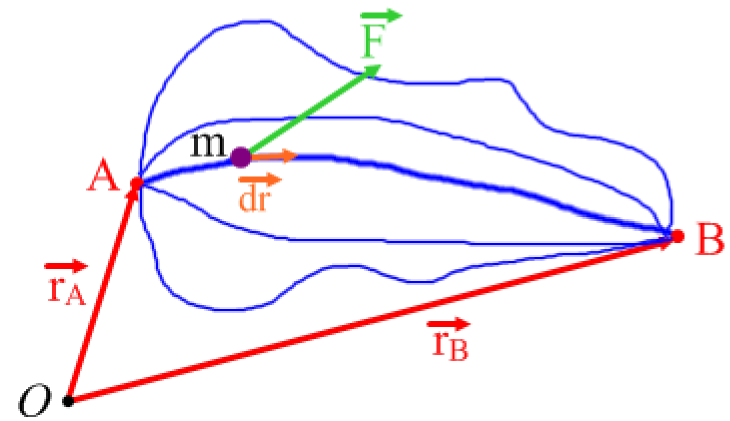
Conservative force fields

Conservative force field: A time-independent force field in which the work done by the force field between two points is independent of the path (this is equivalent to the work being zero for any closed curve).

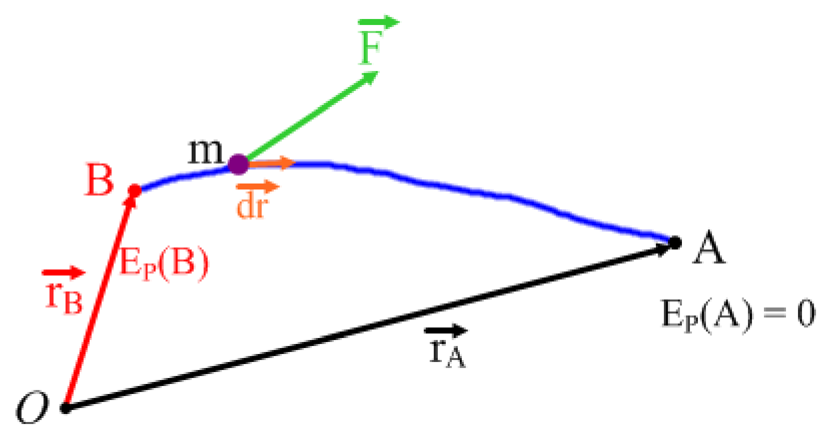


Then we can characterize the points (e.g. B) by the work that the field does while the body moves from there to a selected zero point (e.g. A).

Potential energy:

The potential energy at a point (B) is equal to the work done by the force field as the body moves from there to the zero point (A).

$$E_P(B) = W_{BA} = \int_B^A \vec{F} \cdot d\vec{r}$$



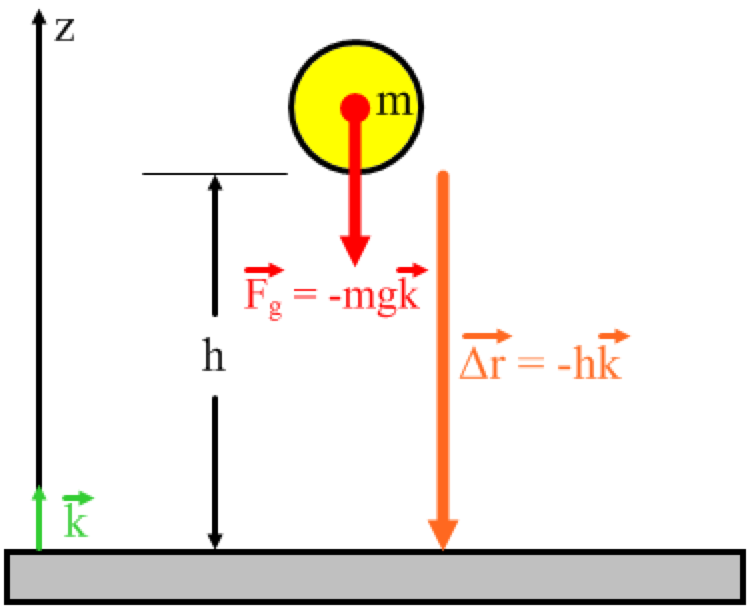
Potential energy of the weight force

Let the floor level be the zero point of the potential energy and drop a body with a weight F_g = 20N from a height of 80m.

Then the work of the weight force (i.e. the potential energy at a height of 80m) is:

$$E_P(h) = W_{h0} = \int_{h\vec{k}}^{0} \vec{F} \cdot d\vec{r} = \int_{h\vec{k}}^{0} \left(-mg\vec{k}\right) \cdot \left(dz\vec{k}\right) =$$

$$= \int_{h}^{0} -mg\, dz = -mg[z]_h^0 = mgh$$

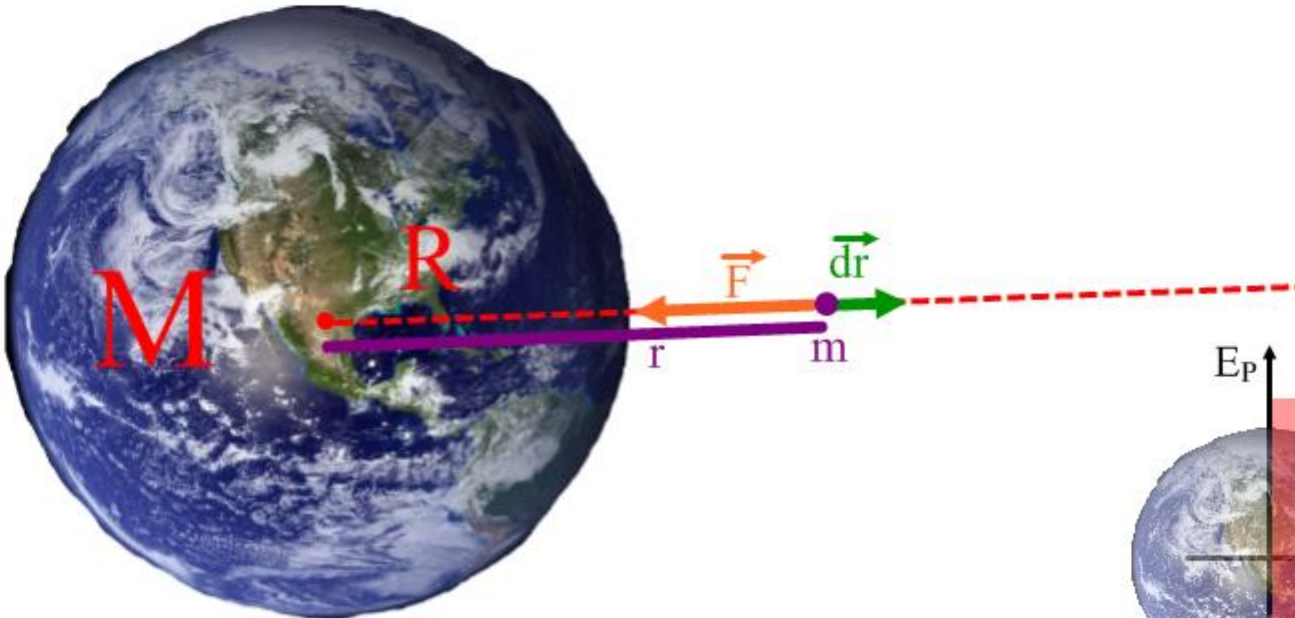


Of course, in this simple case the formula $W = Fs$ can also be used, so the $W = (mg)h = mgh$ is immediately obvious.

That is, in our case: W = (20N)(80m) = 1600J

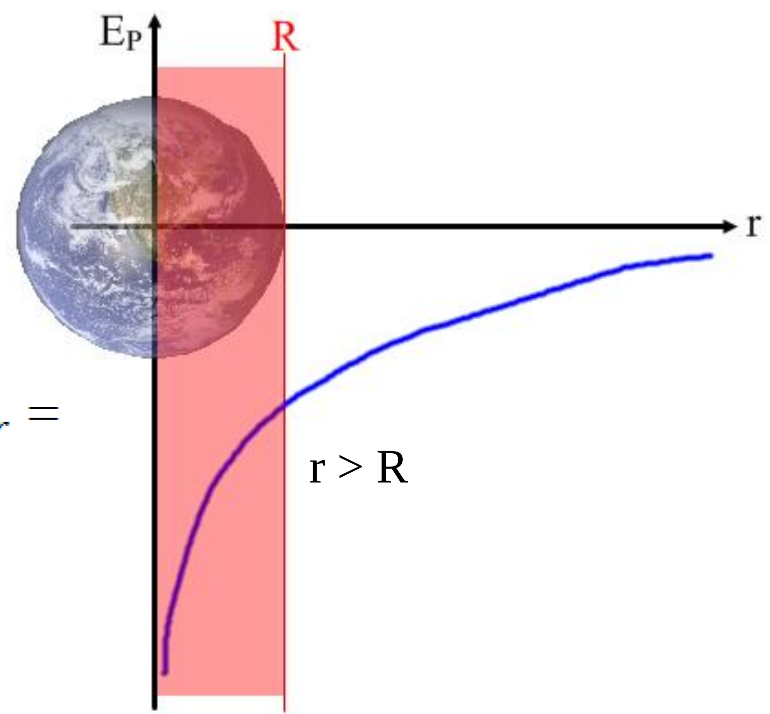
Potential energy in a Newtonian gravitational field

Let the body of mass *M* be fixed, and at a distance *r* from it we calculate the potential energy of the body of mass *m*. The force is radial, so it is advisable to take a radial path. It is advisable to take the zero point at infinity because *r* = 0 is problematic.



$$E_P(\vec{r}) = W_{\vec{r},\infty} = \int_{\vec{r}}^{\infty} \vec{F} \cdot d\vec{r} = \int_{\vec{r}}^{\infty} \left(-\frac{\gamma M m}{r^2}\vec{e}_r\right) dr\vec{e}_r =$$

$$= \int_{r}^{\infty} -\frac{\gamma M m}{r^2} dr = \gamma M m \left[\frac{1}{r}\right]_r^{\infty} = -\frac{\gamma M m}{r}$$

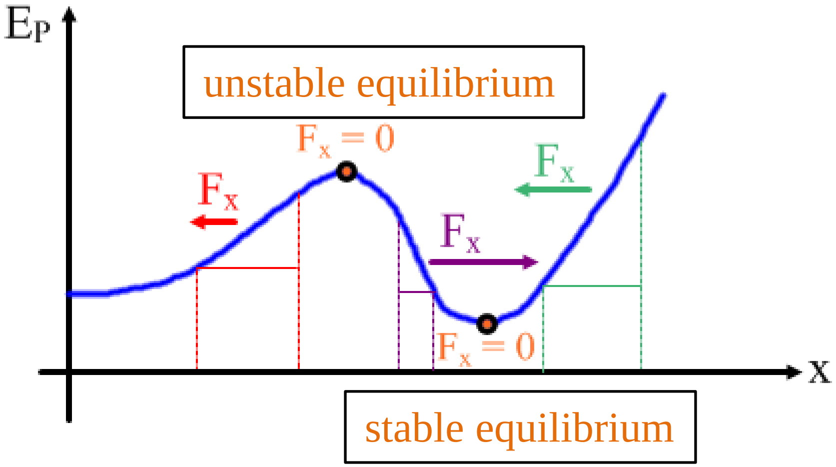


Principle of minimum energy

A greater force means a greater potential energy difference between the same two points. Reversed: The greater the rate at which potential energy changes with changing location, the greater the force exerted **by the force field**.

In general, at an arbitrary point in one dimension: $F_x = -\frac{\partial E_P}{\partial x}$

Principle of minimum energy: The force acts in the direction of decreasing potential energy (negative sign).



In three dimensions:

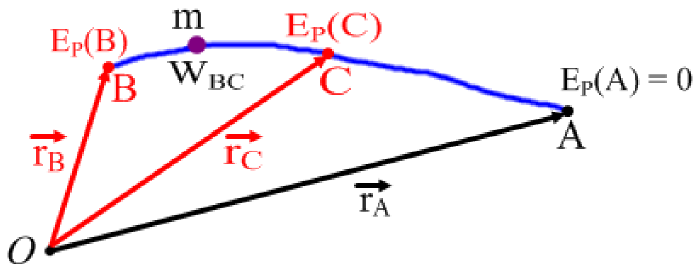
$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k} =$$

$$= -\frac{\partial E_P}{\partial x}\vec{i} - \frac{\partial E_P}{\partial y}\vec{j} - \frac{\partial E_P}{\partial z}\vec{k} =$$

$$= -\nabla E_P = -\mathrm{grad} E_P$$

Mechanical energy

Let's take the special case where **only conservative forces act,** while the body moves from point *B* to point *C*.



Then for any points *B* and *C*: $E_P(B) - E_P(C) = W_{BC} = E_K(C) - E_K(B)$

$$-\Delta E_P = W_{BC} = \Delta E_K$$

Rearranging the original equation: $E_P(B) + E_K(B) = E_P(C) + E_K(C)$
The sum of potential and kinetic energy is the **same at every point**.

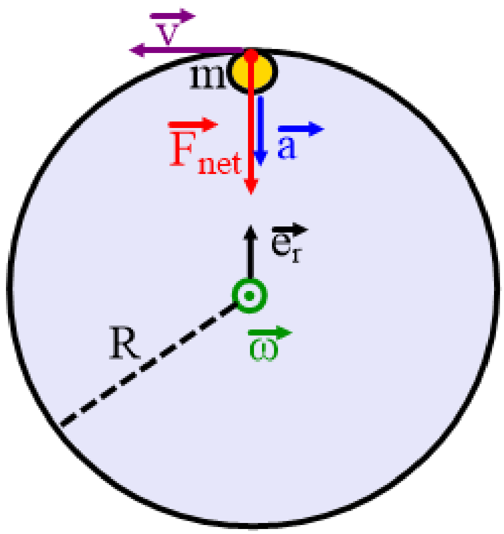
Let's introduce the **mechanical energy**, which is the sum of kinetic and potential energies:

$$E_M = E_P + E_K$$

This mechanical energy is conserved in a conservative force field: $E_M(B) = E_M(C)$

Dynamics of uniform circular motion

<u>Uniform circular motion:</u> During the motion, the magnitude of the velocity is constant, but its direction is constantly changing. So there is acceleration that points towards the center (**centripetal**).
The condition for this is that the net force also points in that direction (centripetal force).



WE DISCUSS IN AN INERTIAL REFERENCE FRAME

Fundamental equation of dynamics: $m\vec{a} = \vec{F}_{net}$

Acceleration has only a centripetal (radial) component.

The magnitude of the net force:

$$F_{net} = ma = ma_{cp} = m\frac{v^2}{R} = m\omega^2 R$$

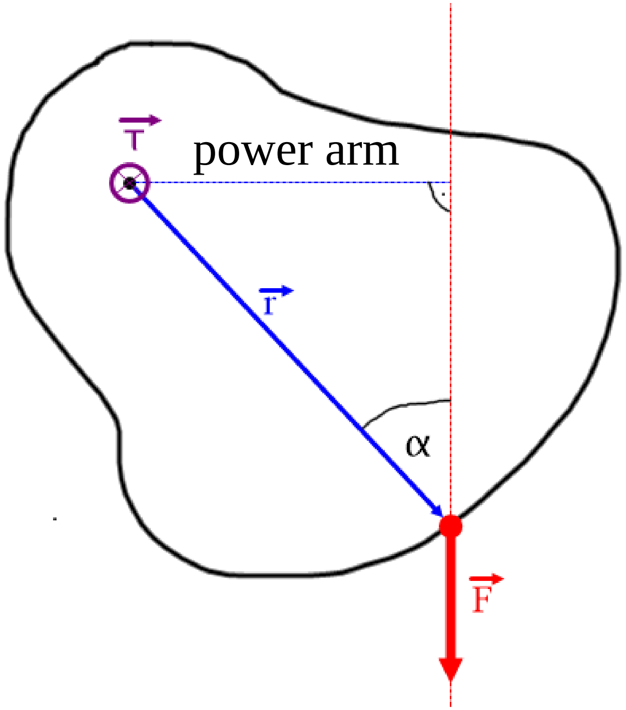
This net force can be provided by many different interactions: e. g. gravitational force, Coulomb force, rope force, normal force, Lorentz force, etc.
Then $\vec{F}_{net} \perp \vec{v}$ the **work done is zero**. The centripetal force does no work.

The direction of the angular velocity vector can be determined using the right-hand rule. E. g. in the figure outwards.

Variable circular motion - torque

The **torque** of a force about the origin (axis of rotation): $\vec{\tau} = \vec{r} \times \vec{F}$

Power arm: the distance of the line of action of the force from the axis of rotation



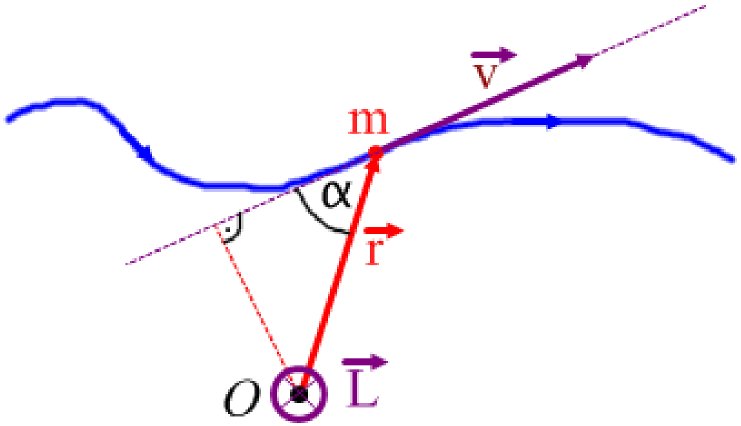
its magnitude: force × power arm
i.e. $\tau = F r_{\perp} = F r \sin\alpha$

The torque is zero if the line of action of the force passes through the axis of rotation, and is maximum if it is perpendicular to the $\vec{r}$ position vector.

its direction: based on the cross product (right-hand rule) perpendicular to the plane defined by the force and position vectors.

Angular momentum

Generally the **angular momentum** of a point-like body is: $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$
(similar to the definition of torque, replacing force with momentum)



If the position and velocity vectors are perpendicular, as for uniform circular motion:

$$L = rmv = mrv = mr\omega r = mr^2\omega$$

The angular momentum vector changes under the influence of torque:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(m\vec{r} \times \vec{v}) = m\left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) = m(\vec{v} \times \vec{v} + \vec{r} \times \vec{a}) =$$

$$= \vec{r} \times (m\vec{a}) = \vec{r} \times \vec{F}_{net} = \vec{\tau}_{net}$$

Torque-angular momentum law: $$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

Moment of inertia

<u>Special case:</u> point mass moves at a constant distance around a **fixed axis** (circular motion)

$$L(t) = mr^2\omega(t)$$

Derivative of the angular momentum: $\frac{dL}{dt} = mr^2\frac{d\omega(t)}{dt} = mr^2\beta(t)$

β is the angular acceleration, and the term mr^2 is the **moment of inertia** of the point mass.

The moment of inertia for the center of mass is therefore: $I = mr^2$,
where r is the distance from the axis.

Using the torque-angular momentum law: $\frac{dL}{dt} = mr^2\frac{d\omega(t)}{dt} = I\beta(t) = \tau$

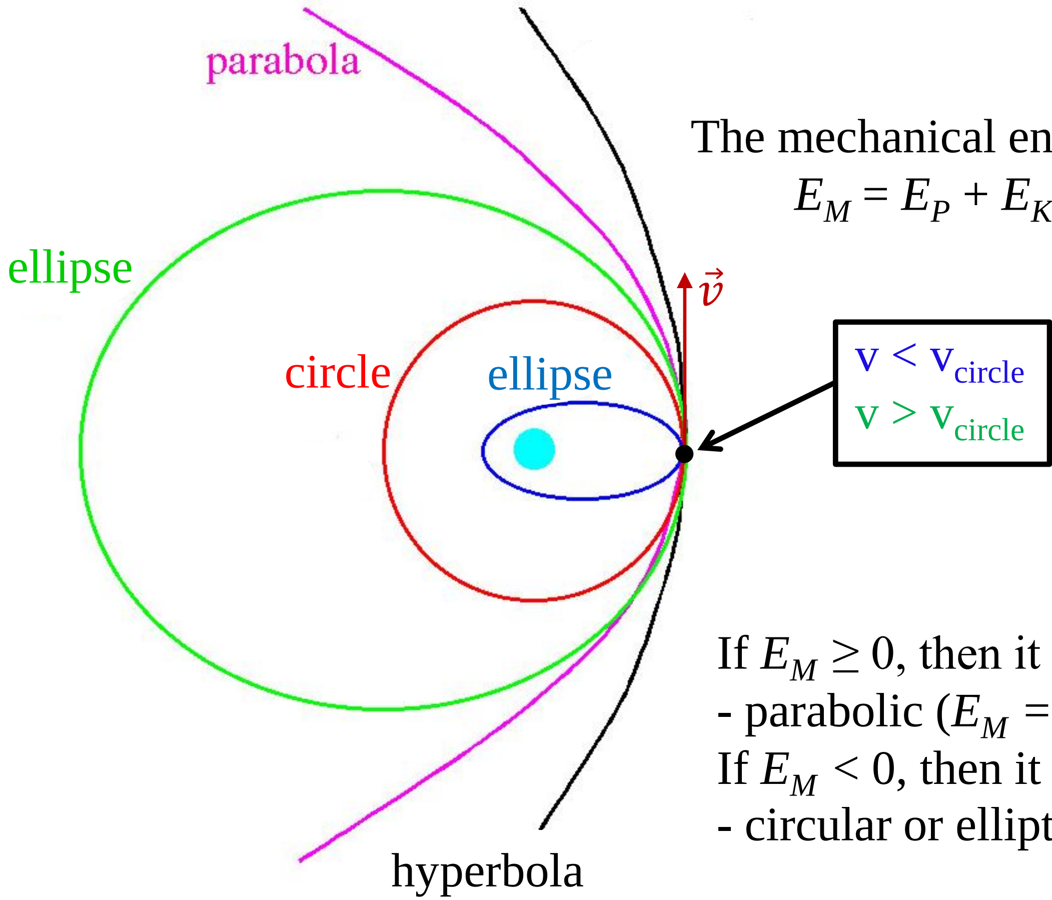
We get the fundamental equation of rotary motion: $\tau = I\beta$

The **kinetic energy** of the point mass: $E_K = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$

Movement of planets and moons

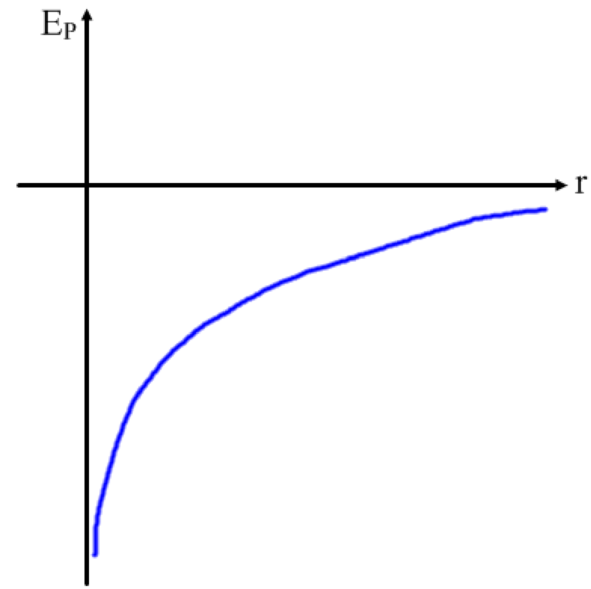
Suppose a body of mass m moves in the gravitational field of a body of much larger mass (M). Since M is much larger than m, it can be considered stationary. e.g. Sun and Earth.

A body of mass m has E_K kinetic energy and E_P potential energy.
We take the zero point at infinity: $E_P = 0$, if $r \rightarrow \infty$



The mechanical energy:

$$E_M = E_P + E_K$$



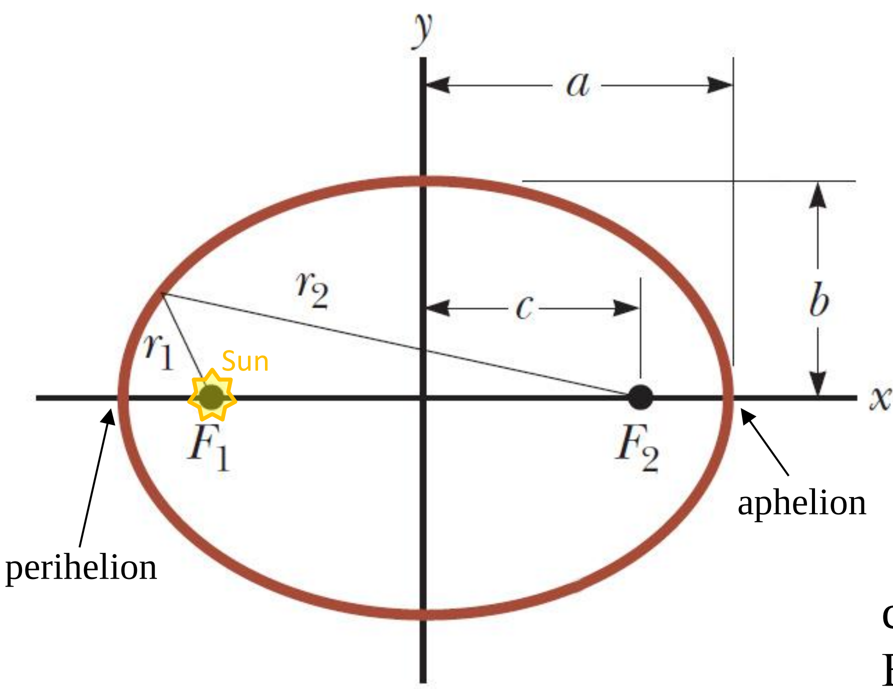
If $E_M \geq 0$, then it is free (can move away to infinity)
- parabolic ($E_M = 0$) or hyperbolic trajectory

If $E_M < 0$, then it is bound (remains at a finite distance)
- circular or elliptical orbit

Kepler's 1st law

For bodies moving in a bound state in the gravitational field of a massive body (e.g. planets).

I. The orbits of the planets are ellipses, and the Sun is at one of its foci.



<u>ellipse:</u> points for which $r_1 + r_2 = \text{constant}$

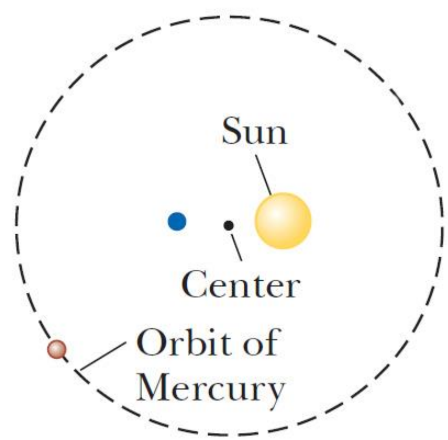
$$a^2 = b^2 + c^2$$

a: semi-major axis
b: semi-minor axis

eccentricity: $e = c/a$

$$0 < e < 1$$

circle: $e = 0$
Earth's orbit: $e = 0.017$
Mercury's orbit: $e = 0.21$
Comet Halley: $e = 0.97$



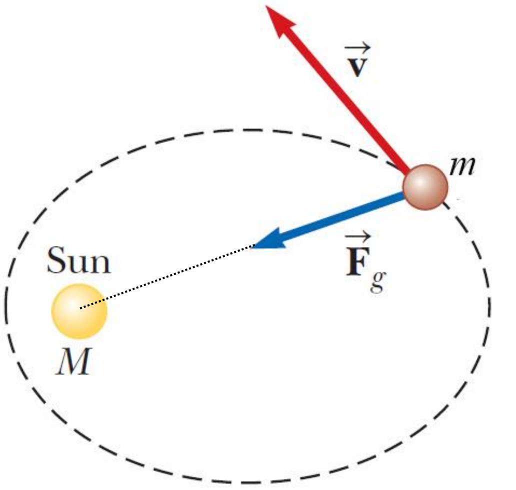
For moons around planets: perigee and apogee

Kepler's 2nd law

II. (Equal area law) The lines between the Sun and the planets sweep out equal areas in equal times. Planets move faster near the perihelion.
It follows from the conservation of angular momentum: there is no torque in a central force field.

$$\vec{L} = \text{constant}$$
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$$

For its magnitude: $L = m|\vec{r} \times \vec{v}|$

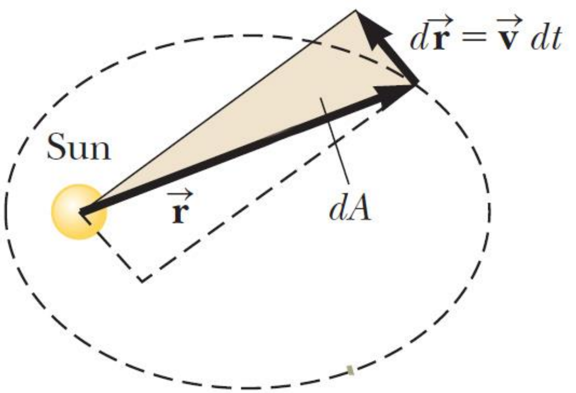


In time dt the area dA is swept out:

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}dt| = \frac{1}{2}|\vec{r} \times \vec{v}|dt$$

$$dA = \frac{L}{2m}dt$$

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

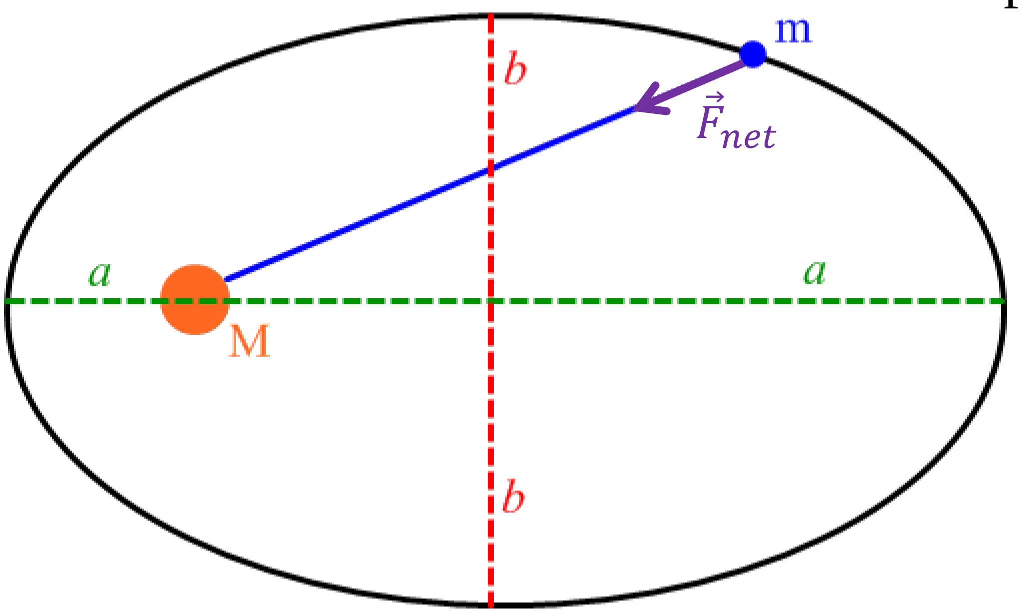


Kepler's 3rd Law

III. The cubes of the semi-major axes (*a*) of elliptical orbits are proportional to the squares of the orbital periods (*T*) of the planets moving in the given orbits.

So for every planet (and any body) orbiting the Sun: $\frac{a^3}{T^2} = \text{constant}$

All three laws can be derived from Newton's laws and the Newtonian force law of gravity.



Proof for a circular orbit: *a* = *b* = *R*

$$F_{net} = ma$$

$$\gamma \frac{Mm}{R^2} = m\omega^2 R$$

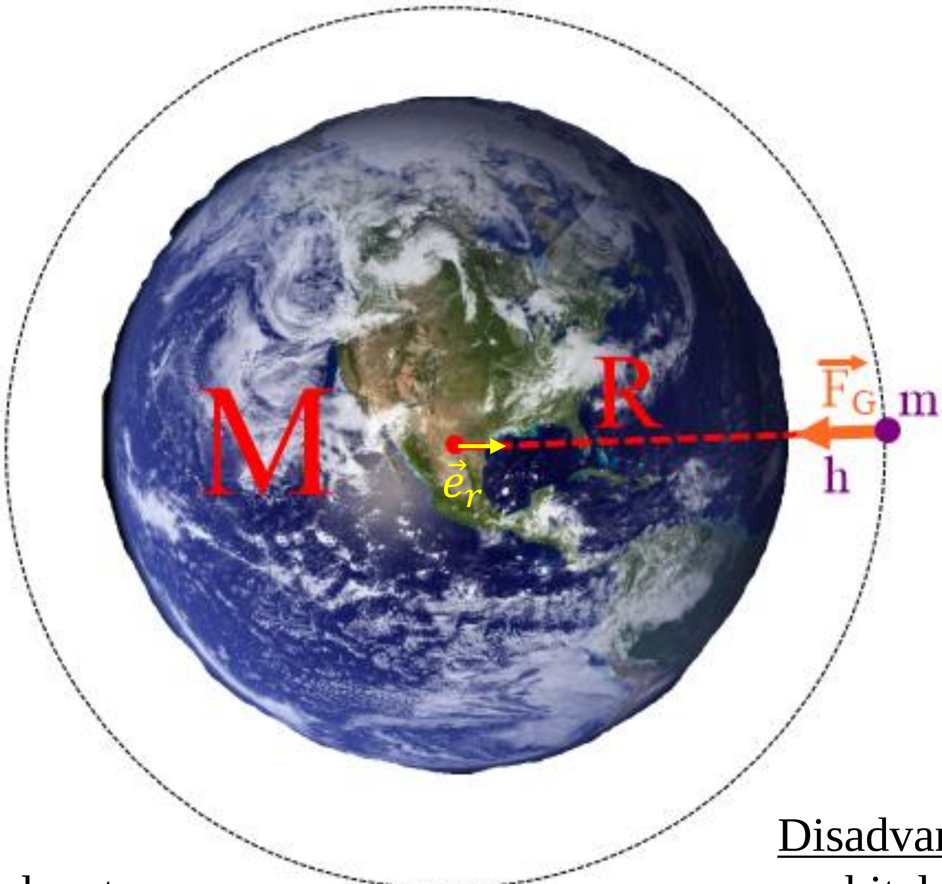
$$\gamma \frac{M}{R^2} = \frac{4\pi^2}{T^2} R$$

$$\frac{\gamma M}{4\pi^2} = \frac{R^3}{T^2} = \text{constant}$$

Orbits of satellites

Low Earth orbit (LEO)

Circular orbits at altitudes between 160km and 2000km. For the ISS it is 408km.



The only force acting is gravity:

$$\vec{F}_{net} = m\vec{a} = -ma_{cp}\vec{e}_r$$

$$\vec{F}_{net} = \vec{F}_G = -G\frac{Mm}{(R+h)^2}\vec{e}_r$$

$$G\frac{M}{(R+h)^2} = a_{cp} = \frac{v^2}{R+h}$$

$$\frac{GM}{(R+h)^2} = \omega^2(R+h) = \frac{4\pi^2}{T^2}(R+h)$$

$$T^2 = \frac{4\pi^2}{GM}(R+h)^3$$

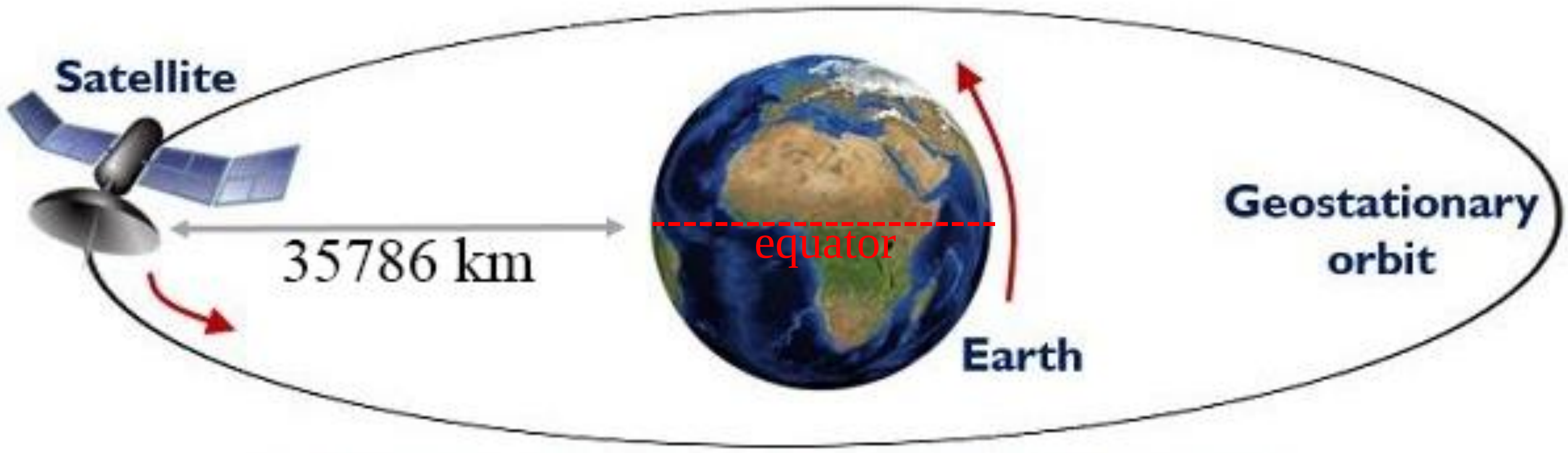
$$T = \frac{2\pi}{\sqrt{GM}}(R+h)^{3/2}$$

Advantages:
- quick communication
- low launch cost
- high-resolution imaging

Disadvantages:
- orbital decay
- space debris
- atomic oxygen
- small viewing area

Geostationary orbit (GEO)

Circular orbit above the equator with exactly one day period.



$$T = \frac{2\pi}{\sqrt{GM}}(R+h)^{3/2}$$

<u>Advantages:</u>
- large viewing area
- always above same spot

<u>Disadvantages:</u>
- lag because of distance
- crowded area