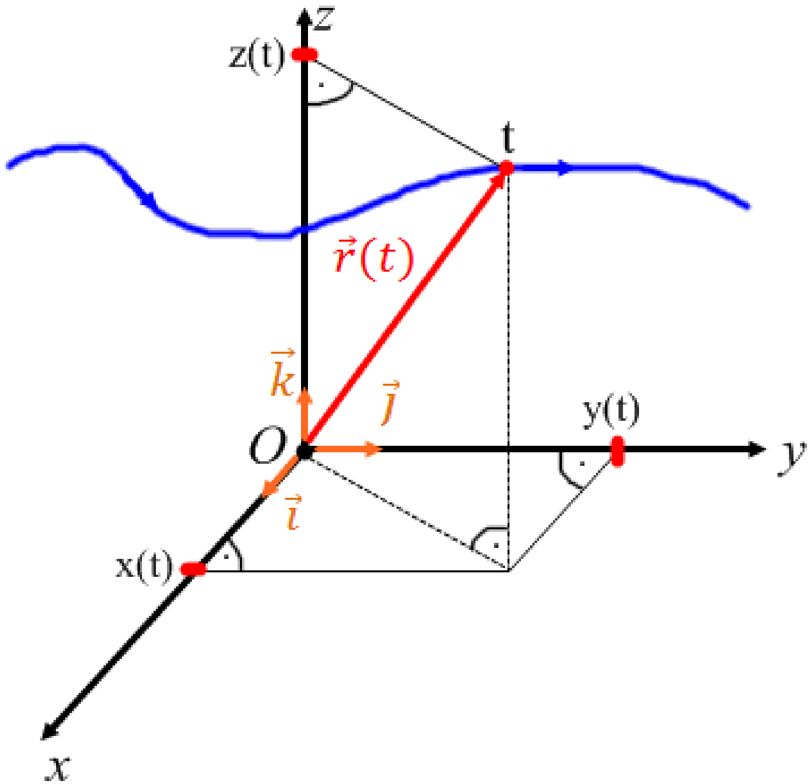
Coordinate-systems

Cartesian coordinate system

The path can be written as:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

The *x(t), y(t), z(t)* functions are the coordinates.



The $\vec{i}, \vec{j}, \vec{k}$ unit vectors are the **base vectors**.

With Pythagorean: $r = \sqrt{x^2 + y^2 + z^2}$

The velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

The speed is then: $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

Acceleration and its magnitude:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

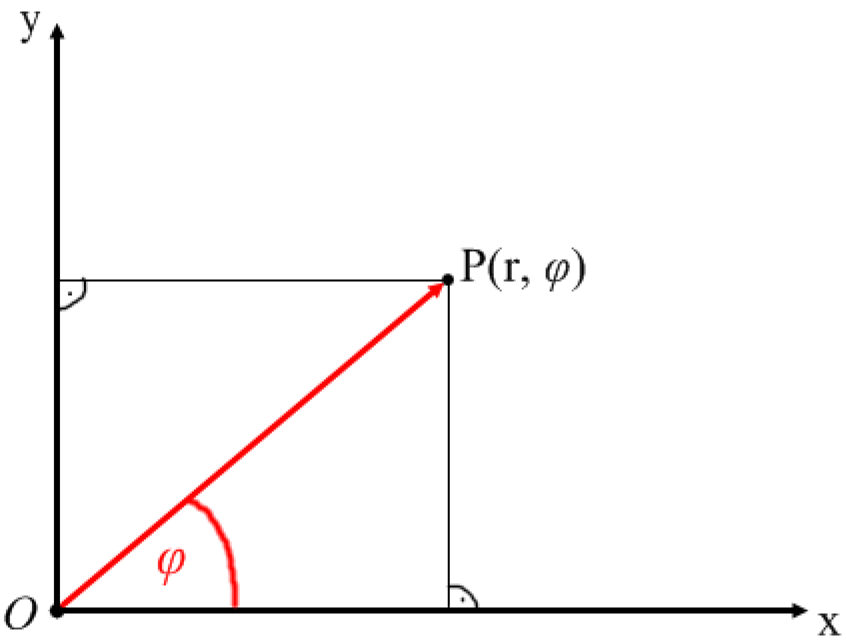
$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

Polar coordinate system

The two coordinates: distance from a point and the angle from a direction.
It can be used well for circular motion if the center is the origin.

Using Pythagorean and the tangent:

$$r = \sqrt{x^2 + y^2} \qquad \tan\varphi = \frac{y}{x}$$



Transforming coordinates the other way:

$$x = r\cos\varphi \qquad y = r\sin\varphi$$

The rate of change of the φ angle is the **angular velocity** (unit: 1/s):

$$\omega = \frac{d\varphi}{dt}$$

The rate of change of the angular velocity is the **angular acceleration** (unit: 1/s^2):

$$\beta = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2}$$

Uniform circular motion

The angular velocity is constant: $\omega = \dfrac{2\pi}{T}$ $\beta = 0$
(where T is the period)

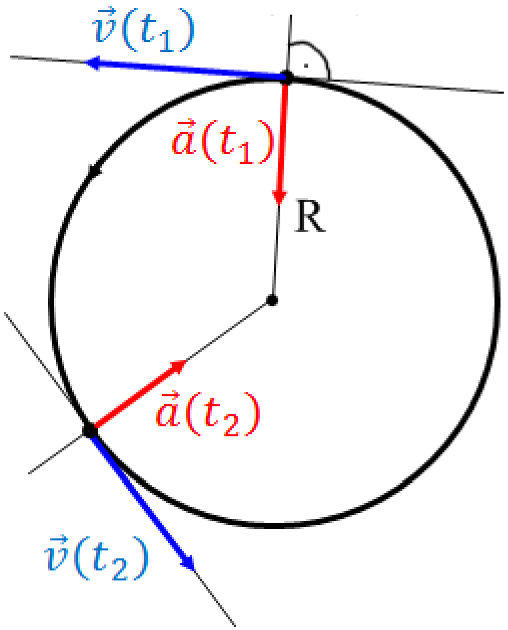
In T time it covers the circumference: $s(T) = 2R\pi$

The magnitude of the velocity (speed) is constant (however its direction is changing!):

$$v = \frac{s(T)}{T} = \frac{2\pi R}{T} = R\omega$$

Since the direction of the velocity keeps changing, the acceleration is not zero. Its magnitude is constant, and its direction is toward the center at all times (centripetal):

$$a = a_{cp} = \frac{v^2}{R} = \frac{R^2\omega^2}{R} = R\omega^2$$



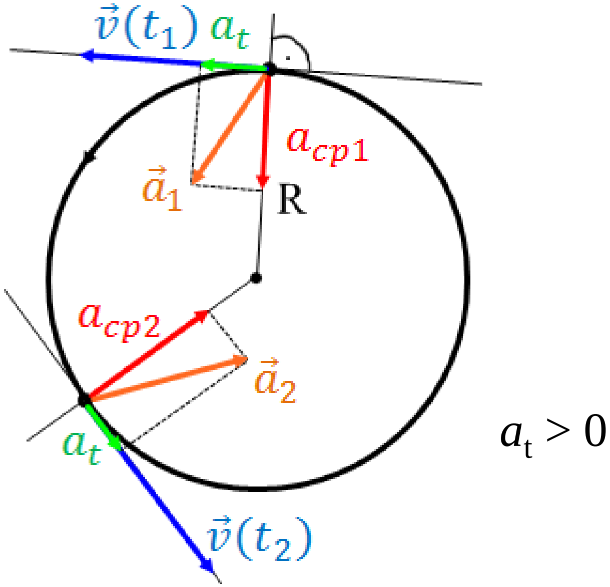
Uniformly changing circular motion

The angular acceleration β = constant, therefore the angular velocity changes linearly:

$$\omega(t) = \beta t + \omega_0$$

Because of the constant angular acceleration the acceleration will have a constant magnitude tangential component. Because of that the speed changes uniformly:

$$a_t = \beta R = \frac{dv}{dt}$$

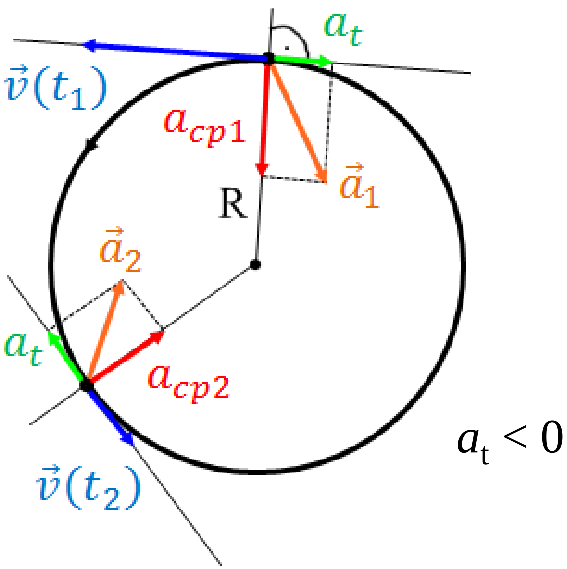


Pythagorean gives the magnitude of the acceleration:

$$a = \sqrt{{a_{cp}}^2 + {a_t}^2}$$

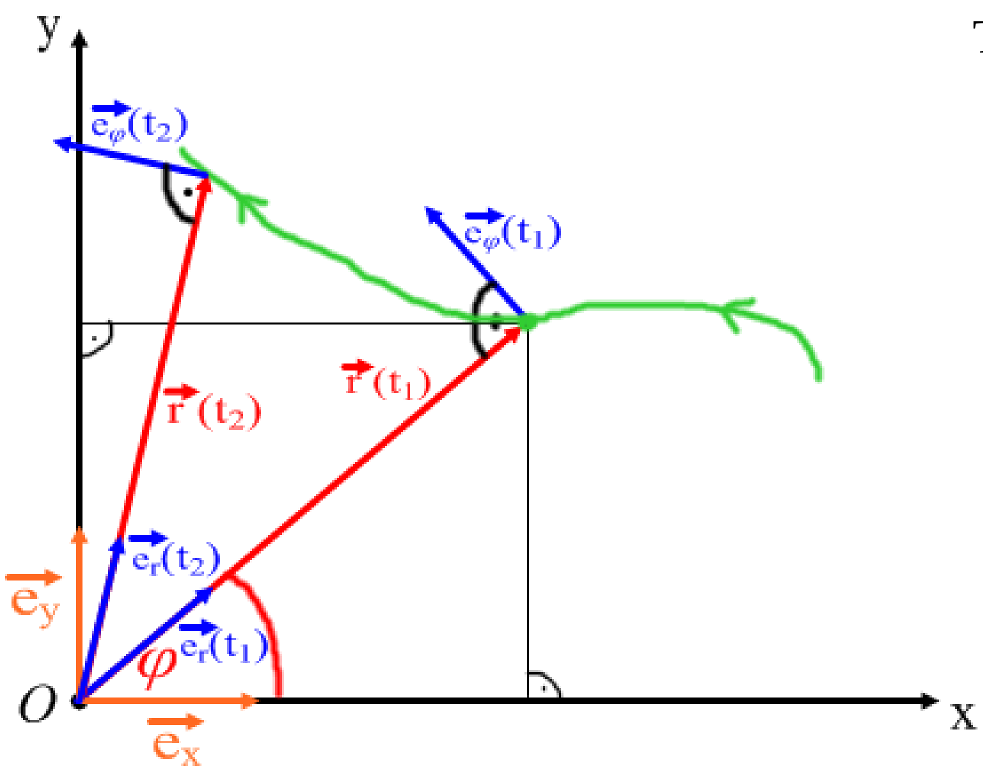
The path traveled only depends on the tangential acceleration component (the other only changes the direction):

$$s(t) = \frac{1}{2} a_t t^2 + v_0 t$$



Describing general motion using polar coordinates

When using the polar coordinates to decribe a motion, we have to keep in mind that the $\vec{e}_r$ és $\vec{e}_\varphi$ base vectors depend on the position of the object, thus on time.
See diagram:



The momentary position of the object:

$$\vec{r} = r\vec{e}_r$$

However, when we calculate the velocity we also have to determine the derivative of the $\vec{e}_r$ base vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\vec{e}_r)}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt}$$

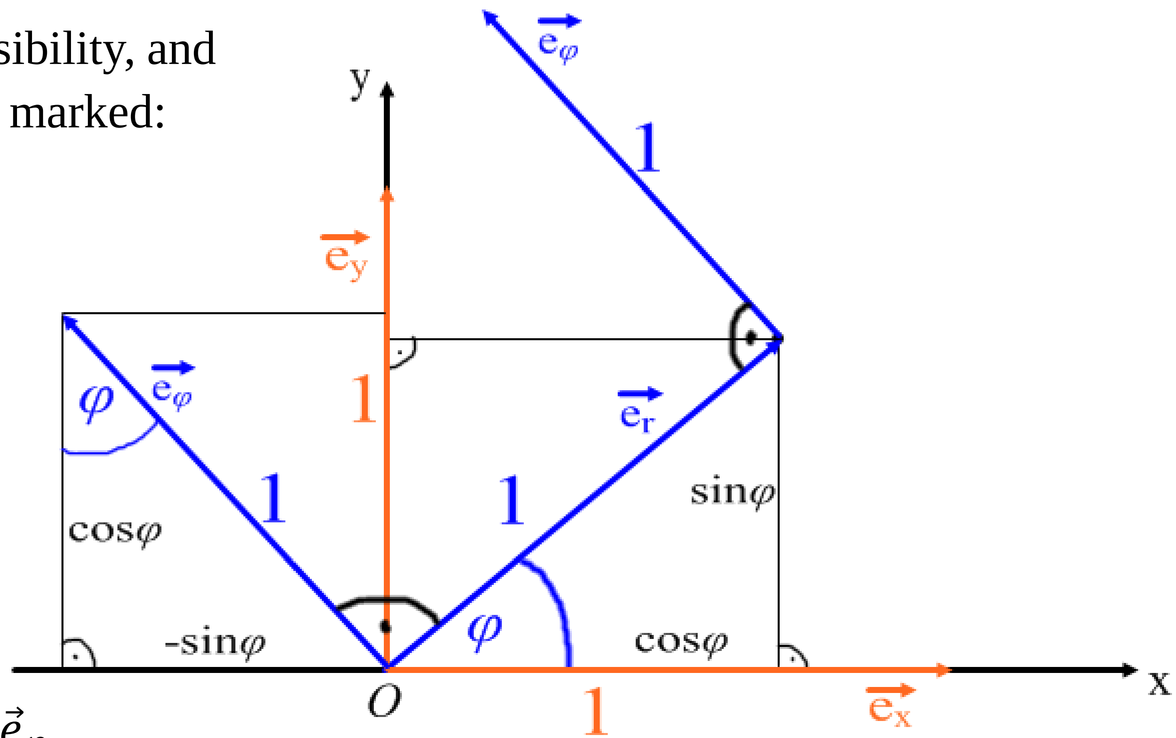
Using the dot for time derivative:

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r$$

In order to determine the derivative of the polar base vectors, let's express them with the constant $\vec{e}_x$ és $\vec{e}_y$ Cartesian base vectors…

Derivatives of the polar base vectors

The $\vec{e}_\varphi$ is shifted to the origin for visibility, and the unit length of the base vectors is marked:



$$\vec{e}_r = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y$$

$$\vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y$$

Since φ also depends on time, using the chain rule we end up with a $\dot{\varphi}$ factor.

Thus the derivatives:

$$\dot{\vec{e}}_r = -\sin\varphi\,\dot{\varphi}\vec{e}_x + \cos\varphi\,\dot{\varphi}\vec{e}_y = \dot{\varphi}\vec{e}_\varphi$$

$$\dot{\vec{e}}_\varphi = -\cos\varphi\,\dot{\varphi}\vec{e}_x - \sin\varphi\,\dot{\varphi}\vec{e}_y = -\dot{\varphi}\vec{e}_r$$

Now the velocity and acceleration can be written:

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi = \dot{r}\vec{e}_r + r\omega\vec{e}_\varphi$$

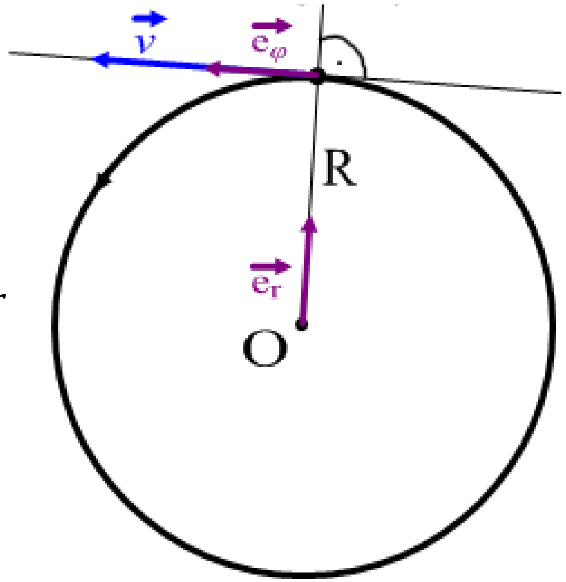
$$\vec{a} = \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi + r\dot{\varphi}\dot{\vec{e}}_\varphi = \ddot{r}\vec{e}_r + \dot{r}\dot{\varphi}\vec{e}_\varphi + \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi - r\dot{\varphi}\dot{\varphi}\vec{e}_r =$$

$$= (\ddot{r} - r\dot{\varphi}^2)\vec{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\vec{e}_\varphi = (\ddot{r} - r\omega^2)\vec{e}_r + (2\dot{r}\omega + r\beta)\vec{e}_\varphi$$

Uniform and uniformly changing circular motion

For circular motion: $\dot{r} = \ddot{r} = 0$

For uniform circular motion: $\ddot{\varphi} = \beta = 0$
Furthermore: $\dot{\varphi} = \omega =$ constant and $r = R$



$$\vec{v} = \dot{r}\vec{e}_r + r\omega\vec{e}_\varphi = R\omega\vec{e}_\varphi = v\vec{e}_\varphi \qquad (v = \text{constant})$$

$$\vec{a} = (\ddot{r} - r\omega^2)\vec{e}_r + (2\dot{r}\omega + r\beta)\vec{e}_\varphi = -R\omega^2\vec{e}_r = -a_{cp}\vec{e}_r$$

$a_{cp} =$ constant magnitude
centripetal acceleration component

For uniformly changing circular motion: $\ddot{\varphi} = \beta =$ constant

$$\vec{v} = \dot{r}\vec{e}_r + r\omega\vec{e}_\varphi = R\omega\vec{e}_\varphi = v\vec{e}_\varphi \qquad (v = \text{momentary speed})$$

$$\vec{a} = (\ddot{r} - r\omega^2)\vec{e}_r + (2\dot{r}\omega + r\beta)\vec{e}_\varphi = -R\omega^2\vec{e}_r + R\beta\vec{e}_\varphi = -\frac{v^2}{R}\vec{e}_r + R\beta\vec{e}_\varphi = -a_{cp}\vec{e}_r + a_t\vec{e}_\varphi$$

$a_{cp} =$ momentary centripetal acceleration component
$a_t =$ constant magnitude tangential acceleration component

Cylindrical coordinate system

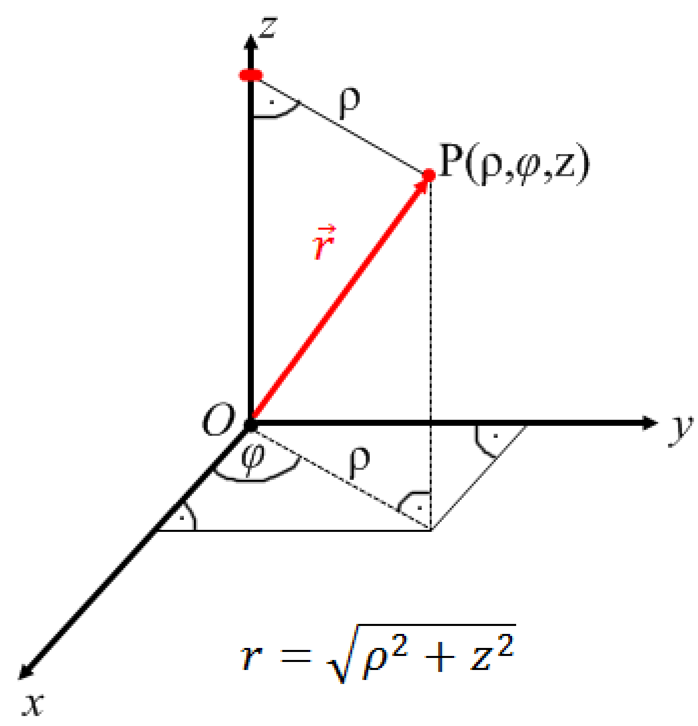
In addition to the two polar coordinates, we also use the *z* coordinate of the Cartesian coordinate system.

It can be used for 3-dimensional motion, especially spiral motion.

As opposed to the polar coordinate system, here instead of *r* we use ρ to give the distance from the axis (this is not the distance from the origin, which is still *r*).

$$x = \rho \cos\varphi \qquad y = \rho \sin\varphi \qquad z = z$$

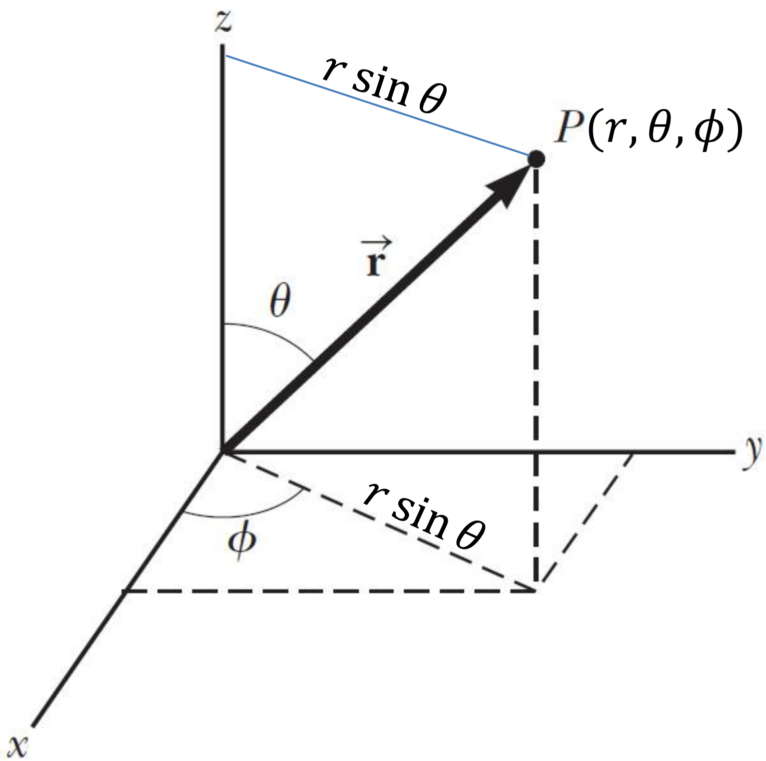
$$\rho = \sqrt{x^2 + y^2} \qquad \tan\varphi = \frac{y}{x}$$



$$r = \sqrt{\rho^2 + z^2}$$

Spherical coordinate system

It can be used to describe spherically symmetric motion, like motion along a sphere:



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

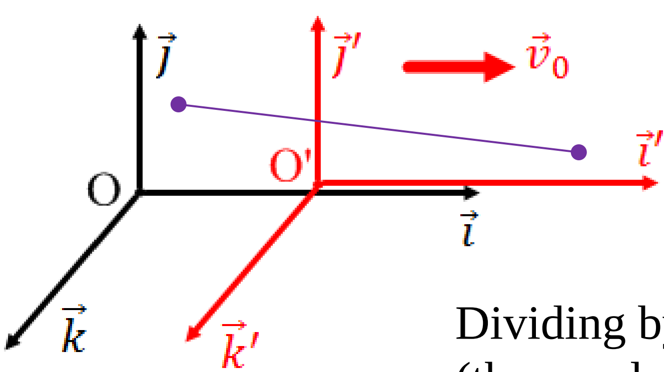
$$\sin\theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\operatorname{tg}\phi = \frac{y}{x}$$

Galilean relativity principle

In any two reference systems moving at constant velocity relative to each other, the **mechanical phenomena** will happen the **same way**.
E.g. on a moving train we do not feel any difference. The dropped coin falls down the same way. Thus none of these systems can be denoted as an absolutely stationary reference system.

Connection between two systems moving relative to each other:
Let *S'* system move relative to *S* in the positive *x* direction at **constant** v_0 speed.



In Δt time the distance between origins: $\overline{OO'} = v_0 \Delta t$
Thus the coordinate differences measured in *S'*:

$$\Delta x' = \Delta x - v_0 \Delta t$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$

Furthermore: $\Delta t' = \Delta t$ (clocks synchronized)

Dividing by Δt (or $\Delta t'$) we get the relation between the velocities (the purple line is a section of the path of a moving object):

$$v_x' = v_x - v_0$$
$$v_y' = v_y$$
$$v_z' = v_z$$

Writing in vector form we get a formula valid in the general case:

$$\boxed{\vec{v}' = \vec{v} - \vec{v}_0}$$

Fundamental equation of dynamics

If we combine Newton's 1st, 2nd, and 4th law, then we get the **fundamental equation of dynamics**:

$$\vec{F}_{net} = \sum_{i=1}^{n} \vec{F}_i = m\vec{a}$$

If we write out each component, include the appropriate force laws, then we get the **equations of motion**. For example in Cartesian coordinate system:

$$\left.\begin{aligned} m\ddot{x} &= F_{netx}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \\ m\ddot{y} &= F_{nety}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \\ m\ddot{z} &= F_{netz}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \end{aligned}\right\} \text{second order coupled differential equations}$$

The forces cannot be the function of the acceleration, because that would be at odds with the principle of superposition.

In order to solve to motion, 6 integration constants must be given. These are generally the 3 coordinates of the **initial position** and the 3 components of the **initial velocity**: $\vec{r}_0$ and $\vec{v}_0$

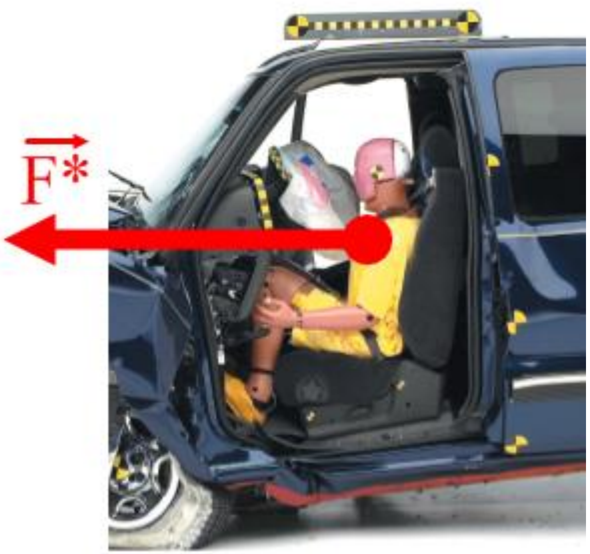
Solving the equations gives the path of the object, i.e. the position of the object as a function of time:

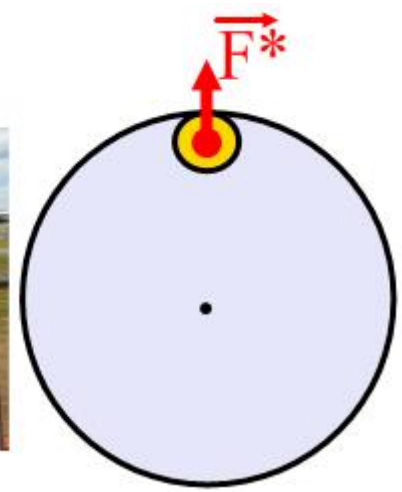
$$\vec{r}(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$$

Inertial forces*

These are **not real** (caused by some interaction) forces, but rather some **fictitious** forces or apparent forces.
These forces occur if the coordinate system is not inertial.
Accelerating systems are not inertial systems.

<u>Examples:</u>
- braking, accelerating of turning car
- spinning carousel
- strictly speaking any planet or moon (rotation and orbiting)



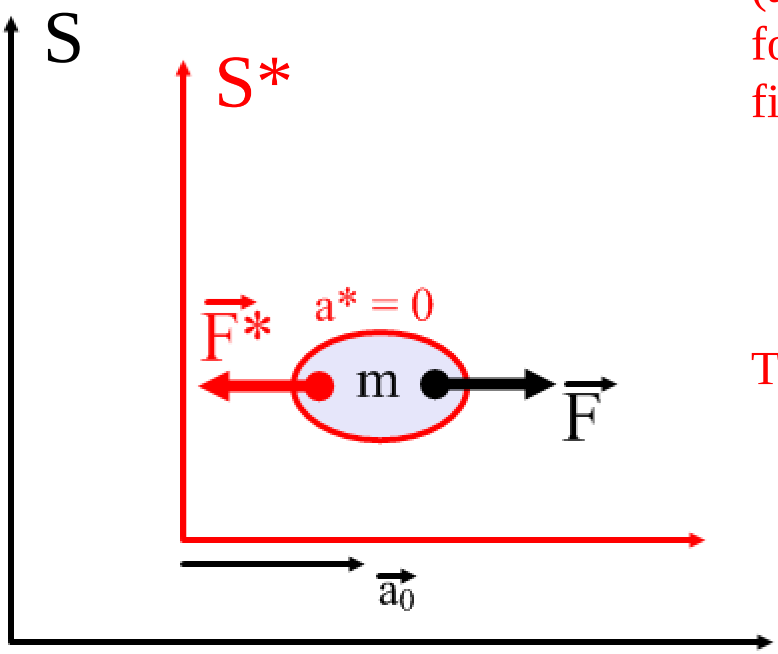


Accelerating reference systems

In certain cases it may be needed to describe the motion in an accelerated reference system (or the inertial forces due to the rotation of the Earth cannot be ignored).

Inertial system (S):
Object moves with $\vec{a}_0$ acceleration (together with the S* system) due to the force $\vec{F}$ acting on it.

$$\vec{F} = m\vec{a}_0$$



Accelerating system (S*):
The object is at rest ($a^* = 0$), because the net force is zero (including the fictitious $\vec{F}^*$ force.)

$$\vec{F} + \vec{F}^* = 0$$

$$m\vec{a}_0 + \vec{F}^* = 0$$

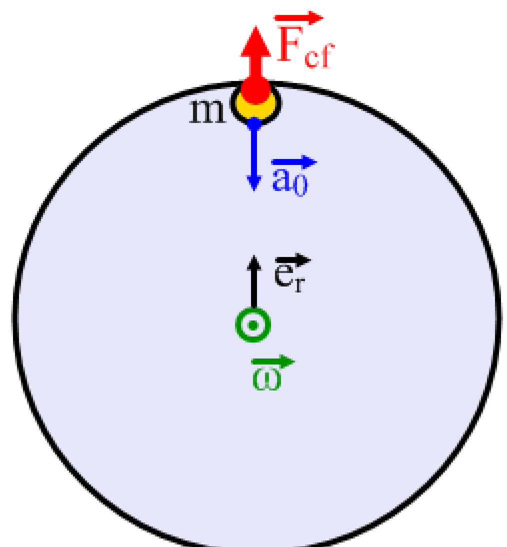
Thus: $$\boxed{\vec{F}^* = -m\vec{a}_0}$$

Centrifugal force

In order to determine the **inertial force**, first we need to determine the acceleration of the object as seen from an inertial reference system.
Then we can use: $\vec{F}^* = -m\vec{a}_0$

Centrifugal force*:
It acts on an object doing circular motion, points radially outward.
(opposite to centripetal acceleration)

$$\vec{F}_{cf} = -m\vec{a}_0 = m\frac{v^2}{R}\vec{e}_r = m\omega^2 R\vec{e}_r$$



Uniformly changing motion along a straight line

If the initial velocity vector and the acceleration vector fall in the same line, then the body will move along that line (one coordinate is enough, e.g. z).

Thus, the body moves in the direction of the *z* axis with constant a_z acceleration and a time-dependent velocity v_z (these components can also be negative!). The other components (*x*, *y*) are zero. The velocity at time t_1 is:

$$v_z(t_1) = \int_{t_0}^{t_1} a_z(t)\,dt + v_z(t_0) = a_z(t_1 - t_0) + v_z(t_0)$$

let $t_0 = 0$

$v_z(t_0) = v_{z0}$

With these: $$\boxed{v_z(t) = a_z t + v_{z0}}$$

$(\Delta v_z = a_z \Delta t\ )$

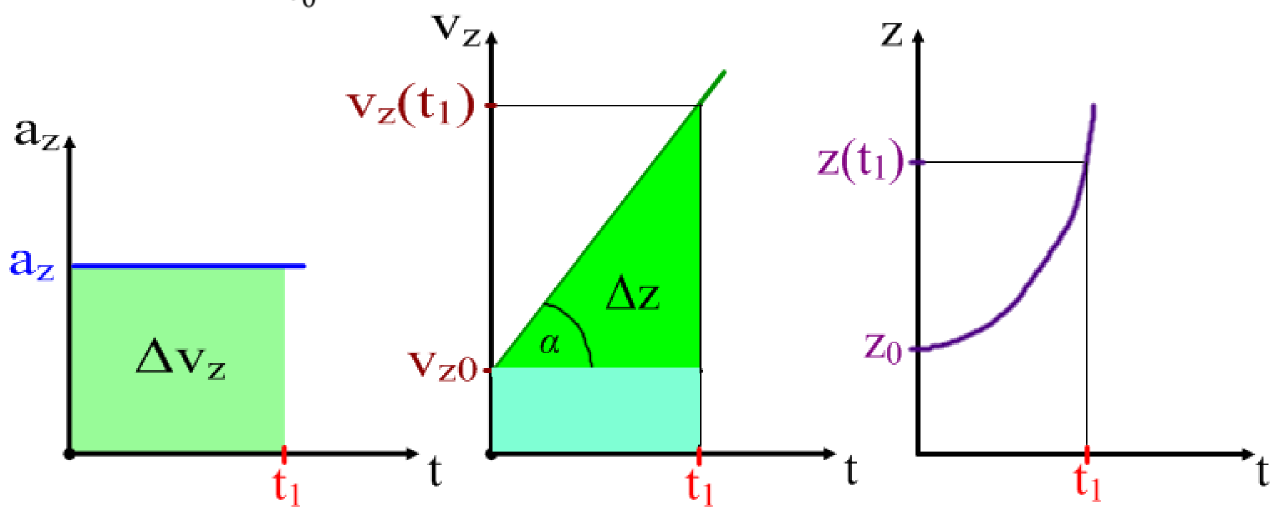
The position:

$$z(t_1) = \int_{t_0}^{t_1} v_z(t)\,dt + z(t_0) = \int_{t_0}^{t_1} (a_z t + v_{z0})\,dt + z(t_0)$$

let $z(t_0) = z_0$

Thus if $t_0 = 0$:

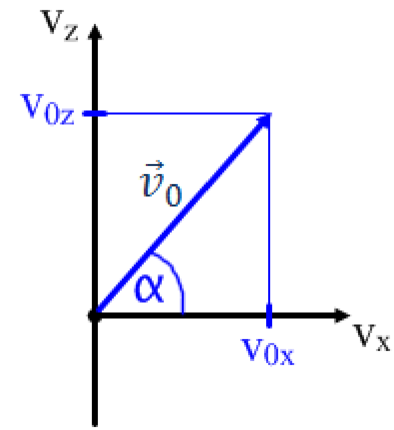
$$\boxed{z(t) = \frac{1}{2} a_z t^2 + v_{z0} t + z_0}$$



Projectile motion

The acceleration is constant (g), but is not parallel with the initial velocity vector:
Projecting the initial velocity (2D – *x* and *z*):

$$v_{0x} = v_0 \cos\alpha \qquad v_{0z} = v_0 \sin\alpha$$



The acceleration: $\vec{a} = -g\vec{k}$

The velocity-time function: $\vec{v}(t) = v_{0x}\vec{i} + 0\vec{j} + (-gt + v_{0z})\vec{k}$

The position vector: $\vec{r}(t) = v_{0x}t\vec{i} + 0\vec{j} + \left(-\frac{g}{2}t^2 + v_{0z}t\right)\vec{k}$

The body hits the ground when *z* = 0:

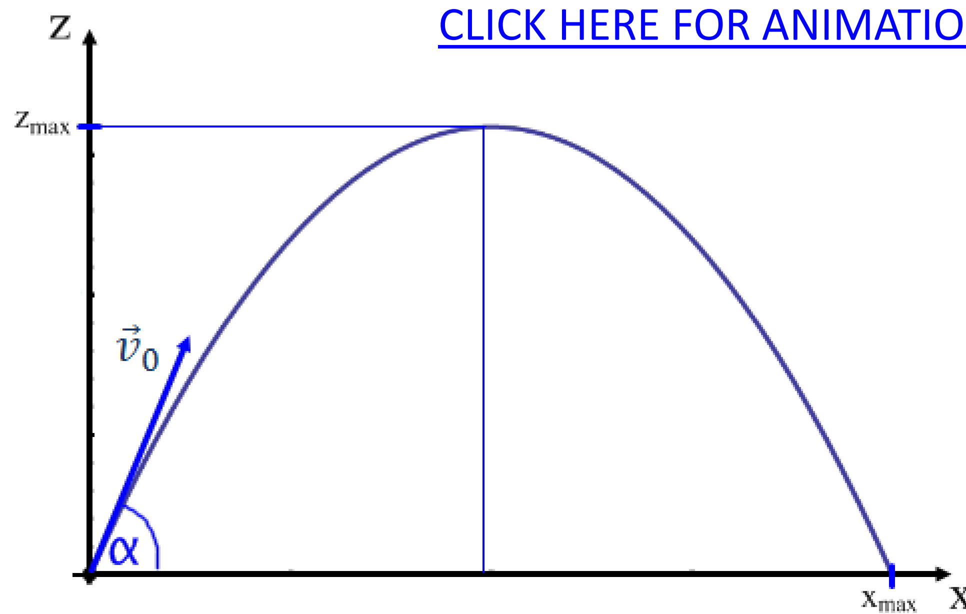
$$-\frac{g}{2}t^2 + v_0 \sin\alpha\, t = 0$$

Solving for time: $t = \dfrac{2v_0 \sin\alpha}{g}$

Substituting into the *x* coordinate we get the range of the throw:

$$x_{max} = \frac{2v^2{}_0 \sin\alpha \cos\alpha}{g}$$

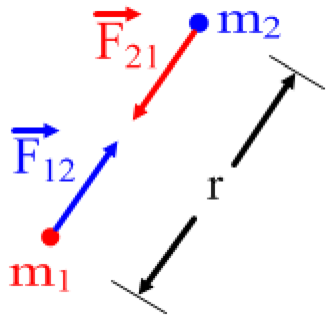
CLICK HERE FOR ANIMATION!



Newtonian gravitational force

The force between two point masses is proportional to the product of the two masses and inversely proportional to the square of their distance.

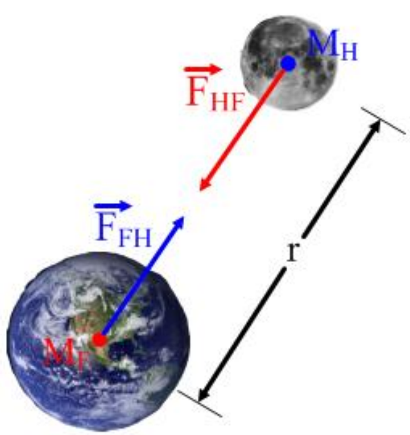
$$F = \frac{\gamma m_1 m_2}{r^2}$$



The interaction is always attractive.

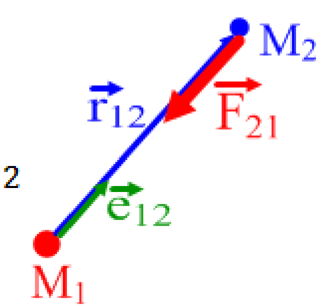
The proportionality factor is the universal gravitational constant: $\gamma = 6{,}67 \cdot 10^{-11}\,\frac{\mathrm{Nm}^2}{\mathrm{kg}^2}$

The simple form of the force law also applies to extended bodies, as long as they are spherically symmetric. The distance is measured between the centers.



The vector form also gives the direction of the force:

$$\vec{F}_{21} = -\frac{\gamma M_1 M_2}{r^2{}_{12}}\frac{\vec{r}_{12}}{r_{12}} = -\frac{\gamma M_1 M_2}{r^2}\vec{e}_{12}$$



Weight force

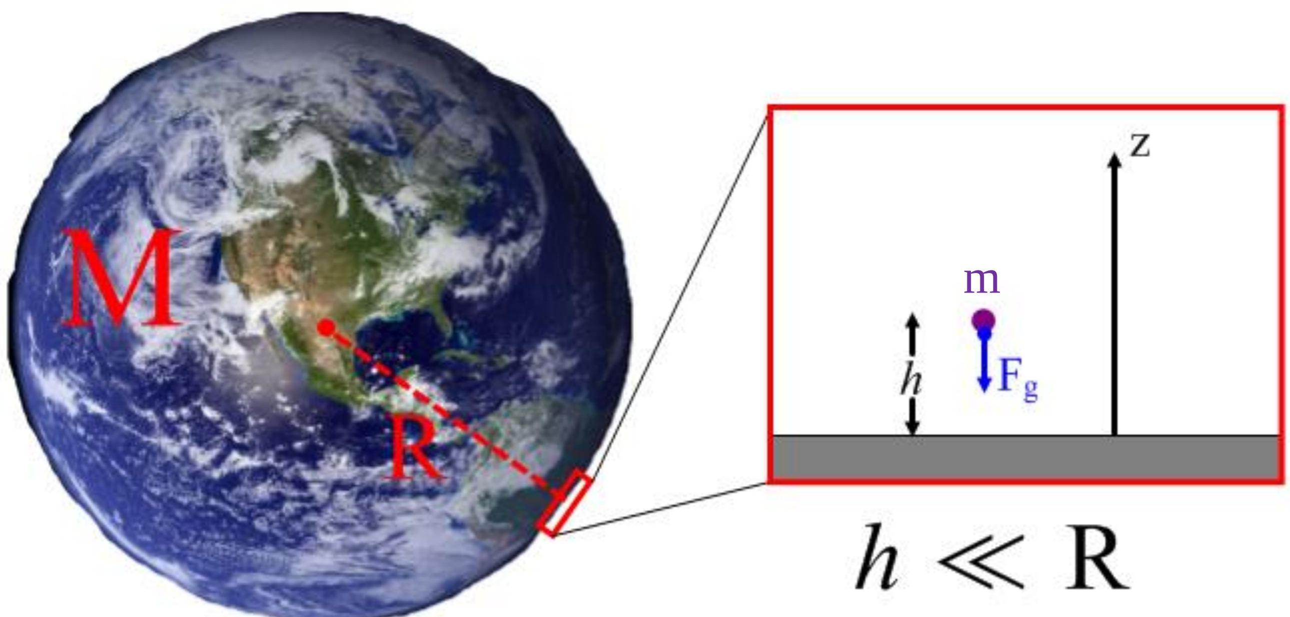
When the displacement of the body is negligible compared to the radius of the planet or moon, the gravitational force can be considered homogeneous (independent of location).

E. g. for movements occurring near the surface of the Earth from the general Newtonian force law we get:

$$\vec{F}_g = -\frac{\gamma M m}{r^2}\vec{e}_r = -\frac{\gamma M m}{(R+h)^2}\vec{e}_r \approx -\frac{\gamma M m}{R^2}\vec{e}_r = -mg\vec{k}$$

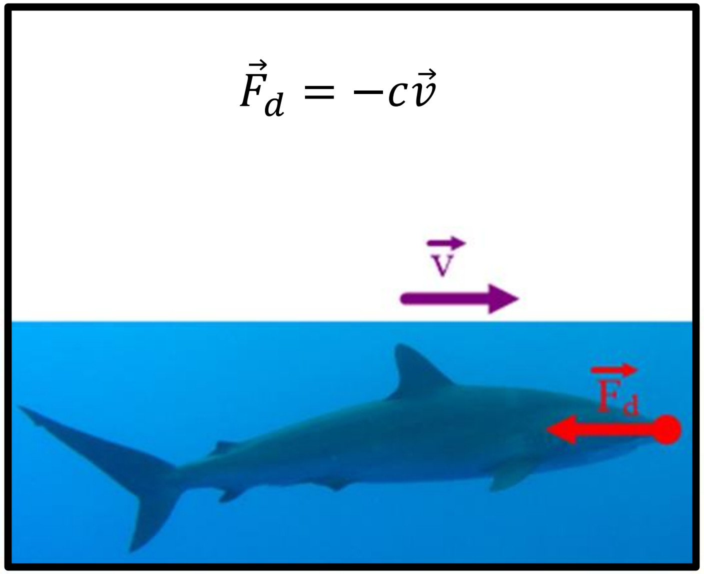
$$g = \frac{\gamma M}{R^2} = 9{,}8\,\frac{\text{m}}{\text{s}^2}$$

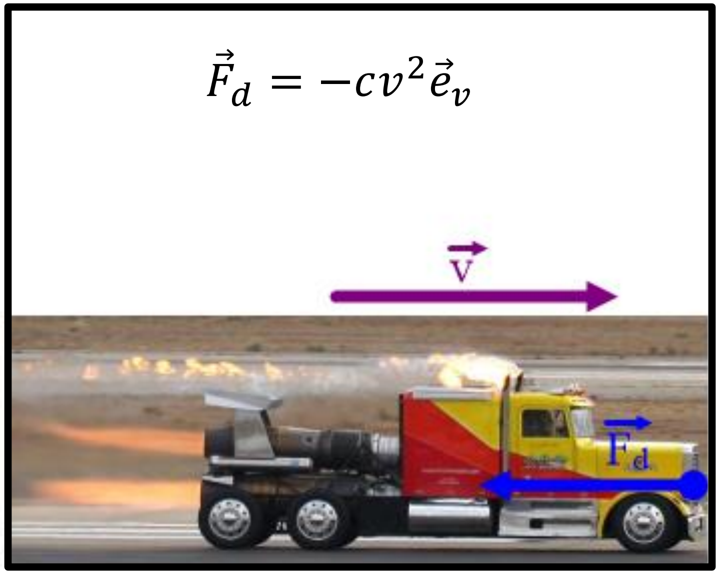
gravitational acceleration on the surface of the Earth



Drag forces

It is proportional to the velocity of the body, or the square of the velocity, and in the opposite direction.



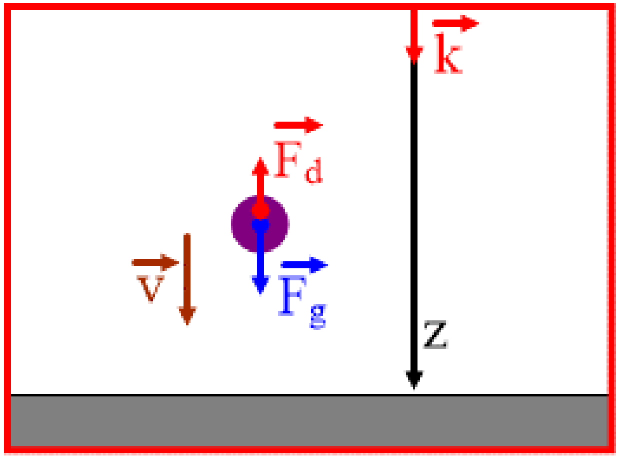


The coefficient c depends on:
- the size of the body's surface perpendicular to the movement
- the shape of the body (how streamlined it is) – drag coefficient C_d
- the density of the medium

$$c = \frac{1}{2}\rho A C_d$$

Free fall with drag

In addition to its weight, the object is also affected by the air resistance (drag).
Starting with relatively large v_0 speed downward.



$$m\vec{a} = \vec{F}_{net} = mg\vec{k} - cv^2\vec{k} \qquad c = \frac{1}{2}\rho A C_d$$

Motion along z axis:

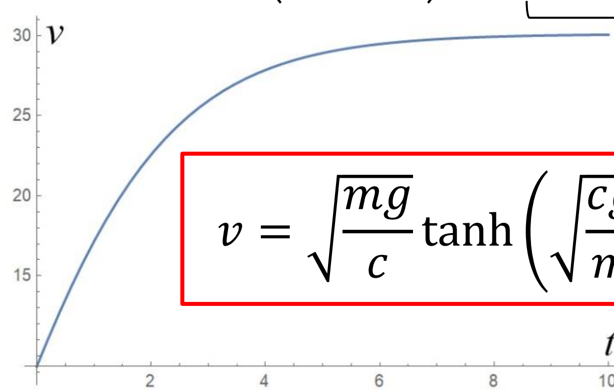
$$m\frac{dv}{dt} = mg - cv^2 \quad \rightarrow \quad \frac{dv}{dt} = g - \frac{c}{m}v^2$$

$$\frac{dv}{g - \frac{c}{m}v^2} = dt \quad \rightarrow \quad \int_{v_0}^{v} \frac{dv'}{1 - \frac{c}{mg}v'^2} = \int_0^t g\,dt'$$

$$\left[\sqrt{\frac{mg}{c}}\tanh^{-1}\left(\sqrt{\frac{c}{mg}}v'\right)\right]_{v_0}^{v} = gt \quad \rightarrow \quad \tanh^{-1}\left(\sqrt{\frac{c}{mg}}v\right) - \underbrace{\tanh^{-1}\left(\sqrt{\frac{c}{mg}}v_0\right)}_{A} = \sqrt{\frac{c}{mg}}gt$$

$$\sqrt{\frac{c}{mg}}v = \tanh\left(\sqrt{\frac{cg}{m}}v + A\right) \quad \rightarrow$$

$$\boxed{v = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}v + A\right)}$$



$C_d = 0.2$
$r = 0.11$
$\rho = 1.225$
$c = 0.004655$
$m = 0.43$
$v_0 = 10$
(soccer ball)