic.



# Rocket propulsion

Using conservation of momentum for the rocket:

- $m$ : mass without the fuel element of mass  $-dm$  ( $dm < 0$ , since total mass decreases!)
- $v$ : exhaust speed compared to rocket



$$(m - dm)v_x = m(v_x + dv_x) - dm(v_x - v)$$

$$mv_x - v_x dm = mv_x + mdv_x - v_x dm + v dm$$

$$0 = mdv_x + v dm$$

$$\frac{dv_x}{dm} = -\frac{v}{m} \rightarrow \int_0^{v_x} dv'_x = -v \int_{\mu+M}^{\mu+m_f} \frac{dm}{m} \rightarrow v_x = -v \ln \frac{\mu + m_f}{\mu + M} = v \ln \frac{\mu + M}{\mu + m_f}$$

- $\mu$ : empty mass of rocket
- $M$ : total mass of fuel
- $m_f$ : momentary fuel mass

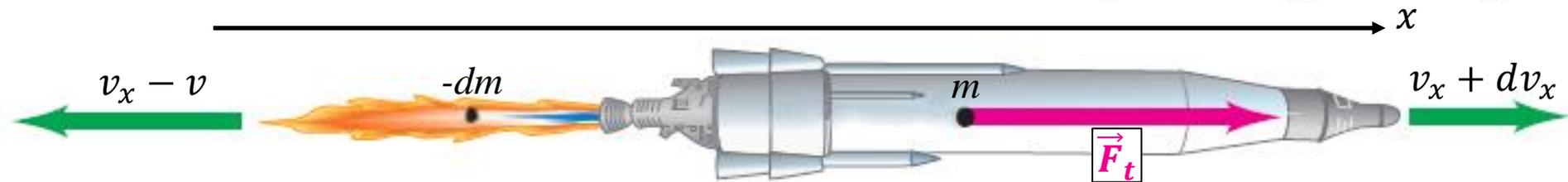
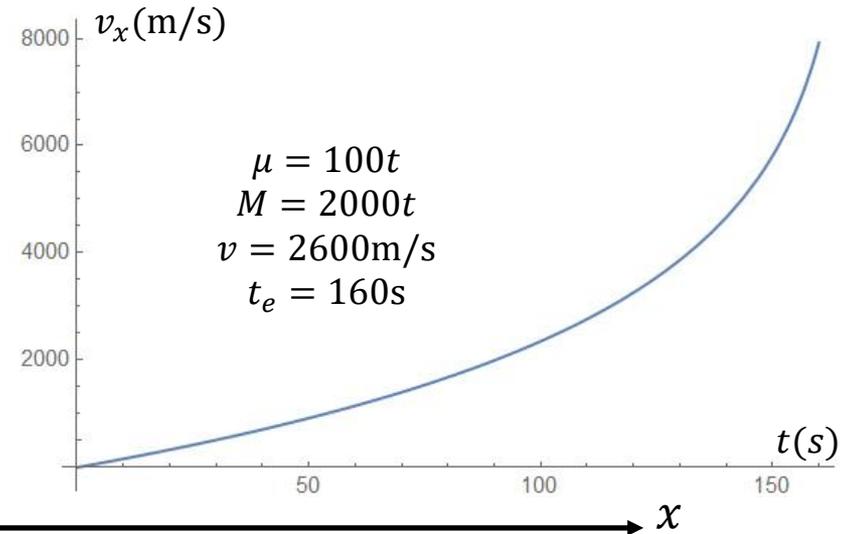
Maximum (terminal) speed once all fuel is used up:  $v_t = v \ln \frac{\mu + M}{\mu}$

# Thrust

Assuming that the rate of fuel burning is constant:

$$R = \frac{M}{t_e} = -\frac{dm}{dt} \quad t_e \text{ is the time to empty the fuel tank}$$

$$m_f = M - Rt \rightarrow v_x(t) = v \ln \frac{\mu + M}{\mu + M - Rt}$$



The magnitude of the **thrust** on the remaining fuel and rocket body with mass  $m$ :

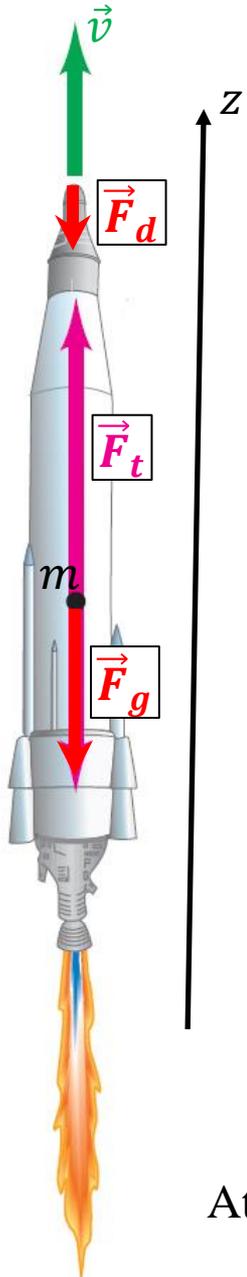
$$F_t = m \frac{dv_x}{dt} = m \frac{dv_x}{dm} \frac{dm}{dt} = m \left( -\frac{v}{m} \right) (-R) = vR$$

Specific impulse:

Momentum increase of the remaining fuel and rocket body with mass  $m$  by unit mass of fuel.

$$I_{sp} = -\frac{dp_x}{dm} = \frac{\frac{dp_x}{dt}}{-\frac{dm}{dt}} = \frac{F_t}{R} = \frac{vR}{R} = v$$

# Lift-off



Equation of motion for the vertical launch of the rocket:

$$m \frac{dv_z}{dt} = F_t - F_g - F_d$$

$$m \frac{dv_z}{dt} = vR - mg - cv_z^2 \quad v: \text{exhaust speed of fuel}$$

At the beginning  $v_z$  is „small”, thus  $cv_z^2$  is „negligible”:

$$\frac{dv_z}{dt} = \frac{vR}{m} - g = \frac{vR}{\mu + M - Rt} - g$$

$$\int_0^{v_z} dv'_z = \int_0^t \left( \frac{vR}{\mu + M - Rt'} - g \right) dt'$$

$$v_z(t) = v \ln \frac{\mu + M}{\mu + M - Rt} - gt$$

Since the drag is  $F_d = \frac{1}{2} \rho C_D A v_z^2$

Taking values  $A = 112\text{m}^2$   $C_D = 0.5$   $\rho = 1.225\text{kg/m}^3$

At  $v_z = 100\text{m/s} \rightarrow F_d = 343000\text{N} = 0.0163mg$  (only 1.63% of the weight)

# Spinning top and gyroscope

Using the torque-angular momentum law for a spinning top:  $\vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \rightarrow d\vec{L} = \vec{\tau}_{ext} dt$   
 $\vec{\tau}_{ext}$  is the net external torque acting on the spinning top

Since torque is in in  $x$ - $y$  plane  $\rightarrow$  it won't fall down but perform precession with  $\omega_p$ . Assuming  $\omega_p \ll \omega$ .

$$\vec{\tau}_{ext} = \vec{r} \times \vec{F}_g \quad \text{thus} \quad \tau_{ext} = r \sin \phi \cdot mg$$

$$d\vec{L} \perp \vec{L} \rightarrow L \text{ remains constant!}$$

$$|d\vec{L}| = L \sin \phi d\theta \rightarrow d\theta = \frac{|d\vec{L}|}{L \sin \phi}$$

$$\omega_p = \frac{d\theta}{dt} = \frac{1}{L \sin \phi} \frac{|d\vec{L}|}{dt} = \frac{1}{L \sin \phi} \left| \frac{d\vec{L}}{dt} \right|$$

$$= \frac{1}{L \sin \phi} \tau_{ext} = \frac{1}{L \sin \phi} r \sin \phi \cdot mg$$

$$= \frac{rmg}{L} = \frac{rmg}{I\omega}$$

