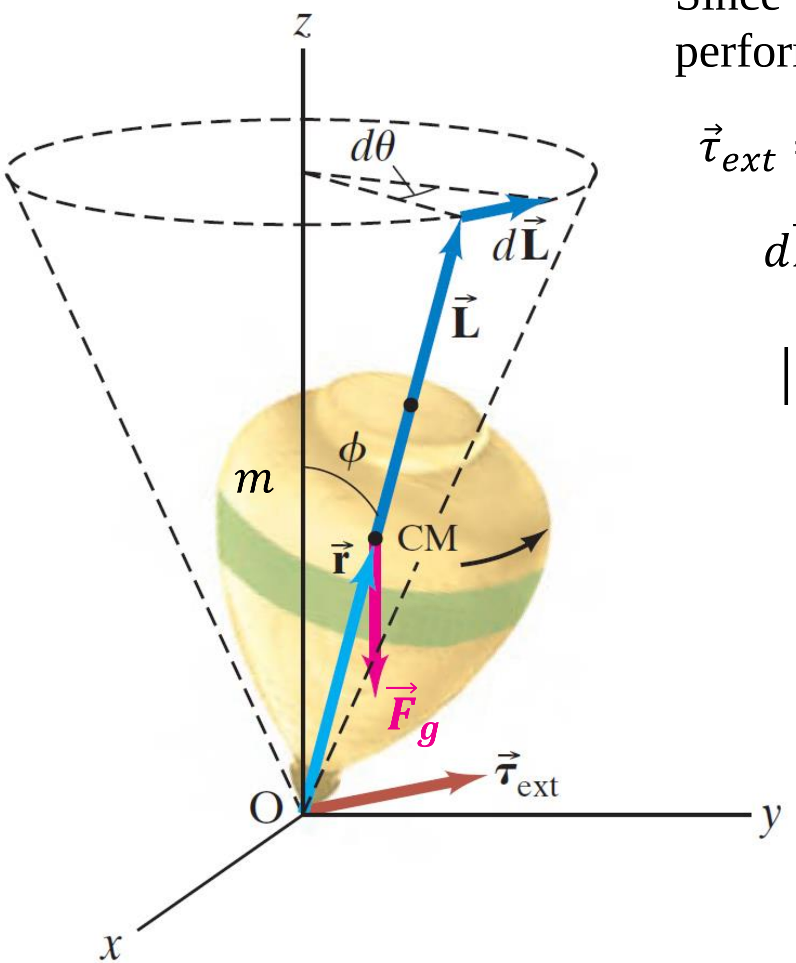
Spinning top and gyroscope

Using the torque-angular momentum law for a spinning top: $\vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \quad \rightarrow \quad d\vec{L} = \vec{\tau}_{ext} dt$

$\vec{\tau}_{ext}$ is the net external torque acting on the spinning top



Since torque is in in *x*-*y* plane → it won't fall down but perform precession with ω_p. Assuming $\omega_p \ll \omega$.

$$\vec{\tau}_{ext} = \vec{r} \times \vec{F}_g \quad \text{thus} \quad \tau_{ext} = r \sin\phi \cdot mg$$

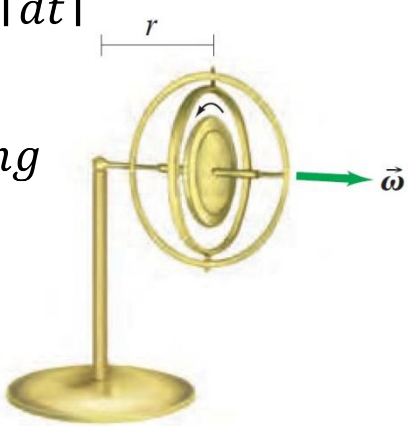
$$d\vec{L} \perp \vec{L} \quad \rightarrow \quad L \text{ remains constant!}$$

$$\left|d\vec{L}\right| = L \sin\phi \, d\theta \quad \rightarrow \quad d\theta = \frac{\left|d\vec{L}\right|}{L \sin\phi}$$

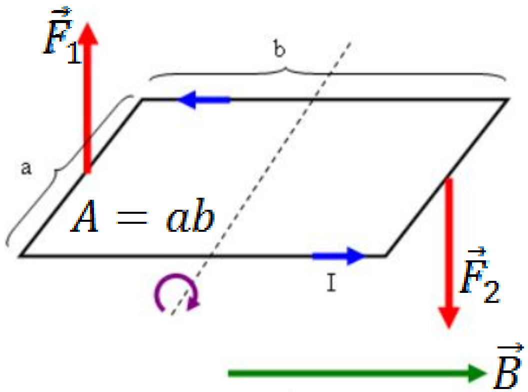
$$\omega_p = \frac{d\theta}{dt} = \frac{1}{L \sin\phi} \frac{\left|d\vec{L}\right|}{dt} = \frac{1}{L \sin\phi} \left|\frac{d\vec{L}}{dt}\right|$$

$$= \frac{1}{L \sin\phi} \tau_{ext} = \frac{1}{L \sin\phi} r \sin\phi \cdot mg$$

$$= \frac{rmg}{L} = \frac{rmg}{I\omega}$$



Torque acting on a current loop



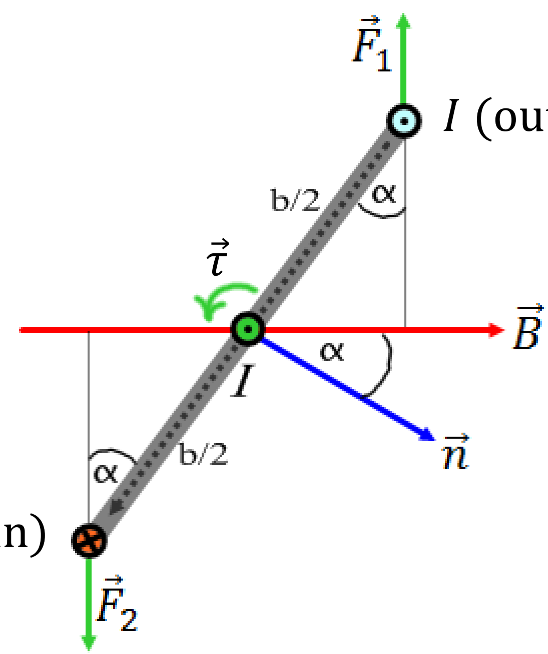
For a straight conductor in a homogeneous magnetic field, when the field is in the plane of the loop: $F_1 = F_2 = F = IaB$

The net force is zero, but the torque is not.

$$\tau = 2F\frac{b}{2} = IaBb = IAB$$

For any orientation, the torque is: $\tau = F_1\frac{b}{2}\sin\alpha + F_2\frac{b}{2}\sin\alpha$

$$F_1 = F_2 = F = IaB$$



$$\tau = Fb\sin\alpha = IaBb\sin\alpha = IAB\sin\alpha$$

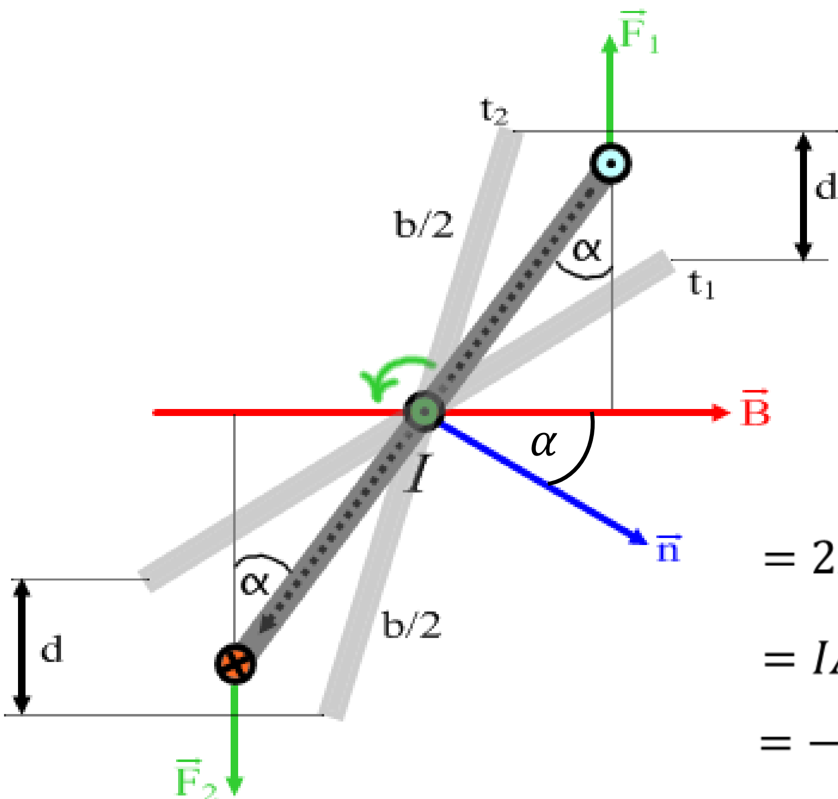
Also taking into account the directions:

$$\vec{\tau} = IA\vec{n}\times\vec{B} = I\vec{A}\times\vec{B} = \vec{m}\times\vec{B}$$

$\vec{m} = I\vec{A}$ the **magnetic dipole moment** $[m] = \mathrm{Am}^2$

The torque ceases when the dipole has turned into the direction of the magnetic induction (stable equilibrium, and unstable equilibrium in the opposite direction!).
A current-carrying loop can also be used as a compass.

Potential energy of a current loop



Calculating the work done on the loop between times t_1 and t_2 while the angle between the normal vector and the magnetic induction changes from α_1 to α_2 (decreases):

$$F = F_1 = F_2 = IaB$$

$$W_{12} = 2Fd = 2IaB\left(\frac{b}{2}\cos\alpha_2 - \frac{b}{2}\cos\alpha_1\right) =$$

$$= 2IaB\frac{b}{2}(\cos\alpha_2 - \cos\alpha_1) = IabB(\cos\alpha_2 - \cos\alpha_1) =$$

$$= IAB(\cos\alpha_2 - \cos\alpha_1) = mB(\cos\alpha_2 - \cos\alpha_1) =$$

$$= -mB\cos\alpha_1 + mB\cos\alpha_2$$

It can be seen that if: $E_P = -mB\cos\alpha = -\vec{m}\cdot\vec{B}$

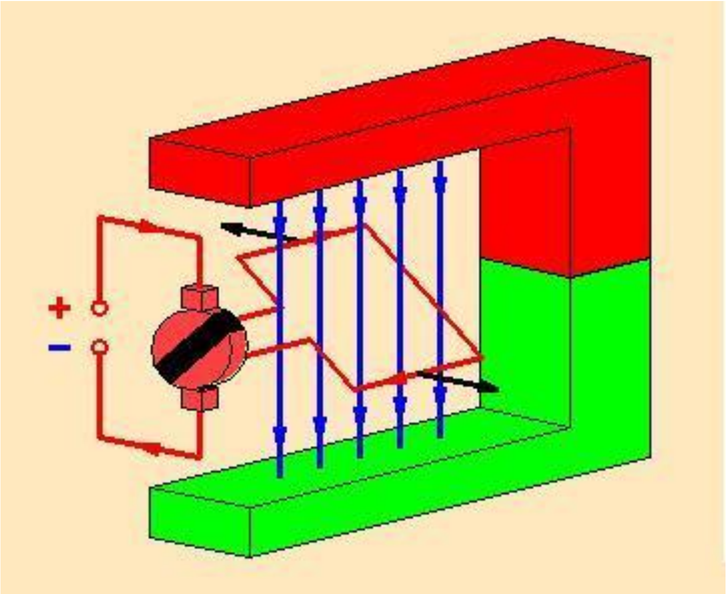
then the work done can be written in the form typical of conservative force fields:

$$W_{12} = E_{P1} - E_{P2}$$

DC electric motor

The two terminals of the rotating loop are connected to the half-cylinder separated by an insulator.

The brushes under DC voltage connect to the other half-cylinder after each half-turn.



The homogeneous magnetic field tries to rotate the loop into the stable equilibrium position.

However, as the loop would reach the stable equilibrium position, the polarity reverses.

Since the current flows in the opposite direction, the stable equilibrium becomes the unstable equilibrium.

Having turned beyond the unstable equilibrium position due to its momentum, the loop tries to turn further to the stable equilibrium position, but there the polarity is reversed again, so it keeps turning indefinitely.

Hydrostatic pressure

A common property of liquids and gases is that they can change their shape freely.
If we consider them to have a continuous mass distribution, we speak of a **continuum**.

<u>**Hydrostatics:**</u> mechanics of fluids at rest

Pressure: At a point, pressure is the perpendicular component of the force acting on the (infinitely) small surface area surrounding the point, divided by the size of the surface. It is a scalar quantity.

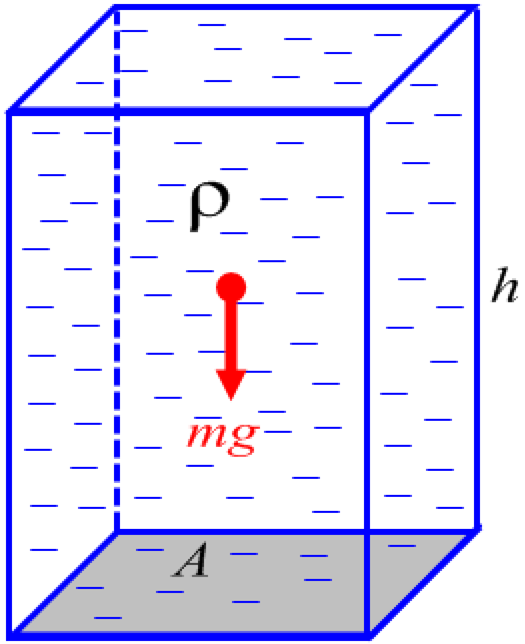
$$p = \lim_{A\to 0}\frac{F_\perp(A)}{A}$$

Unit of measurement: $[p] = \frac{\mathrm{N}}{\mathrm{m}^2} = \mathrm{Pa}$ (Pascal)

Hydrostatic pressure is the uniformly distributed pressure caused by the weight of the liquid (column of height *h*):

$$p = \frac{F_\perp(A)}{A} = \frac{mg}{A} = \frac{V\rho g}{A} = \frac{Ah\rho g}{A} = h\rho g$$

ρ: density

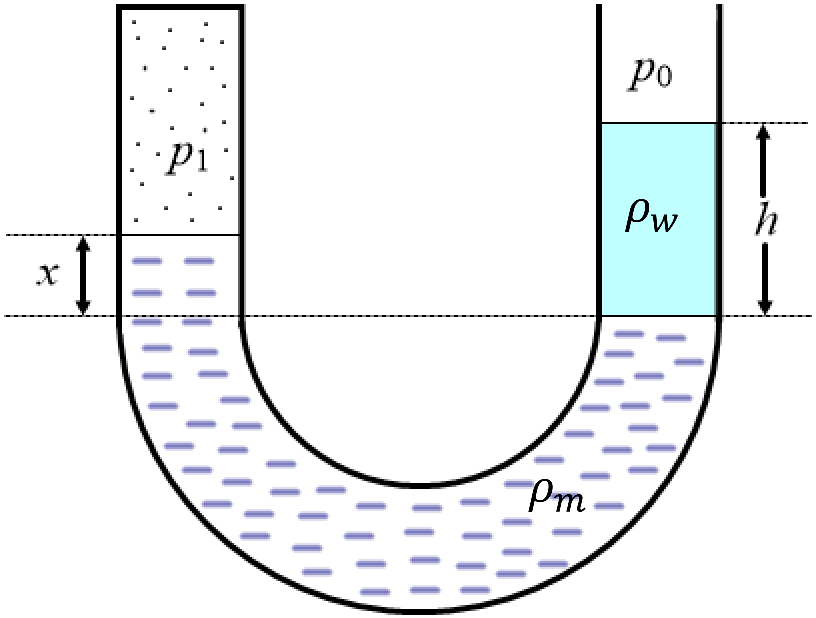


Since the shape of the liquid can change freely, the pressure of a liquid at rest at a given depth does not depend on the orientation of the surface, and the force exerted is always perpendicular to the surface.
Pascal's law: At the same height in the same kind of stationary fluid the pressure is the same.

Pascal's law - Example

The left end of a U-shaped glass tube is closed, the other is open. There is mercury at the bottom with density 13.6g/cm^3, above it on the right there is 50cm water. The atmospheric pressure is 1bar, and the air pressure above mercury on the left is 0.9bar. What is the height difference between the two mercury levels?

$p_0 = 10^5$Pa, $p_1 = 0.9 \cdot 10^5$Pa, $h = 50$cm



In mercury, as a homogeneous liquid at rest, the pressure on the left and right sides must be equal at the level marked by the dashed line:

$$p_l = p_r$$

$$p_1 + x\rho_m g = p_0 + h\rho_w g$$

$$x = \frac{p_0 + h\rho_w g - p_1}{\rho_m g} = \cdots = 11.17\text{cm}$$

Buoyant force

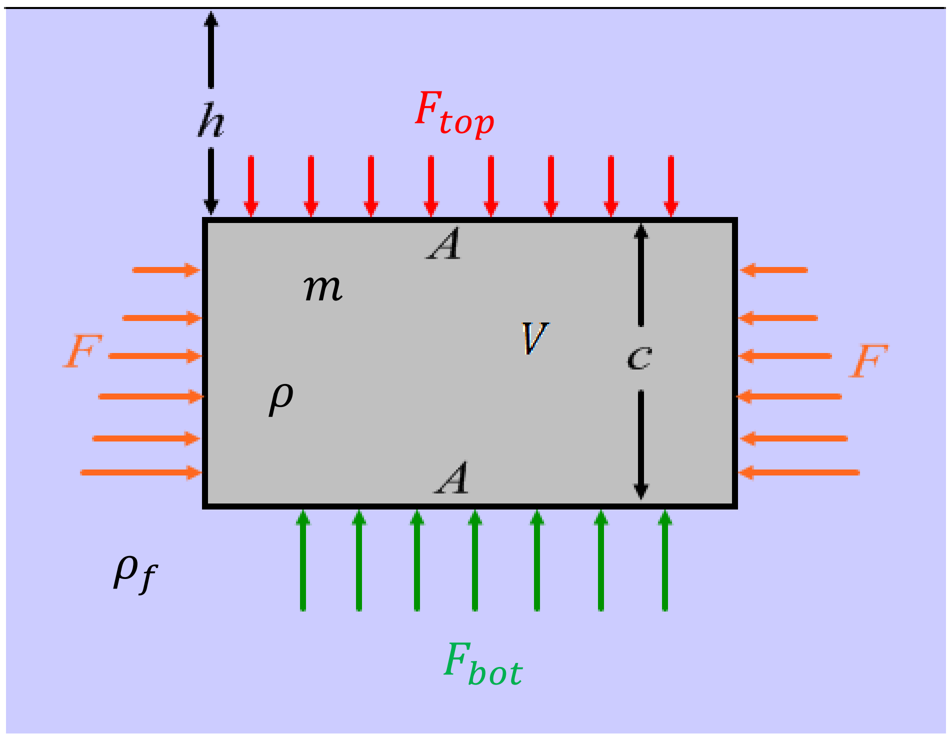
Buoyant force is the net force exerted by the fluid on the entire surface of the body.
For the faces of the cuboid body:
- the resultant of the front and back is zero
- the resultant of the left and right is zero
- the resultant of the lower and upper...

$$F_b = F_{bot} - F_{top} = p_{bot}A - p_{top}A =$$

$$= \rho_f g(h+c)A - \rho_f ghA = \rho_f gcA =$$

$$= \rho_f Vg = m_f g$$

V is the volume of fluid displaced by the body, whose mass is m_f



Thus the buoyant force equals the weight of the displaced fluid.
This is true for other shapes as well!

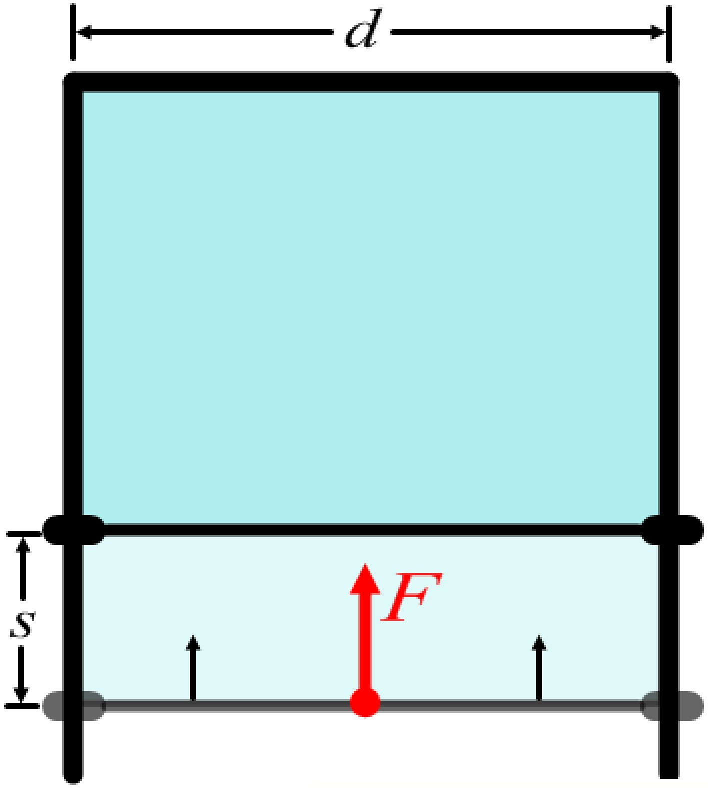
Archimedes' principle: Every body immersed in a liquid is subjected to a buoyant force that equals the weight of the liquid displaced (by the immersed part).

If its density is greater than that of the fluid, it sinks because the buoyant force is less than its weight. If the density of the liquid is greater, then part of the body does not sink, the body floats.

Surface tension

The stretched film exerts a contracting force on the sides of a wire frame dipped in soapy water.
For the bottom side of length *d*:

$F = 2\alpha d$ where α is the **surface tension**.
The factor of 2 is because of the front and rear surfaces.



If the bottom side is a movable rod that moves upward a distance *s*, the change in surface area:

$$\Delta A = -2ds \text{ (still 2 sides)}$$

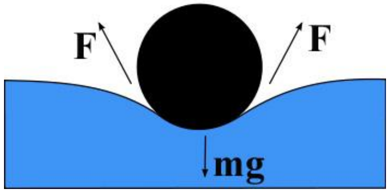
And the work by the force of the soap film:

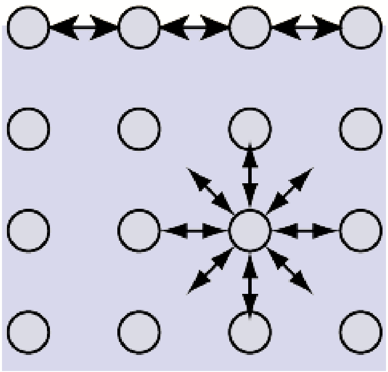
$$W = Fs = 2\alpha ds = -\alpha\Delta A$$

Since work is proportional to the change in surface area, the fluid has energy proportional to its surface area:

$$E = \alpha A$$

The reason for this lies in the attractive interaction between molecules.
The liquid tries to minimize the surface area: shape of droplet





Conservation of mass

Hydrodynamics is the field describing the flow of liquids as continuous media.

Two descriptions:

1. Lagrange-method: writing Newton's equations of motion for the selected liquid elements, then solving those using the initial conditions.

2. Euler-method: measuring the properties of the flowing liquid at selected points. (e.g. pressure, velocity, density).
If these are constants at all points, then the flow is **stationary**.

Continuity equation: mass is conserved, it cannot appear or disappear.
Let's consider a volume *V* bounded by closed surface *S*. In *dt* time the mass flowing out through *dA* is $dm = \rho dV = \rho dA v \cos\alpha\, dt = \rho\vec{v}\cdot d\vec{A} dt$

Thus in unit time: $\frac{dm}{dt} = \rho\vec{v}\cdot d\vec{A}$

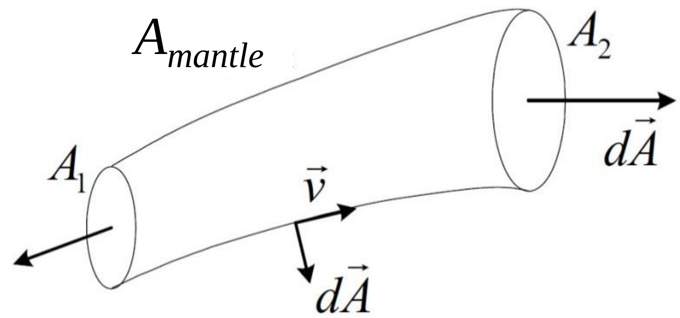
The mass flowing out from *V* through the whole *S* in unit time equals the negative change in the mass in *V* per unit time (negative derivative of the total mass in *V*):

$$-\frac{d}{dt}\int_V \rho\, dV = \oint_S \rho\vec{v}\cdot d\vec{A}$$

Continuity equation for stationary flow

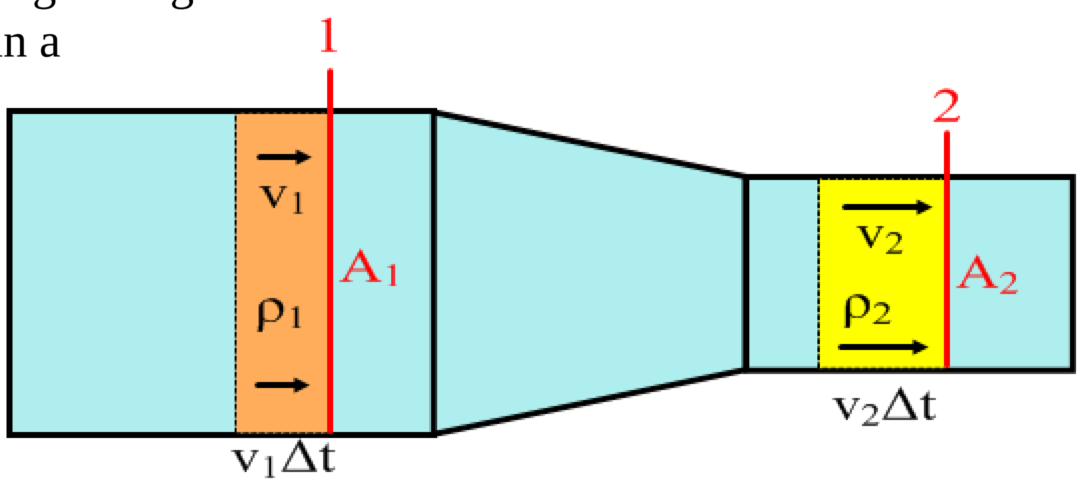
Stationary flow: every time derivative is zero.

$$0 = \oint_A \rho\vec{v}\cdot d\vec{A} = \int_{A_1} \rho\vec{v}\cdot d\vec{A} + \int_{A_2} \rho\vec{v}\cdot d\vec{A} + \int_{A_m} \rho\vec{v}\cdot d\vec{A}$$



The integral over the mantle is zero because the velocity is parallel to the surface. So the inflow and outflow at the two ends must cancel each other out.
The result is that the mass of fluid flowing through the cross-sections at any two locations in a pipe is the same.

For the cross sections A_1 and A_2 during time Δt:



$$m_1 = m_2$$
$$\rho_1 V_1 = \rho_2 V_2$$
$$\rho_1 A_1 v_1 \Delta t = \rho_2 A_2 v_2 \Delta t$$

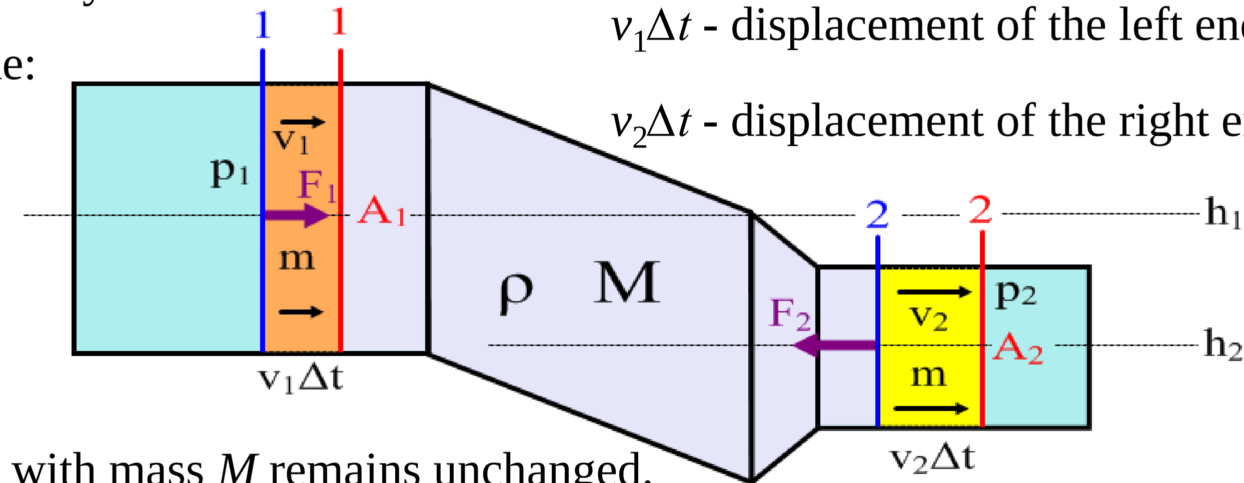
$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$ the **mass flow** is the same along the pipe.

For incompressible fluid ($\rho_1 = \rho_2$): The $A_1 v_1 = A_2 v_2$ **volume flow** is also the same along the pipe.

Bernoulli equation

Let us apply the work-energy theorem $W = \Delta E_K$ to the incompressible fluid of mass $m + M$ with density ρ between the cross-section A_1 at height h_1 and the cross-section A_2 at height h_2, in the case of stationary flow.

During a small Δt time:



$v_1\Delta t$ - displacement of the left end

$v_2\Delta t$ - displacement of the right end

The intermediate part with mass M remains unchanged.
The work is done by the adjacent fluid and gravity:

$$W = W_f + W_g = F_1 v_1 \Delta t - F_2 v_2 \Delta t + mg(h_1 - h_2) = p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t + mg(h_1 - h_2) =$$
$$= p_1 \Delta V - p_2 \Delta V + \rho \Delta V g(h_1 - h_2) = \Delta V(p_1 - p_2 + \rho g h_1 - \rho g h_2)$$

The change in kinetic energy: $\Delta E_K = E_{K2}(m) + E_K(M) - E_{K1}(m) - E_K(M)$

$$\Delta E_K = \frac{1}{2} m v_2{}^2 - \frac{1}{2} m v_1{}^2 = \Delta V \left(\frac{1}{2}\rho v_2{}^2 - \frac{1}{2}\rho v_1{}^2\right)$$

Thus:

$$p_1 - p_2 + \rho g h_1 - \rho g h_2 = \frac{1}{2}\rho v_2{}^2 - \frac{1}{2}\rho v_1{}^2 \rightarrow \boxed{p_1 + \frac{1}{2}\rho v_1{}^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2{}^2 + \rho g h_2}$$

Bernoulli equation - Example

At what speed does water flow out of a hole in the bottom of a bucket with water up to height *h*?

Assuming that the water level decreases very slowly: $v_1 \approx 0$

Using the Bernoulli equation:

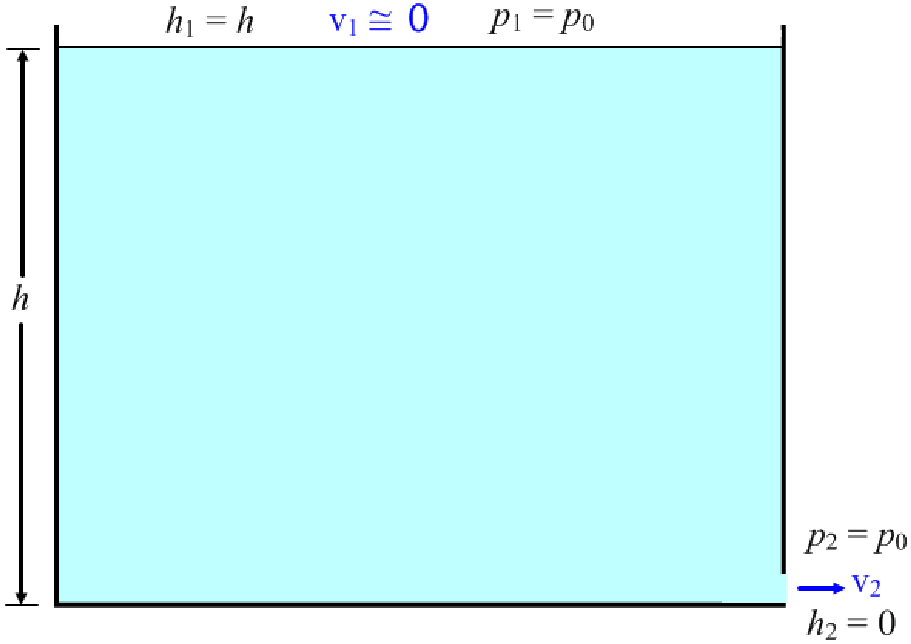
$$p_1 + \frac{1}{2}\rho {v_1}^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho {v_2}^2 + \rho g h_2$$

$$p_0 + 0 + \rho g h = p_0 + \frac{1}{2}\rho {v_2}^2 + 0$$

$$\rho g h = \frac{1}{2}\rho {v_2}^2$$

$$2gh = {v_2}^2$$

$$v_2 = \sqrt{2gh}$$



The speed is the same as for a body falling freely from height *h*.