Standing waves

Reaching the boundary of the medium, the wave is reflected. When the incoming and reflected waves meet, their displacements add up (they **interfere**). In some cases, a **standing wave** may be created. Consider a wave traveling in the *x* direction and a reflected wave traveling in the –*x* direction:

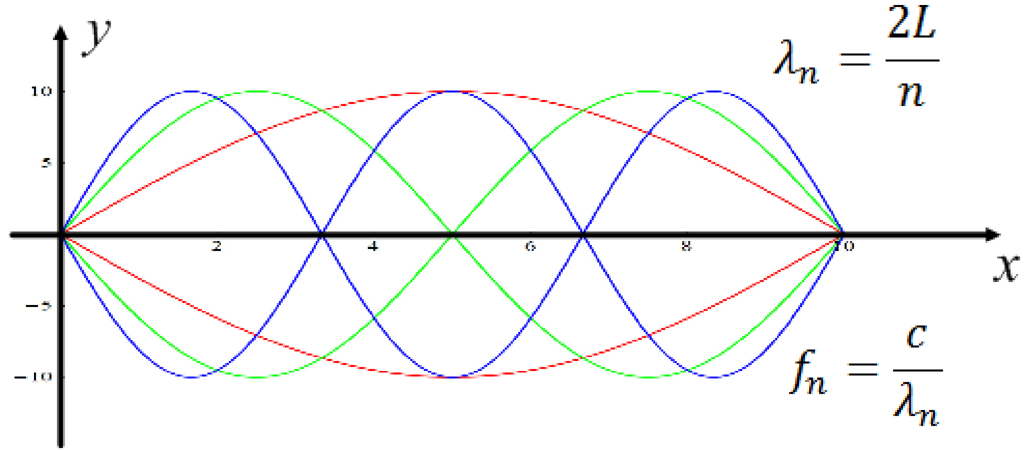
$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$y(x,t) = A\sin(\omega t - kx) + A\sin(\omega t + kx)$$

$$y(x,t) = A\sin(\omega t)\cos(kx) - A\cos(\omega t)\sin(kx) + A\sin(\omega t)\cos(kx) + A\cos(\omega t)\sin(kx)$$

$$y(x,t) = 2A\sin(\omega t)\cos(kx) = 2A\cos(kx)\sin(\omega t) = 2A(x)\sin(\omega t)$$

The amplitude becomes position dependent: **nodes** and **antinodes**.
However, the phase no longer depends on time (position and time dependence are decoupled).
The wavelength of the standing wave can only be the wavelengths allowed by the given geometry: **boundary conditions**
E.g. node at the end of fixed-end string, antinode at the end of open-end whistle.
Therefore, the frequency cannot be arbitrary either: fundamental frequency and harmonics.



$$\lambda_n = \frac{2L}{n}$$

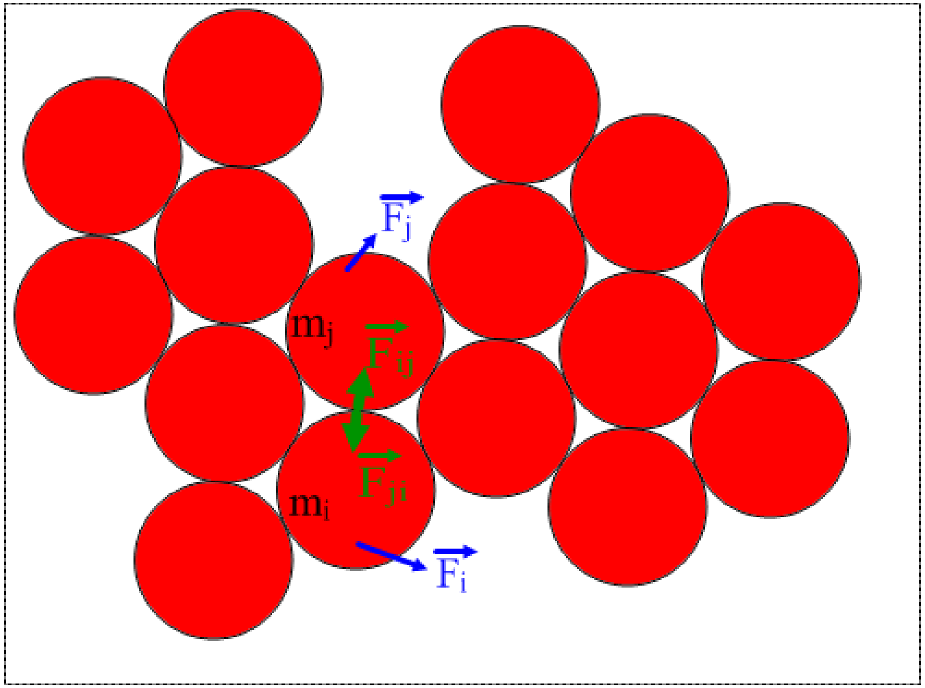
$$f_n = \frac{c}{\lambda_n}$$

Momentum-force law for mass point systems

For a system of points, let's write the **momentum-force law** for one of the points (*i*):

The net external force acting on it: $\vec{F}_i$

The force exerted by point *j*: $\vec{F}_{ji}$



Using the basic equation of dynamics for mass point *i*:

$$\frac{d}{dt}\vec{p}_i = m_i \vec{a}_i = \vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji}$$

Summarized for all the points:

$$\sum_{i=1}^{N} \frac{d}{dt}\vec{p}_i = \sum_{i=1}^{N} \vec{F}_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \vec{F}_{ji}$$

Internal forces cancel (Newton's 3rd axiom): $\vec{F}_{ji} = -\vec{F}_{ij}$

Momentum-force law for mass point systems:

$$\frac{d}{dt}\vec{p} = \sum_{i=1}^{N} \vec{F}_i$$

If the net external force acting on a point system is zero, then the **momentum is conserved**.

Collisions

If the colliding bodies form a closed system (no external forces act on them), then **momentum conservation** is **always** satisfied during the collision.
The total momentum of the members of the system is the same before and after collision:

$$\vec{p}_{A1} + \vec{p}_{B1} + \vec{p}_{C1} + \cdots = \vec{p}_{A2} + \vec{p}_{B2} + \vec{p}_{C2} + \cdots$$

$$m_A\vec{v}_{A1} + m_B\vec{v}_{B1} + m_C\vec{v}_{C1} + \cdots = m_A\vec{v}_{A2} + m_B\vec{v}_{B2} + m_C\vec{v}_{C2} + \cdots$$

This usually means 3 independent equations for the *x*, *y*, and *z* components of momentum.
This is necessary if the collision occurs in space and is not central
(e.g. two balls smashed together whose centers don't line up, or it can also be used in the case of an exploding firework).
The collision of billiard balls in the plane of the table gives two equations.
The central collision of two bodies moving along a straight line gives only one:

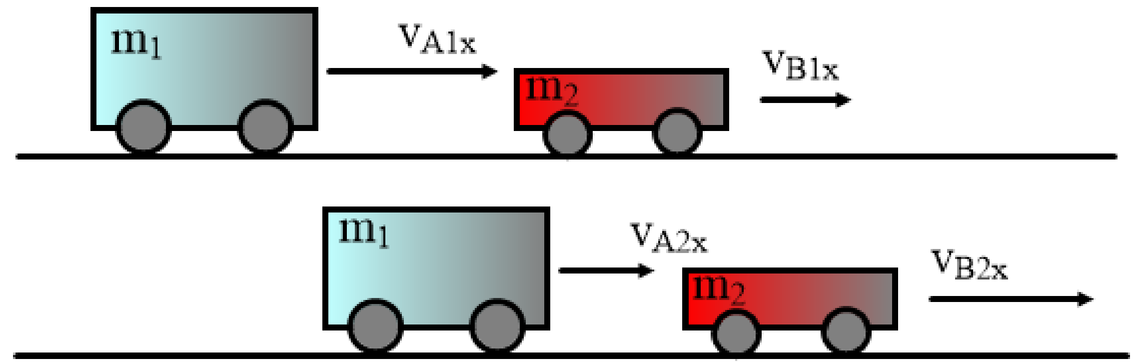
$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

If the two bodies move in opposite directions, then one velocity is negative.
Extreme cases:
In perfectly inelastic collision, the two bodies stick together: $v_{A2x} = v_{B2x}$ and $m = m_A + m_B$
In a perfectly elastic collision, kinetic energy is also conserved:

$$\frac{1}{2}m_A {v_{A1}}^2 + \frac{1}{2}m_B {v_{B1}}^2 = \frac{1}{2}m_A {v_{A2}}^2 + \frac{1}{2}m_B {v_{B2}}^2$$

Coefficient of restitution

Real collisions are neither perfectly inelastic nor perfectly elastic.
The **coefficient of restitution** shows how elastic the collision is.
This number is the ratio of the velocity of separation and the velocity of approach.



The coefficient of restitution:

$$k = \frac{v_{B2x} - v_{A2x}}{v_{A1x} - v_{B1x}}$$

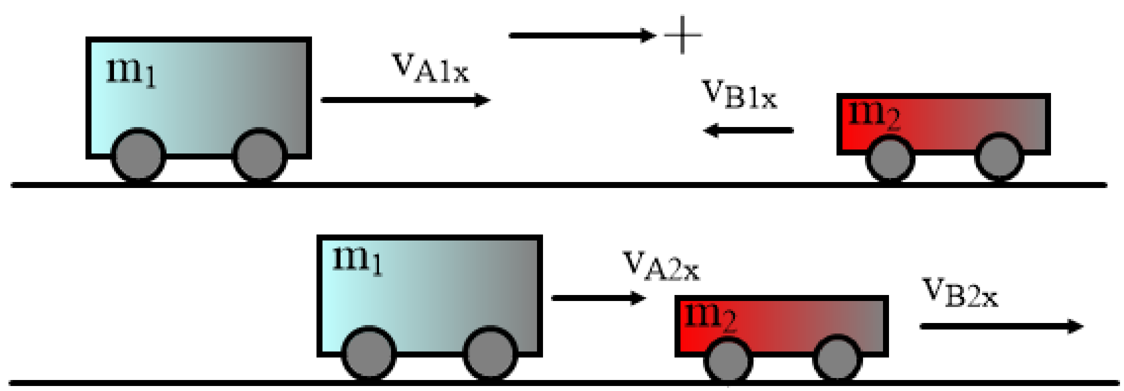
Extreme cases:

perfectly inelastic: $k = 0$

perfectly elastic: $k = 1$

in general: $0 \leq k \leq 1$

This is easier to use instead of conservation of kinetic energy, because this is not quadratic.



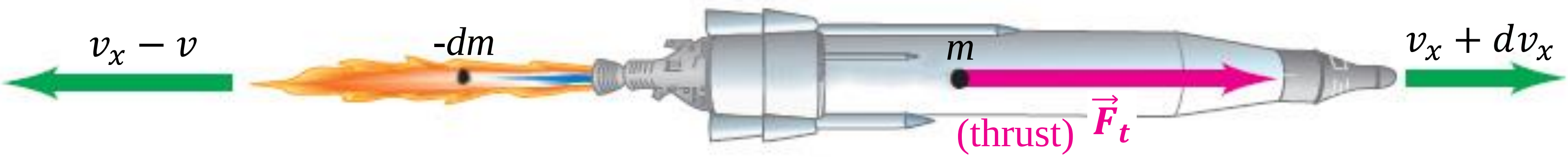
Rocket propulsion

Using conservation of momentum for the rocket:

- m: mass without the fuel element of mass $-dm$ ($dm < 0$, since total mass decreases!)
- v: exhaust speed compared to rocket

BEFORE: $p_x = (m - dm)v_x$

x



AFTER: $p_x = m(v_x + dv_x) - dm(v_x - v)$

$$(m - dm)v_x = m(v_x + dv_x) - dm(v_x - v)$$

$$mv_x - v_x dm = mv_x + mdv_x - v_x dm + vdm$$

$$0 = mdv_x + vdm$$

- μ: empty mass of rocket
- M: total mass of fuel
- m_f: momentary fuel mass

$$\frac{dv_x}{dm} = -\frac{v}{m} \quad \rightarrow \quad \int_0^{v_x} dv'_x = -v\int_{\mu+M}^{\mu+m_f} \frac{dm}{m} \quad \rightarrow v_x = -v \ln\frac{\mu + m_f}{\mu + M} = v \ln\frac{\mu + M}{\mu + m_f}$$

Maximum (terminal) speed once all fuel is used up: $$\boxed{v_t = v \ln\frac{\mu + M}{\mu}}$$

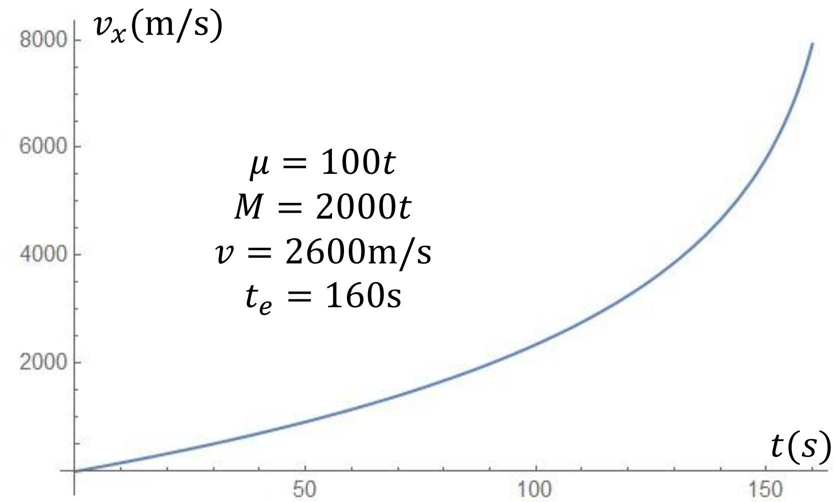
Thrust

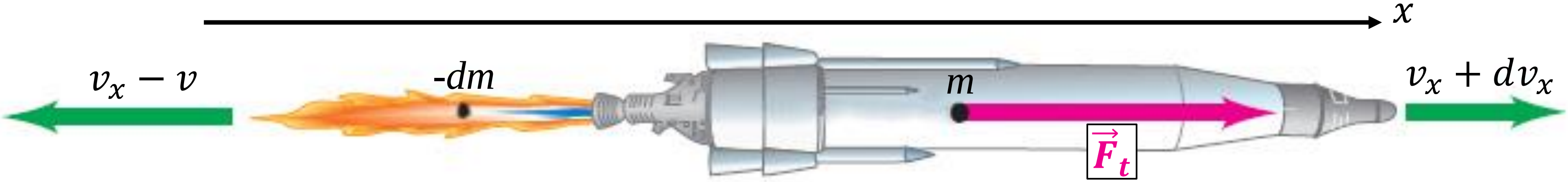
Assuming that the rate of fuel burning is constant:

$$R = \frac{M}{t_e} = -\frac{dm}{dt}$$

t_e is the time to empty the fuel tank

$$m_f = M - Rt \quad \rightarrow \quad v_x(t) = v \ln \frac{\mu + M}{\mu + M - Rt}$$





The magnitude of the **thrust** on the remaining fuel and rocket body with mass *m*:

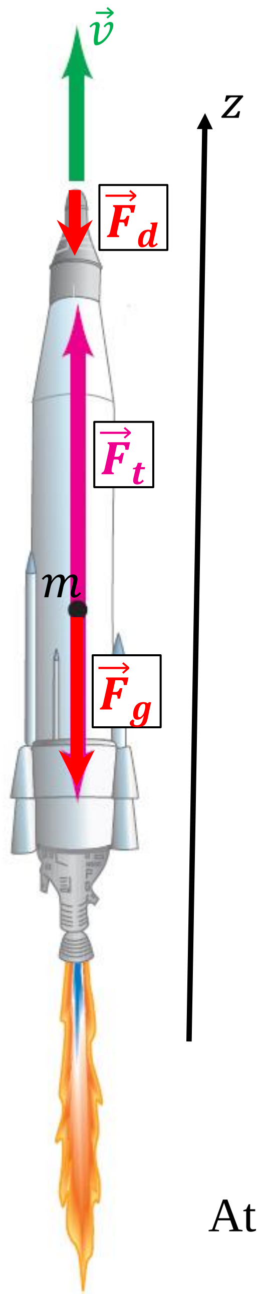
$$F_t = m\frac{dv_x}{dt} = m\frac{dv_x}{dm}\frac{dm}{dt} = m\left(-\frac{v}{m}\right)(-R) = vR$$

<u>Specific impulse:</u>

Momentum increase of the remaining fuel and rocket body with mass *m* by unit mass of fuel.

$$I_{sp} = -\frac{dp_x}{dm} = \frac{\frac{dp_x}{dt}}{-\frac{dm}{dt}} = \frac{F_t}{R} = \frac{vR}{R} = v$$

Lift-off



Equation of motion for the vertical launch of the rocket:

$$m\frac{dv_z}{dt} = F_t - F_g - F_d$$

$$m\frac{dv_z}{dt} = vR - mg - cv_z^2 \qquad v\text{: exhaust speed of fuel}$$

At the beginning v_z is „small", thus cv_z^2 is „negligible":

$$\frac{dv_z}{dt} = \frac{vR}{m} - g = \frac{vR}{\mu + M - Rt} - g$$

$$\int_0^{v_z} dv'_z = \int_0^t \left(\frac{vR}{\mu + M - Rt'} - g\right) dt'$$

$$v_z(t) = v \ln \frac{\mu + M}{\mu + M - Rt} - gt$$

Since the drag is $F_d = \frac{1}{2}\rho C_D A {v_z}^2$

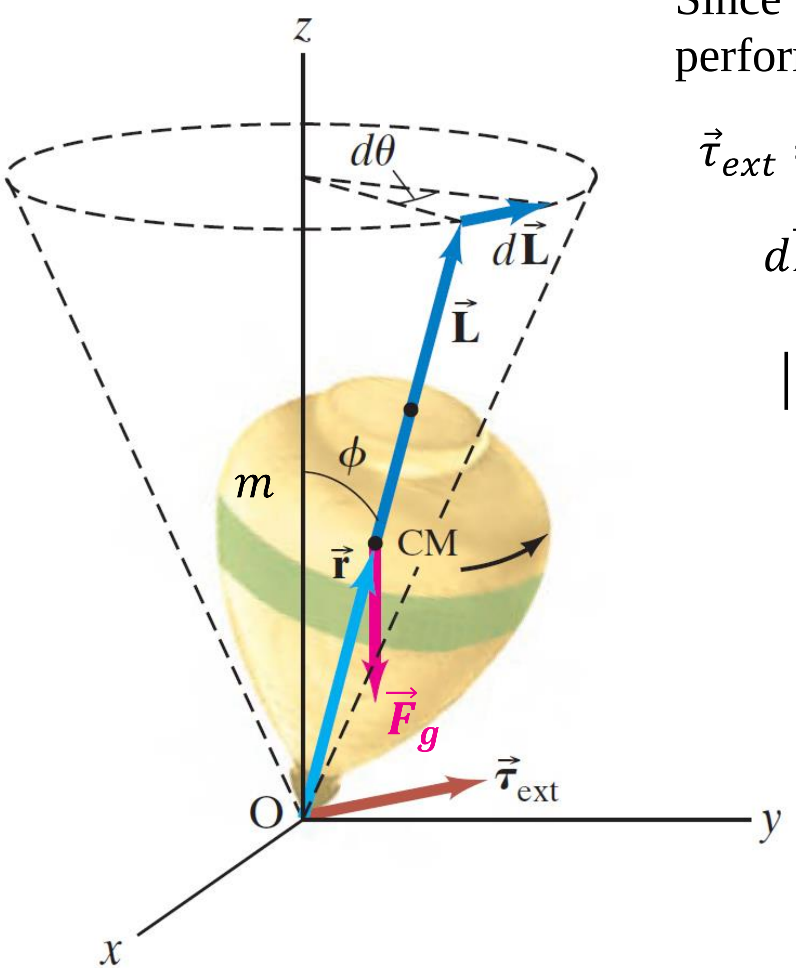
Taking values $A = 112\mathrm{m}^2$ $\quad C_D = 0.5$ $\quad \rho = 1.225\mathrm{kg/m}^3$

At $v_z = 100\mathrm{m/s}$ $\rightarrow$ $F_d = 343000\mathrm{N} = 0.0163mg$ (only 1.63% of the weight)

Spinning top and gyroscope

Using the torque-angular momentum law for a spinning top: $\vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \;\rightarrow\; d\vec{L} = \vec{\tau}_{ext} dt$

$\vec{\tau}_{ext}$ is the net external torque acting on the spinning top



Since torque is in in *x*-*y* plane → it won't fall down but perform precession with ω_p. Assuming $\omega_p \ll \omega$.

$$\vec{\tau}_{ext} = \vec{r} \times \vec{F}_g \quad \text{thus} \quad \tau_{ext} = r \sin\phi \cdot mg$$

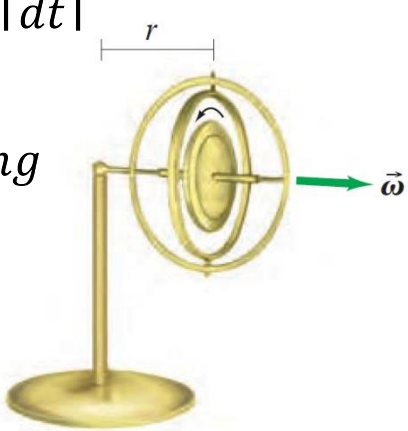
$$d\vec{L} \perp \vec{L} \quad \rightarrow \quad L \text{ remains constant!}$$

$$\left|d\vec{L}\right| = L \sin\phi \, d\theta \quad \rightarrow \quad d\theta = \frac{\left|d\vec{L}\right|}{L \sin\phi}$$

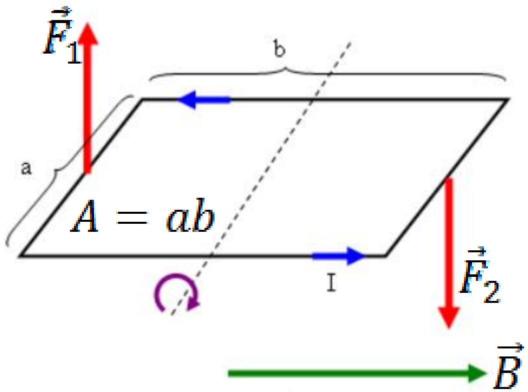
$$\omega_p = \frac{d\theta}{dt} = \frac{1}{L\sin\phi}\frac{\left|d\vec{L}\right|}{dt} = \frac{1}{L\sin\phi}\left|\frac{d\vec{L}}{dt}\right|$$

$$= \frac{1}{L\sin\phi}\tau_{ext} = \frac{1}{L\sin\phi} r \sin\phi \cdot mg$$

$$= \frac{rmg}{L} = \frac{rmg}{I\omega}$$



Torque acting on a current loop

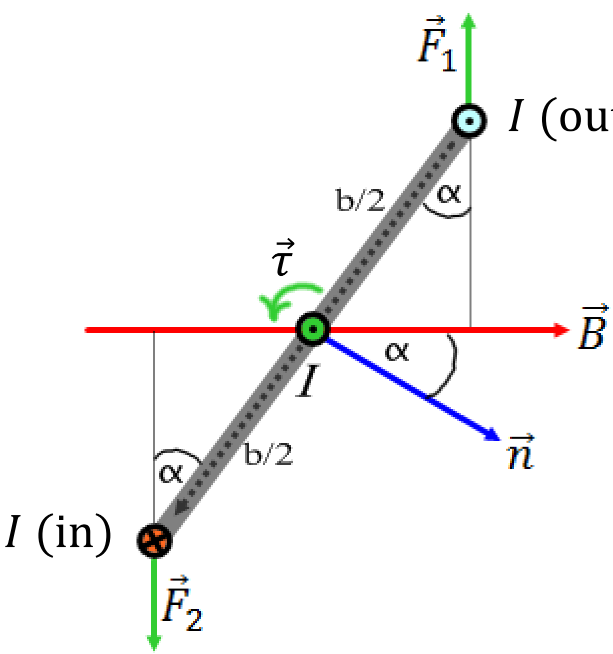


For a straight conductor in a homogeneous magnetic field, when the field is in the plane of the loop: $F_1 = F_2 = F = IaB$

The net force is zero, but the torque is not.

$$\tau = 2F\frac{b}{2} = IaBb = IAB$$

For any orientation, the torque is: $\tau = F_1\frac{b}{2}\sin\alpha + F_2\frac{b}{2}\sin\alpha$



$$F_1 = F_2 = F = IaB$$

$$\tau = Fb\sin\alpha = IaBb\sin\alpha = IAB\sin\alpha$$

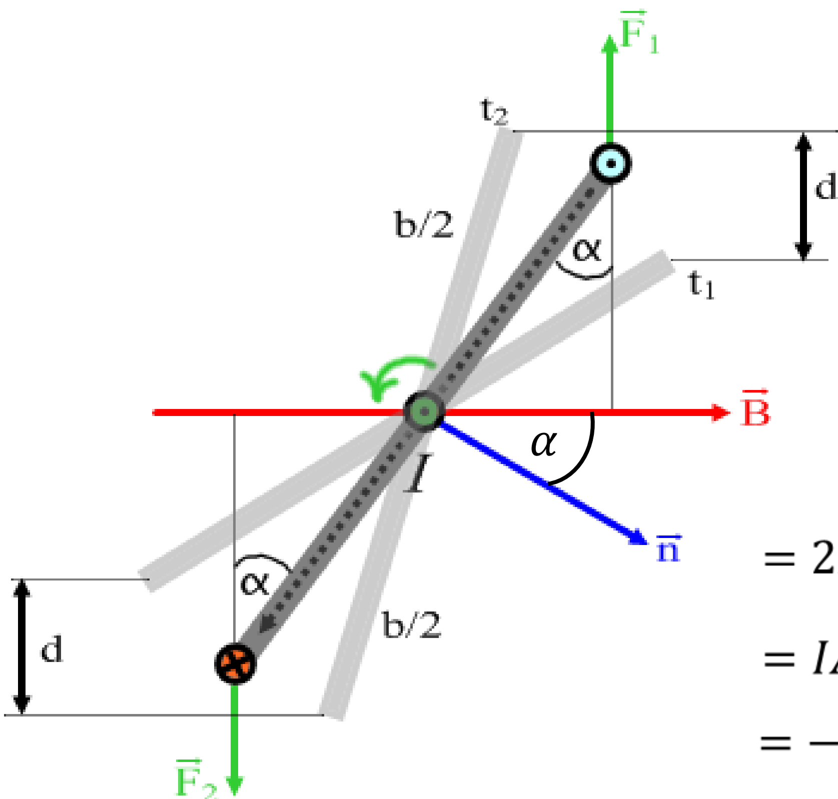
Also taking into account the directions:

$$\vec{\tau} = IA\vec{n}\times\vec{B} = I\vec{A}\times\vec{B} = \vec{m}\times\vec{B}$$

$\vec{m} = I\vec{A}$ the **magnetic dipole moment** $[m] = \mathrm{Am}^2$

The torque ceases when the dipole has turned into the direction of the magnetic induction (stable equilibrium, and unstable equilibrium in the opposite direction!).
A current-carrying loop can also be used as a compass.

Potential energy of a current loop



Calculating the work done on the loop between times t_1 and t_2 while the angle between the normal vector and the magnetic induction changes from α_1 to α_2 (decreases):

$$F = F_1 = F_2 = IaB$$

$$W_{12} = 2Fd = 2IaB\left(\frac{b}{2}\cos\alpha_2 - \frac{b}{2}\cos\alpha_1\right) =$$

$$= 2IaB\frac{b}{2}(\cos\alpha_2 - \cos\alpha_1) = IabB(\cos\alpha_2 - \cos\alpha_1) =$$

$$= IAB(\cos\alpha_2 - \cos\alpha_1) = mB(\cos\alpha_2 - \cos\alpha_1) =$$

$$= -mB\cos\alpha_1 + mB\cos\alpha_2$$

It can be seen that if: $E_P = -mB\cos\alpha = -\vec{m}\cdot\vec{B}$

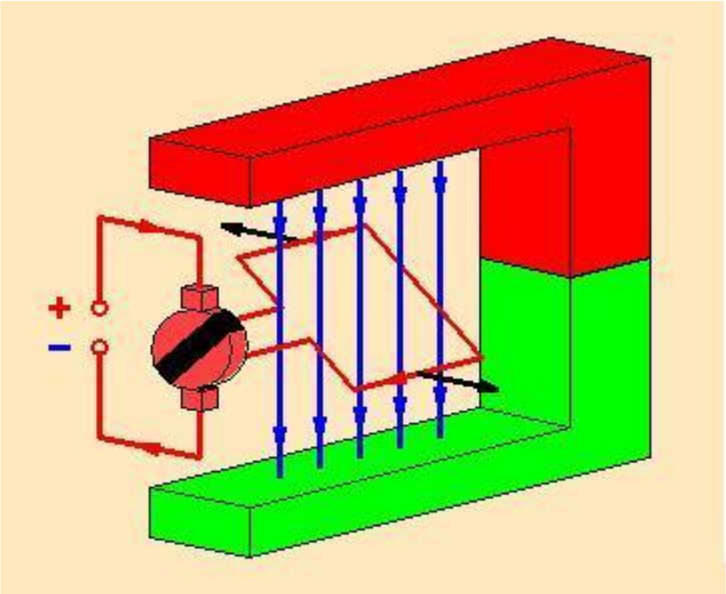
then the work done can be written in the form typical of conservative force fields:

$$W_{12} = E_{P1} - E_{P2}$$

DC electric motor

The two terminals of the rotating loop are connected to the half-cylinder separated by an insulator.

The brushes under DC voltage connect to the other half-cylinder after each half-turn.



The homogeneous magnetic field tries to rotate the loop into the stable equilibrium position.

However, as the loop would reach the stable equilibrium position, the polarity reverses.

Since the current flows in the opposite direction, the stable equilibrium becomes the unstable equilibrium.

Having turned beyond the unstable equilibrium position due to its momentum, the loop tries to turn further to the stable equilibrium position, but there the polarity is reversed again, so it keeps turning indefinitely.