Conservation of mass

Hydrodynamics is the field describing the flow of liquids as continuous media.

Two descriptions:

1. Lagrange-method: writing Newton's equations of motion for the selected liquid elements, then solving those using the initial conditions.

2. Euler-method: measuring the properties of the flowing liquid at selected points.
(e.g. pressure, velocity, density).
If these are constants at all points, then the flow is **stationary**.

Continuity equation: mass is conserved, it cannot appear or disappear.
Let's consider a volume *V* bounded by closed surface *S*. In *dt* time the mass flowing out through *dA* is $dm = \rho dV = \rho dA v \cos\alpha\, dt = \rho\vec{v}\cdot d\vec{A}dt$

Thus in unit time: $\frac{dm}{dt} = \rho\vec{v}\cdot d\vec{A}$

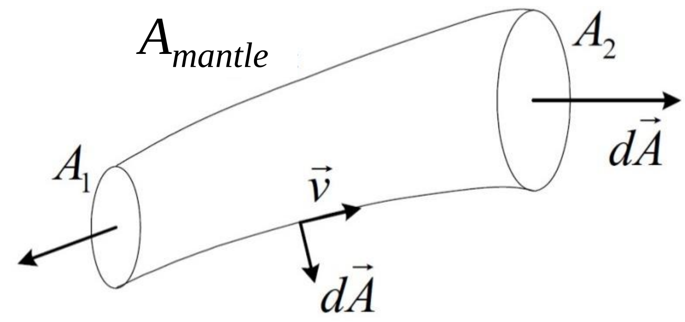
The mass flowing out from *V* through the whole *S* in unit time equals the negative change in the mass in *V* per unit time (negative derivative of the total mass in *V*):

$$-\frac{d}{dt}\int_V \rho\, dV = \oint_S \rho\vec{v}\cdot d\vec{A}$$

Continuity equation for stationary flow

Stationary flow: every time derivative is zero.

$$0 = \oint_A \rho\vec{v}\cdot d\vec{A} = \int_{A_1} \rho\vec{v}\cdot d\vec{A} + \int_{A_2} \rho\vec{v}\cdot d\vec{A} + \int_{A_m} \rho\vec{v}\cdot d\vec{A}$$

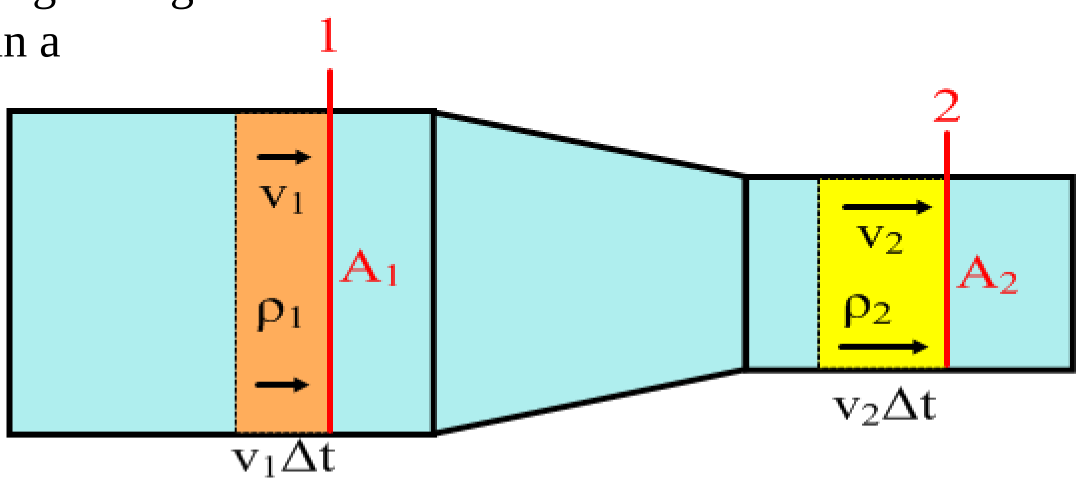


The integral over the mantle is zero because the velocity is parallel to the surface. So the inflow and outflow at the two ends must cancel each other out.

The result is that the mass of fluid flowing through the cross-sections at any two locations in a pipe is the same.

For the cross sections A_1 and A_2 during time Δt:

$$m_1 = m_2$$
$$\rho_1 V_1 = \rho_2 V_2$$
$$\rho_1 A_1 v_1 \Delta t = \rho_2 A_2 v_2 \Delta t$$



$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$ the **mass flow** is the same along the pipe.

For incompressible fluid ($\rho_1 = \rho_2$): The $A_1 v_1 = A_2 v_2$ **volume flow** is also the same along the pipe.

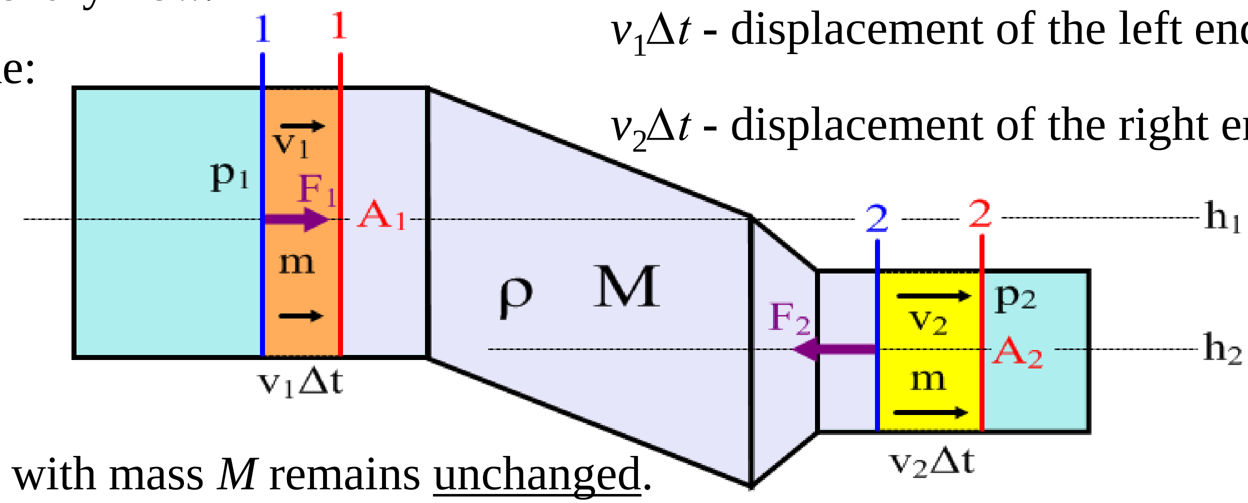
Bernoulli equation

Let us apply the work-energy theorem $W = \Delta E_K$ to the incompressible fluid of mass $m + M$ with density ρ between the cross-section A_1 at height h_1 and the cross-section A_2 at height h_2, in the case of stationary flow.

During a small Δt time:

$v_1\Delta t$ - displacement of the left end

$v_2\Delta t$ - displacement of the right end



The intermediate part with mass M remains unchanged.
The work is done by the adjacent fluid and gravity:

$$W = W_f + W_g = F_1 v_1 \Delta t - F_2 v_2 \Delta t + mg(h_1 - h_2) = p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t + mg(h_1 - h_2) =$$
$$= p_1 \Delta V - p_2 \Delta V + \rho \Delta V g(h_1 - h_2) = \Delta V(p_1 - p_2 + \rho g h_1 - \rho g h_2)$$

The change in kinetic energy: $\Delta E_K = E_{K2}(m) + E_K(M) - E_{K1}(m) - E_K(M)$

$$\Delta E_K = \frac{1}{2} m v_2{}^2 - \frac{1}{2} m v_1{}^2 = \Delta V \left(\frac{1}{2}\rho v_2{}^2 - \frac{1}{2}\rho v_1{}^2\right)$$

Thus:

$$p_1 - p_2 + \rho g h_1 - \rho g h_2 = \frac{1}{2}\rho v_2{}^2 - \frac{1}{2}\rho v_1{}^2 \rightarrow \boxed{p_1 + \frac{1}{2}\rho v_1{}^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2{}^2 + \rho g h_2}$$

Bernoulli equation - Example

At what speed does water flow out of a hole in the bottom of a bucket with water up to height *h*?

Assuming that the water level decreases very slowly: $v_1 \approx 0$

Using the Bernoulli equation:

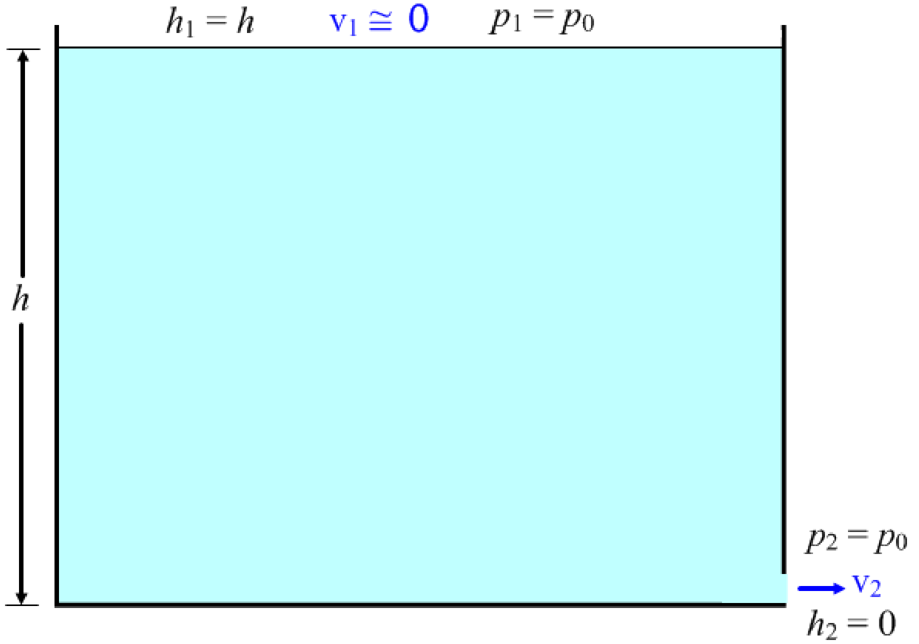
$$p_1 + \frac{1}{2}\rho v_1{}^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2{}^2 + \rho g h_2$$

$$p_0 + 0 + \rho g h = p_0 + \frac{1}{2}\rho v_2{}^2 + 0$$

$$\rho g h = \frac{1}{2}\rho v_2{}^2$$

$$2gh = v_2{}^2$$

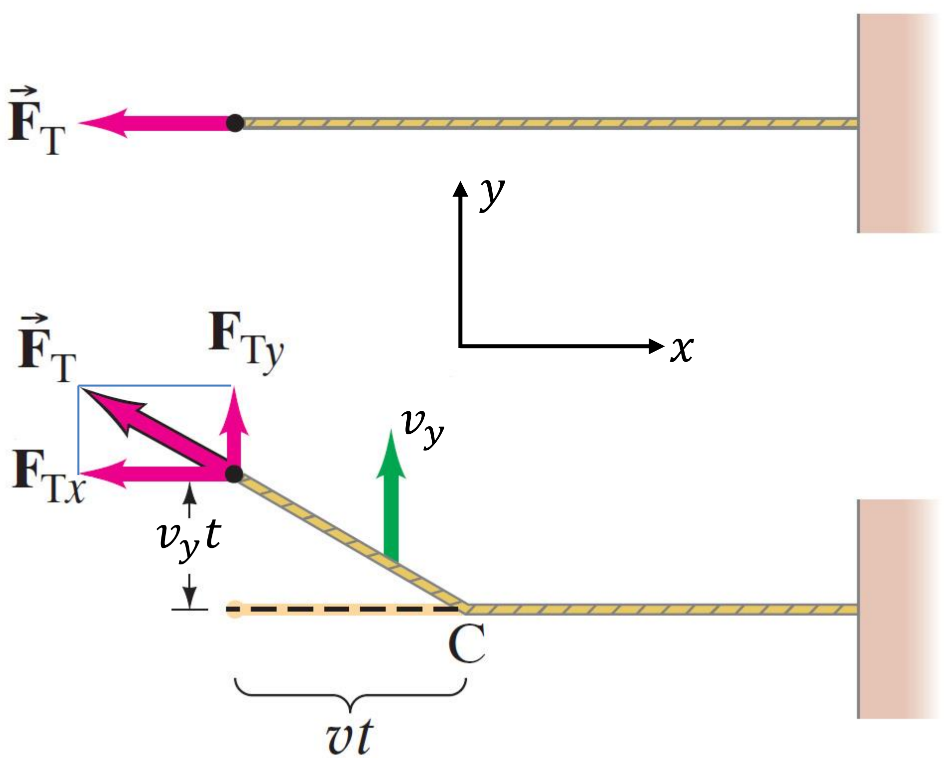
$$v_2 = \sqrt{2gh}$$



The speed is the same as for a body falling freely from height *h*.

Speed of transversal wave on a string

String is being pulled by a force to the negative *x*-direction, creating tension in the string. End is suddenly moved up at v_y by the force component F_{Ty} for a very short t time.



Pulse ends at C at *vt* distance after time *t*, and travels at *v* speed.

Using similar triangles:

$$\frac{F_{Ty}}{F_T} \approx \frac{F_{Ty}}{F_{Tx}} = \frac{v_y t}{vt} = \frac{v_y}{v}$$

(for small amplitudes: $v_y \ll v$)

From momentum-force law:

$$F_{Ty}t = F_T \frac{v_y}{v} t = \Delta p_y = \mu v t v_y$$

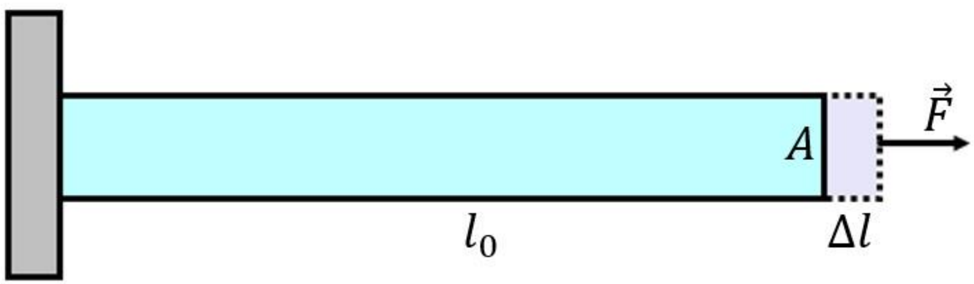
$$F_T \frac{1}{v} = \mu v$$

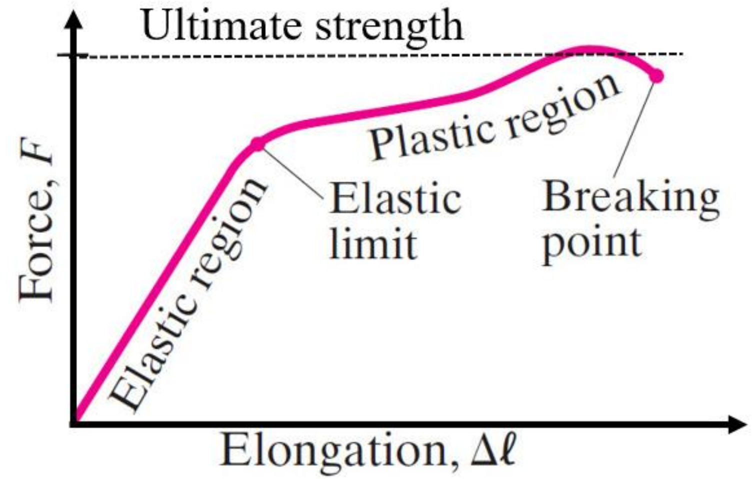
$\mu = \dfrac{m}{l}$ linear mass density of string

Speed of wave: $v = \sqrt{\dfrac{F_T}{\mu}}$

Microscopic Hooke's law

When a metal rod/bar/wire is being pulled by a force:





In the elastic region Hooke's law is valid ($\Delta l > 0$):

$$F = D\Delta l$$

Dividing by cross section and original length:

$$\frac{F}{Al_0} = \frac{D\Delta l}{Al_0} \quad \rightarrow \quad \frac{F}{A} = \frac{l_0 D}{A}\frac{\Delta l}{l_0}$$

$$\frac{F}{A} = \sigma \quad \text{stress} \qquad [\sigma] = \frac{\text{N}}{\text{m}^2} = \text{Pa}$$

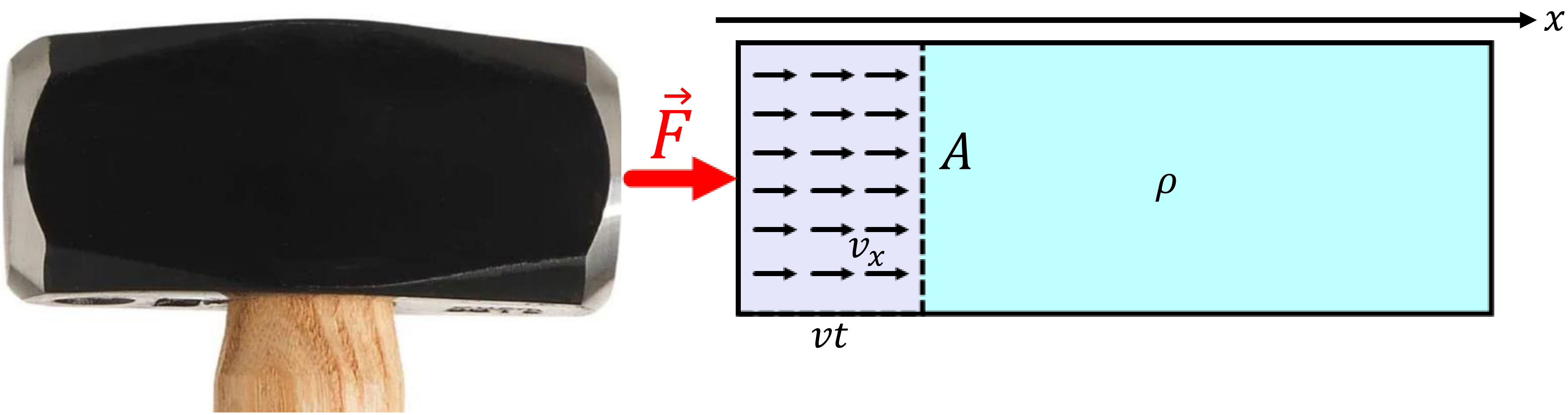
$$\boxed{\sigma = E\varepsilon}$$

$$\frac{l_0 D}{A} = E \quad \text{elastic modulus (Young's modulus)}$$

$$\frac{\Delta l}{l_0} = \varepsilon \quad \text{strain}$$

Speed of a longitudinal wave in metals

End of a metal rod/bar pushed by a force at v_x speed for a very short t time:



During t the pulse travels vt distance, and all parts within Avt volume move at v_x speed.
From momentum-force law:

$$Ft = \Delta p_x = Avt\rho v_x \qquad v_x \ll v$$

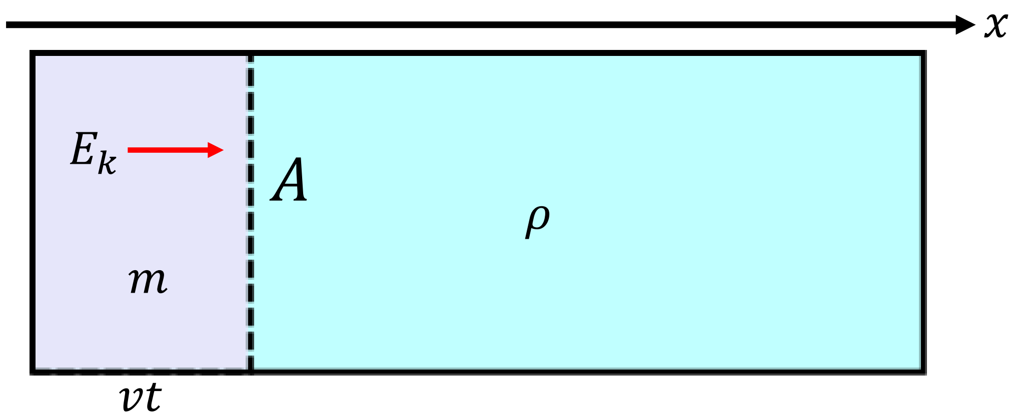
For the force: $$F = D|\Delta l| = Dv_x t = \frac{EA}{l_0} v_x t$$

Thus: $$Avt\rho v_x = \frac{EA}{l_0} v_x t^2 = \frac{EA}{vt} v_x t^2 = \frac{EA}{v} v_x t$$

Simplifying: $v\rho = \dfrac{E}{v}$ $\rightarrow$ Speed of wave: $v = \sqrt{\dfrac{E}{\rho}}$

Intensity of the wave

In the wave there is energy flowing in the *x*-direction.



This E_k energy in the *Avt* volume is the kinetic energy of the segment moving at v_x speed.
Intensity is the energy flow per unit area in unit time:

$$I = \frac{E_k}{tA} = \frac{\frac{1}{2} m {v_x}^2}{tA} = \frac{\frac{1}{2} A v t \rho {v_x}^2}{tA} = \frac{1}{2} v \rho {v_x}^2$$

This is the energy density multiplied by the speed of the wave:

$$I = \frac{1}{2} v \rho {v_x}^2 = \frac{1}{2} v \frac{m}{V} {v_x}^2 = \frac{E_k}{V} v$$ general result for any wave!