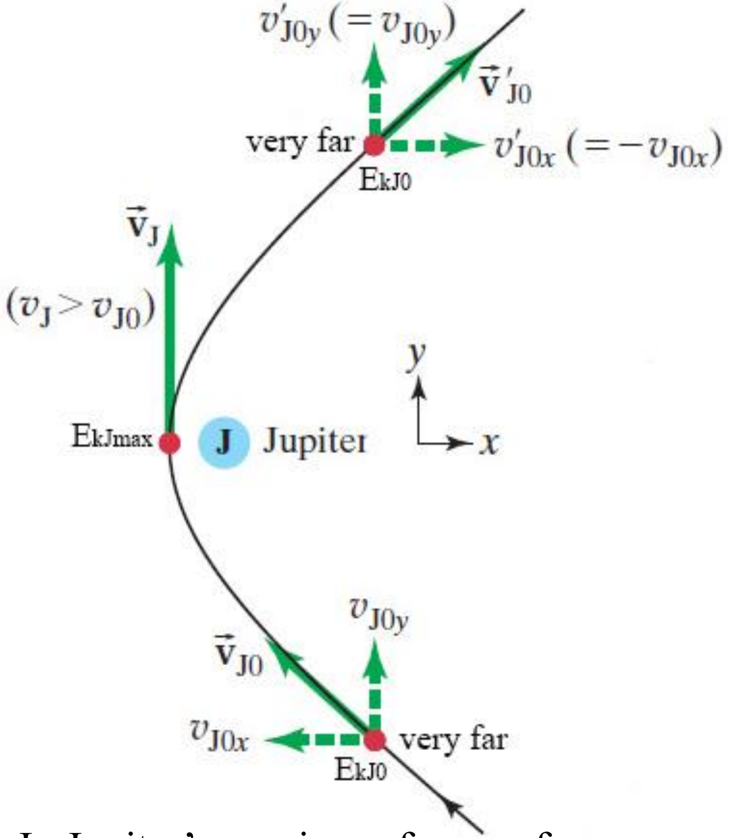
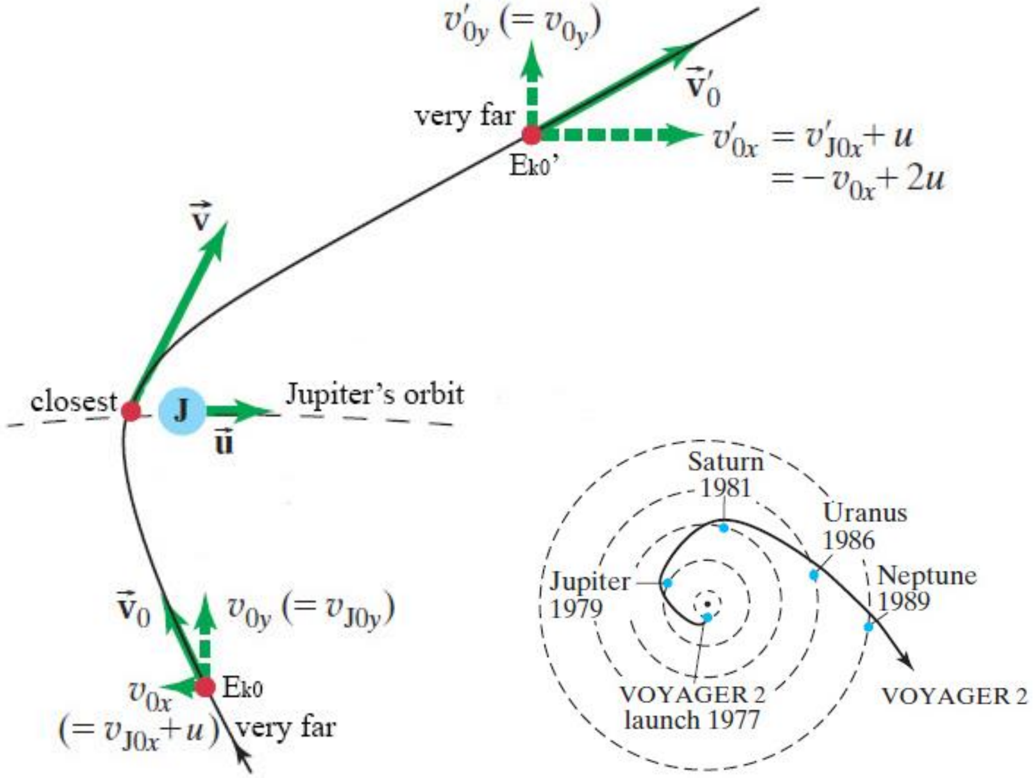
Gravity assist (slingshot, swing-by)



In Jupiter's moving reference frame
When $r \to \infty$ then $E_P \to 0$



In Sun's frame of reference:

Using Galilean transformation: $\vec{v} = \vec{v}_J + \vec{u}$

$$v'_{0x} = v'_{J0x} + u = -v_{J0x} + u = -(v_{0x} - u) + u = -v_{0x} + 2u$$

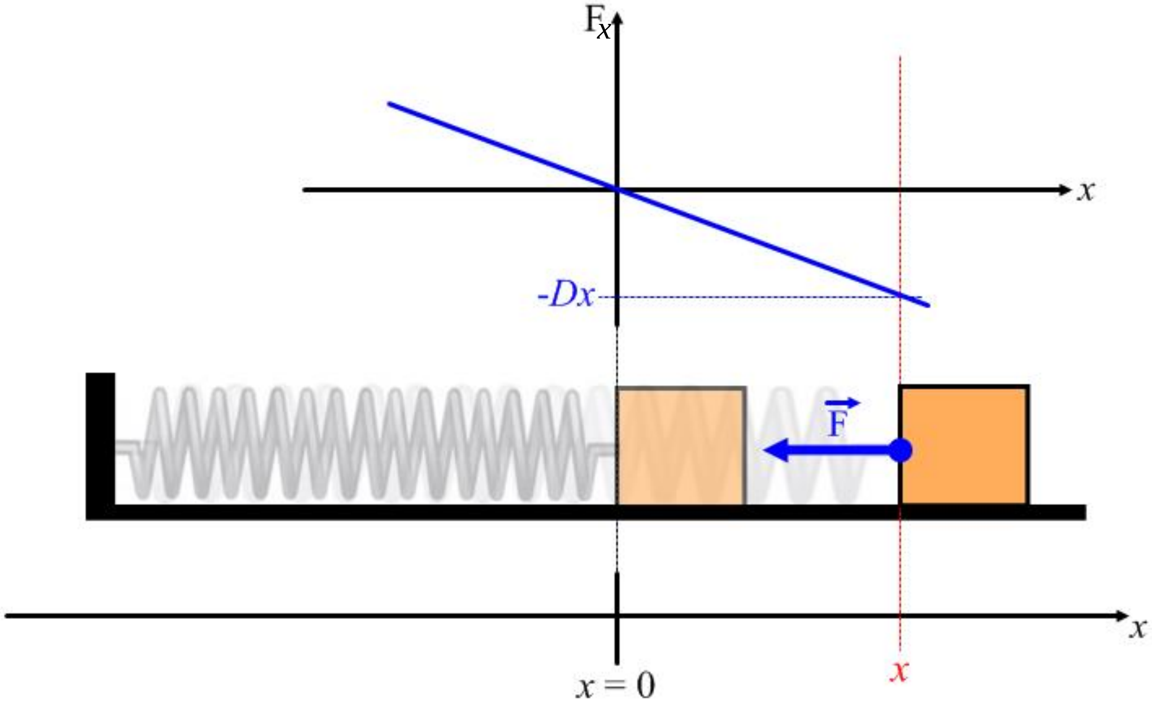
Increase in kinetic energy:

$$E'_{k0} - E_{k0} = \frac{1}{2}m\left[(-v_{0x} + 2u)^2 + {v_{0y}}^2\right] - \frac{1}{2}m\left({v_{0x}}^2 + {v_{0y}}^2\right) =$$

$$= \frac{1}{2}m\left({v_{0x}}^2 - 4v_{0x}u + 4u^2 + {v_{0y}}^2 - {v_{0x}}^2 - {v_{0y}}^2\right) = 2mu(u - v_{0x}) > 0 \qquad (v_{0x} < 0)$$

Equation of motion of harmonic oscillation

Harmonic oscillation: The force acting on the body must be harmonic: $F_x = -Dx$ (Hooke's law). For example, a body attached to a spring (if all other forces are negligible or cancel each other out, or are constant, such as the weight force).



Writing the equation of motion:

$$ma_x = -Dx$$

$$m\ddot{x} = -Dx$$

$$\frac{d^2x}{dt^2} = -\frac{D}{m}x$$

General solution
(position vs. time):

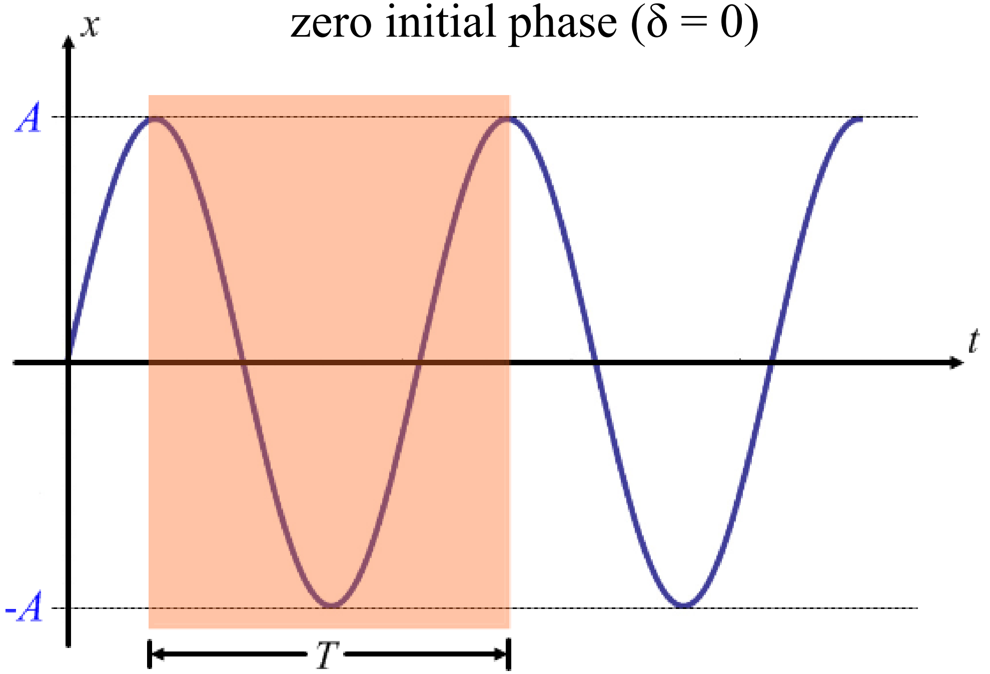
$$x(t) = A\sin(\omega t + \delta)$$

determined by initial conditions {
A: amplitude (maximum deviation)
δ: initial phase

ω: angular frequency (see later)

Position vs. time for harmonic oscillation

Sinusoidal harmonic vibration with zero initial phase (δ = 0)



$$x(t) = x(t + T)$$

$$\omega(t + T) = \omega t + 2\pi$$

$$\omega T = 2\pi$$

$$\omega = \frac{2\pi}{T}$$ angular frequency

$$\omega = 2\pi f$$

The displacement-time function:

$$x(t) = A\sin(\omega t + \delta)$$

Differentiating this, we get the velocity:

$$v_x(t) = \frac{dx}{dt} = A\omega\cos(\omega t + \delta)$$

The derivative of the velocity is the acceleration:

$$a_x(t) = \frac{dv_x}{dt} = -A\omega^2\sin(\omega t + \delta)$$

We can use: $x(t) = A\sin(\omega t + \delta)$

Thus the acceleration: $a_x(t) = -\omega^2 x$

We had in the equation of motion: $a_x = -\frac{D}{m}x$

Thus: $\omega^2 = \frac{D}{m}$

Kinetic and potential energy

Kinetic energy: Using the velocity-time function ($\delta = 0$)

$$E_K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t) = \frac{1}{2}DA^2\cos^2(\omega t)$$

Potential energy: Using the displacement-time function ($\delta = 0$) – elastic force field

$$E_P = \frac{1}{2}Dx^2 = \frac{1}{2}DA^2\sin^2(\omega t)$$

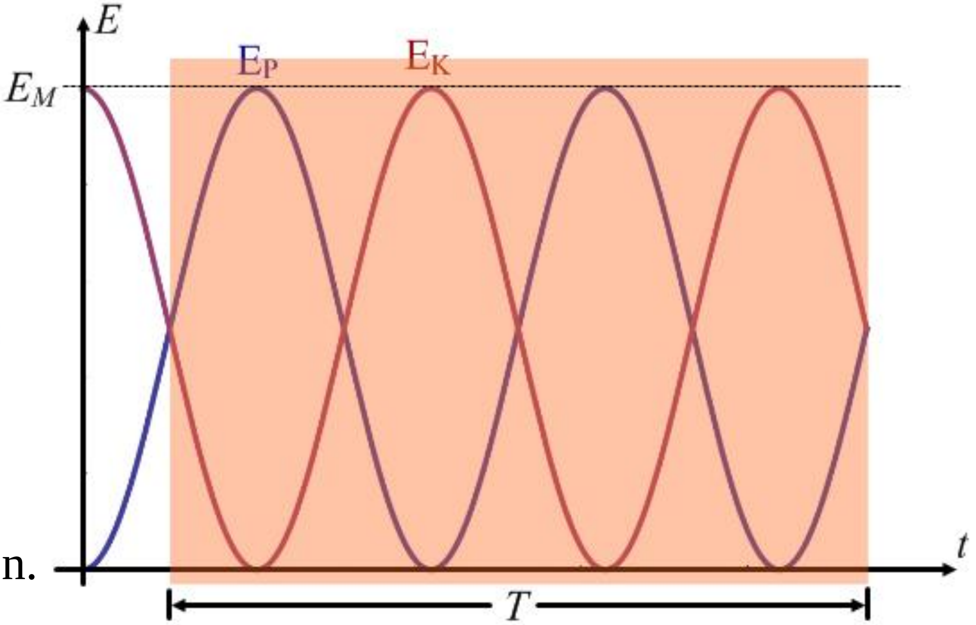
Mechanical energy:

The sum of potential and kinetic energy

$$E_M = E_K + E_P$$

$$= \frac{1}{2}DA^2\cos^2(\omega t) + \frac{1}{2}DA^2\sin^2(\omega t)$$

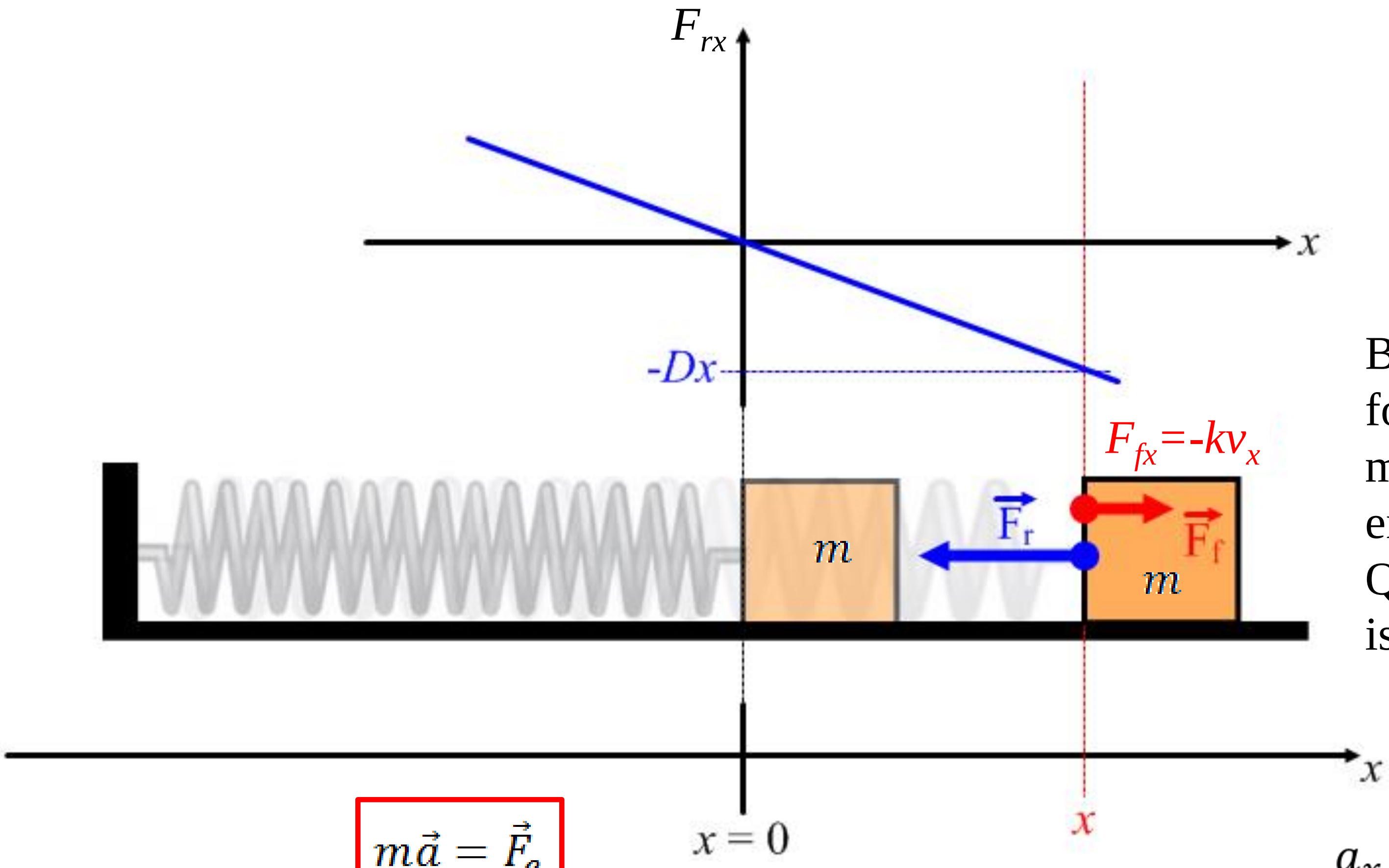
$$= \frac{1}{2}DA^2[\cos^2(\omega t) + \sin^2(\omega t)] = \frac{1}{2}DA^2$$



Potential and kinetic energy are converted back and forth into each other during motion.

Damped vibration

Damped vibration: In reality, sooner or later vibrations die down.
Besides the elastic force, taking into account a braking force proportional to the speed:



Because of the braking force the energy of the motion (mechanical energy) dissipates. Quasiperiodic motion is created.

$$\boxed{m\vec{a} = \vec{F}_e}$$

$$a_x = \ddot{x} \qquad v_x = \dot{x}$$

The equation of motion (rectilinear motion along *x*): $m\ddot{x} = -Dx - k\dot{x}$

Position vs. time for damped vibration

Starting from the equation of motion: $m\ddot{x} = -Dx - k\dot{x}$

$$\ddot{x} + \frac{k}{m}\dot{x} + \frac{D}{m}x = 0 \qquad \frac{D}{m} = \omega_0{}^2$$

ω_0 - would be the angular frequency of undamped vibration

$$\ddot{x} + 2\alpha\dot{x} + \omega_0{}^2 x = 0 \qquad \alpha = \frac{k}{2m}$$

α - the damping coefficient

Homogeneous, linear, second-order differential equation. Solution is exponential: $x = e^{\lambda t}$

Plugging in: $\lambda^2 e^{\lambda t} + 2\alpha\lambda e^{\lambda t} + \omega_0{}^2 e^{\lambda t} = 0$ and $e^{\lambda t} \neq 0$

Simplifying, we get the characteristic equation: $\lambda^2 + 2\alpha\lambda + \omega_0{}^2 = 0$

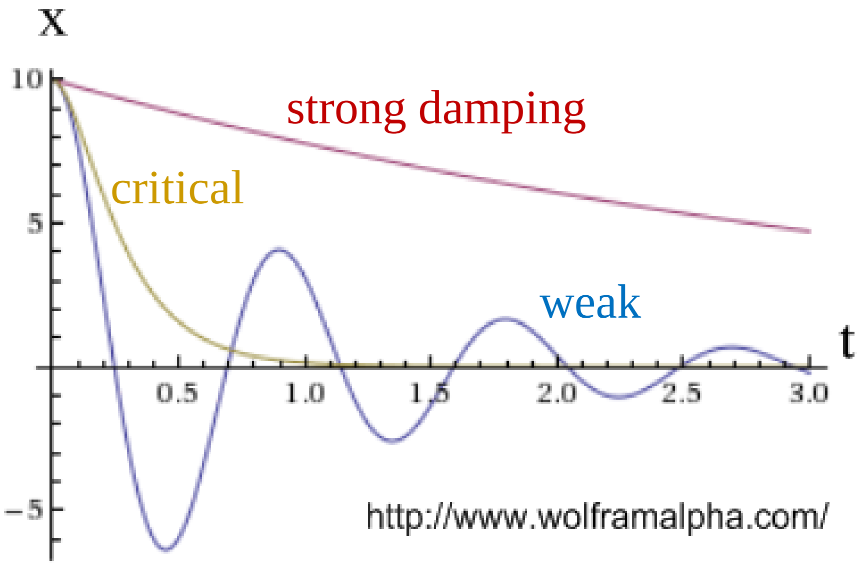
Solutions: $\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$

<u>Three possible cases:</u>

1. weak damping: $\alpha^2 - \omega_0{}^2 < 0$

2. critical damping: $\alpha^2 - \omega_0{}^2 = 0$

3. strong damping: $\alpha^2 - \omega_0{}^2 > 0$



http://www.wolframalpha.com/

Weakly damped vibration

$\alpha^2 - {\omega_0}^2 < 0$

Transforming the negative discriminant:

$$\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2} = -\alpha \pm i\sqrt{{\omega_0}^2 - \alpha^2} = -\alpha \pm i\omega$$

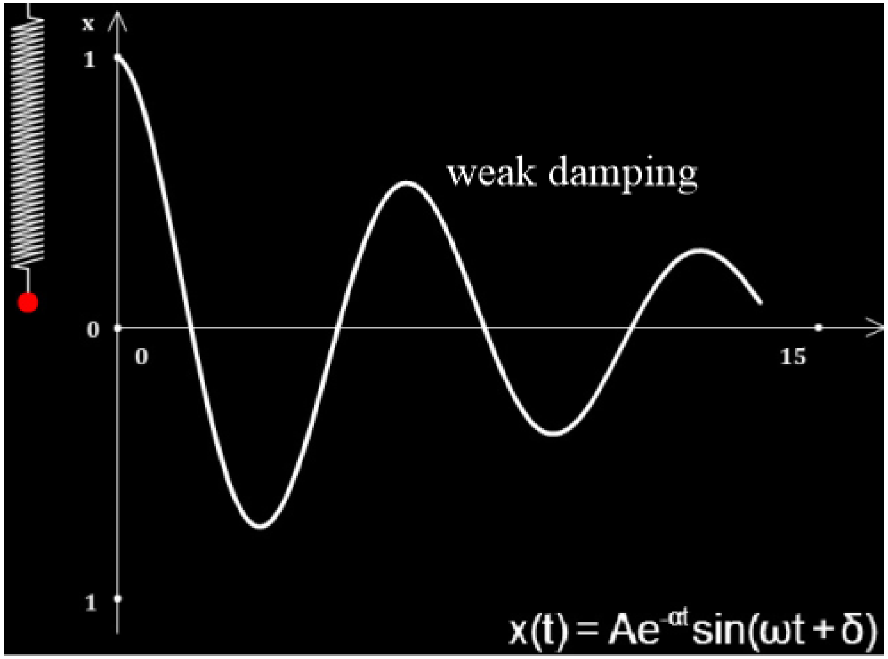
General solution of the differential equation:

$$x(t) = Ce^{-\alpha t}\sin(\omega t + \delta)$$

Differentiating, we get the general form of the velocity:

$$v_x(t) = -\alpha Ce^{-\alpha t}\sin(\omega t + \delta) + \omega Ce^{-\alpha t}\cos(\omega t + \delta)$$

The constants ***C*** and **δ** can be determined from the **boundary conditions**.
E. g. $x(0)$ and $v_x(0)$ can be given, and solving the two equations the constants can be calculated.



Forced vibration

A periodic force restores the dissipated energy:

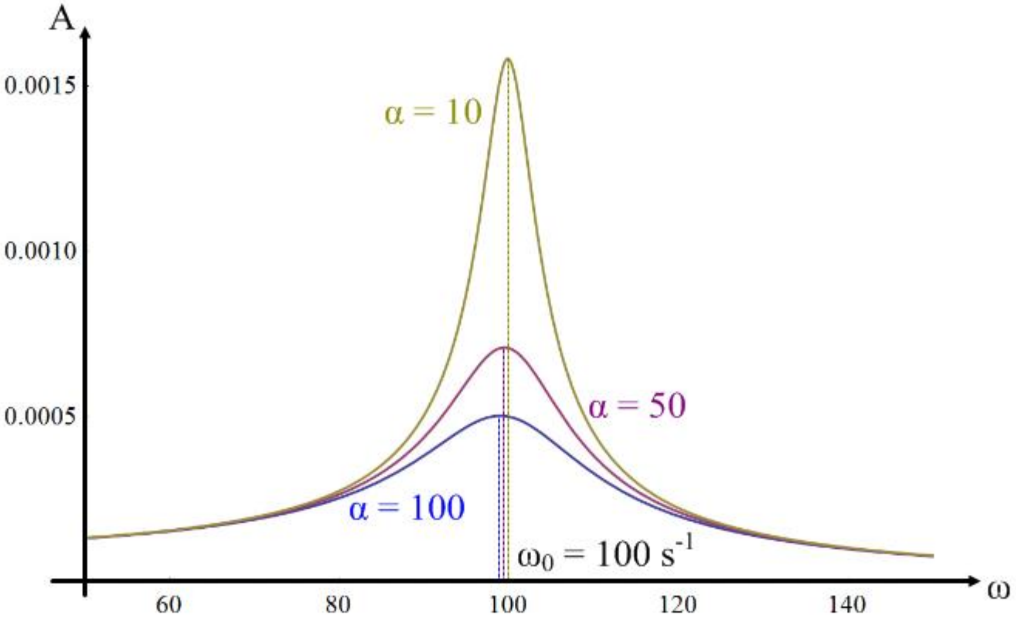
$$m\ddot{x} = -Dx - k\dot{x} + F_0 \sin(\omega t)$$

Solution: a decaying vibration (similar to the previous one) and a steady vibration at the excitation frequency.
Thus, after some time, the position vs. time:

$$x(t) = \frac{F_0/m}{\sqrt{(\omega_0{}^2 - \omega^2)^2 + 4\alpha^2\omega^2}} \sin(\omega t - \delta) \qquad \frac{D}{m} = \omega_0{}^2$$

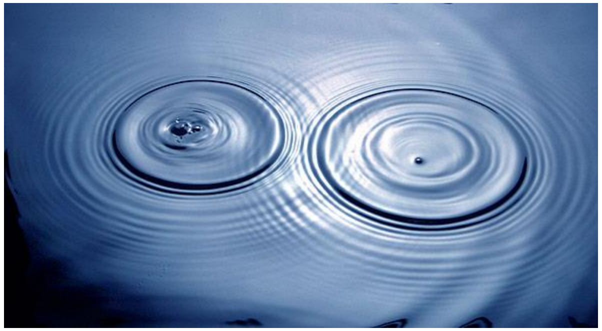
ω_0 - undamped
δ - phase delay

<u>Resonance:</u> The ω_r angular frequency, for which the amplitude of the vibration the greatest.
If the damping is weak (α is small), then $\omega_r \approx \omega_0$ and the amplitude can grow beyond all limits (until the system falls apart…) – resonance catastrophe.



Waves

Waves are created when the vibration of a part of an elastic medium propagates through the medium, with neighboring points also taking over the vibration.
E.g. guitar string (1D), water surface (2D), sound or light (3D)



The speed of propagation is the **phase velocity** of the wave (c).
This determines the time delay between the vibrations of two distant points.
Consider a plane wave propagating in the x direction
(or a wave propagating on a 1D string).
The vibration at $x = 0$ is the usual harmonic function: $y(t) = A\sin(\omega t)$
In comparison, at location x, the vibration is delayed by x/c time:

$$y(x,t) = A\sin\left[\omega\left(t - \frac{x}{c}\right)\right] = A\sin\left(\omega t - \frac{\omega x}{c}\right)$$

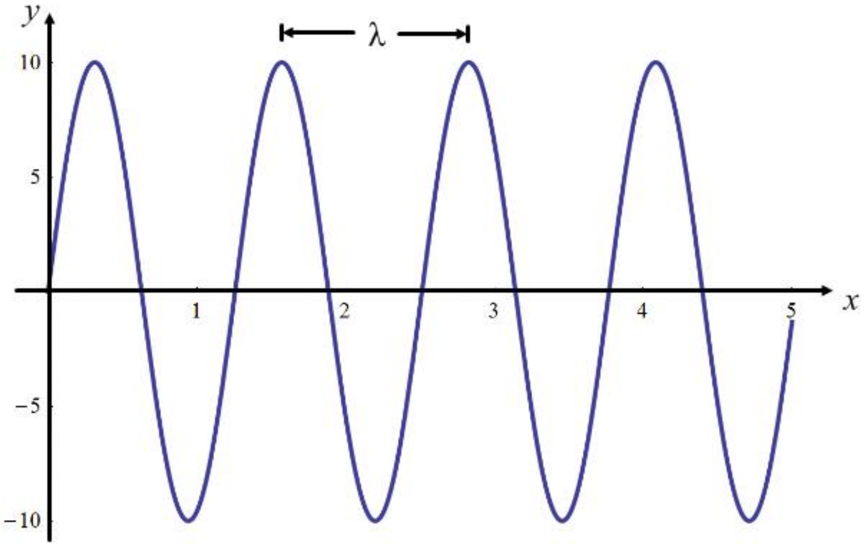
$$= A\sin\left(\omega t - \frac{2\pi x}{Tc}\right) = A\sin\left(\omega t - \frac{2\pi x}{\lambda}\right) = A\sin(\omega t - kx)$$

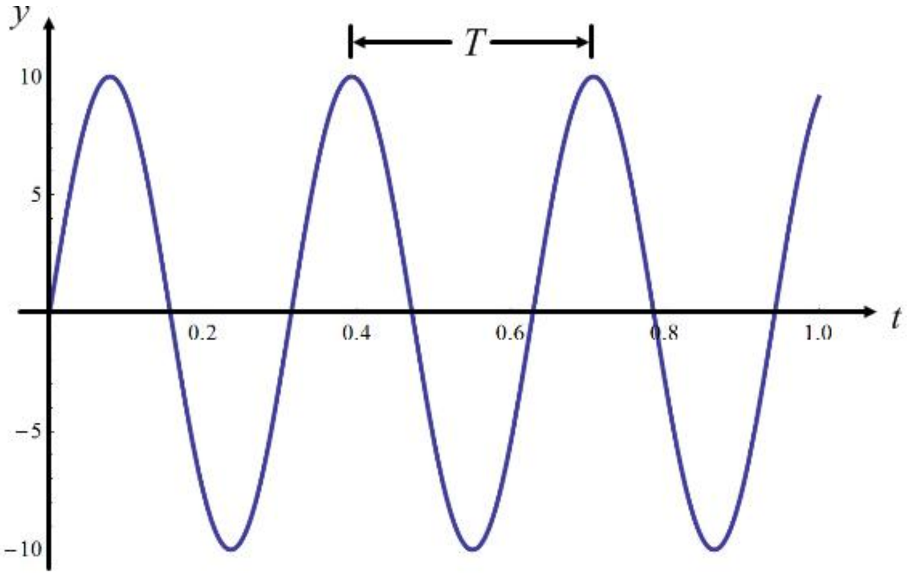
T: **period** of vibration ω: **angular frequency** of the vibration $\omega = 2\pi f$ **=** $2\pi/T$
λ: **wavelength** (distance traveled during one period) $\lambda = Tc$
k: **angular wave number** $k = 2\pi/\lambda$ Since: $T = 1/f$ therefore $\boxed{c = \lambda f}$

Waves: spatial and temporal dependence

For a wave, both the spatial and temporal dependences are periodic functions:

$$y(x,t) = A\sin(\omega t - kx)$$

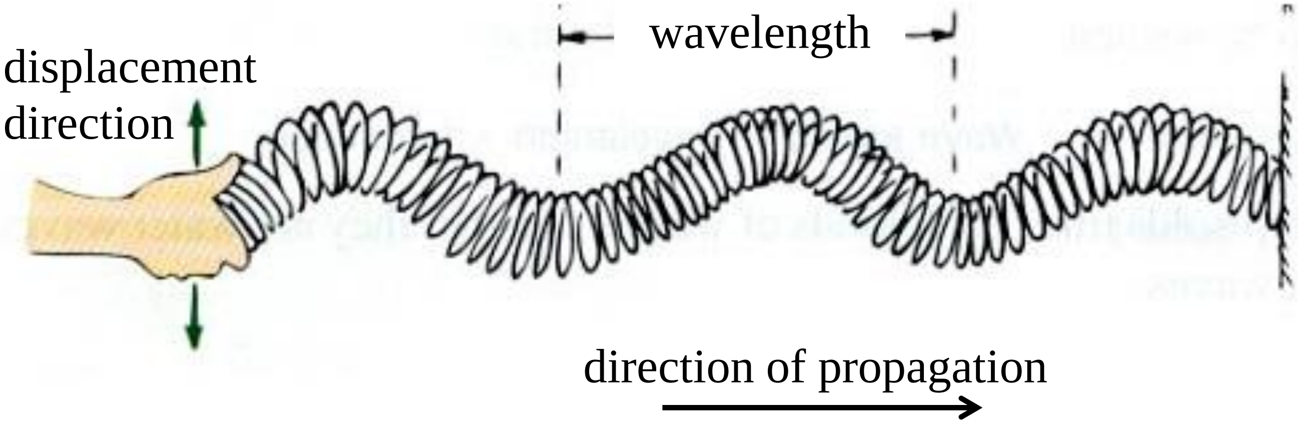




Spatial periodicity is the wavelength
(snapshot at a given time)

Temporal periodicity is the period
(time dependence at a given location)

Transversal and longitudinal waves

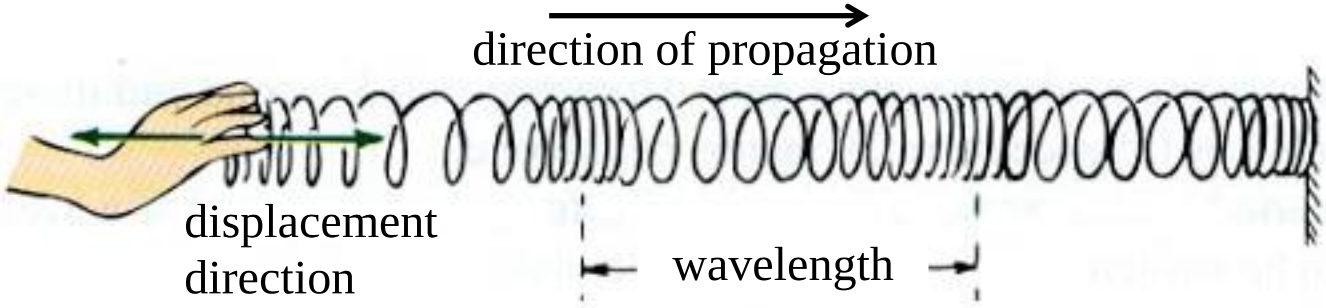


displacement direction

wavelength

direction of propagation

transversal wave:
the displacement is perpendicular
to the direction of propagation



direction of propagation

displacement direction

wavelength

longitudinal wave:
the displacement is parallel with the direction of propagation

sound is also
a longitudinal
wave
(20Hz – 20kHz)

Standing waves

Reaching the boundary of the medium, the wave is reflected. When the incoming and reflected waves meet, their displacements add up (they **interfere**). In some cases, a **standing wave** may be created. Consider a wave traveling in the *x* direction and a reflected wave traveling in the –*x* direction:

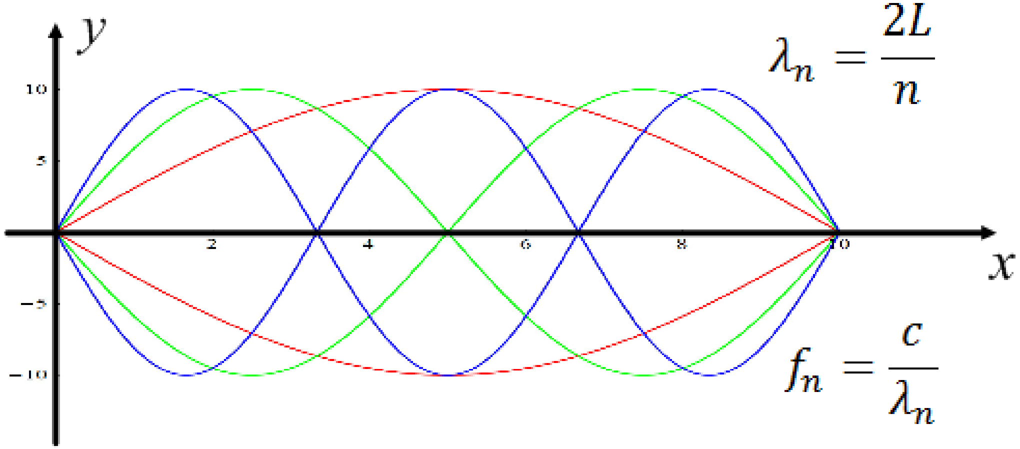
$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$y(x,t) = A\sin(\omega t - kx) + A\sin(\omega t + kx)$$

$$y(x,t) = A\sin(\omega t)\cos(kx) - A\cos(\omega t)\sin(kx) + A\sin(\omega t)\cos(kx) + A\cos(\omega t)\sin(kx)$$

$$y(x,t) = 2A\sin(\omega t)\cos(kx) = 2A\cos(kx)\sin(\omega t) = 2A(x)\sin(\omega t)$$

The amplitude becomes position dependent: **nodes** and **antinodes**.
However, the phase no longer depends on time (position and time dependence are decoupled).
The wavelength of the standing wave can only be the wavelengths allowed by the given geometry: **boundary conditions**
E.g. node at the end of fixed-end string, antinode at the end of open-end whistle.
Therefore, the frequency cannot be arbitrary either: fundamental frequency and harmonics.



$$\lambda_n = \frac{2L}{n}$$

$$f_n = \frac{c}{\lambda_n}$$

Doppler effect

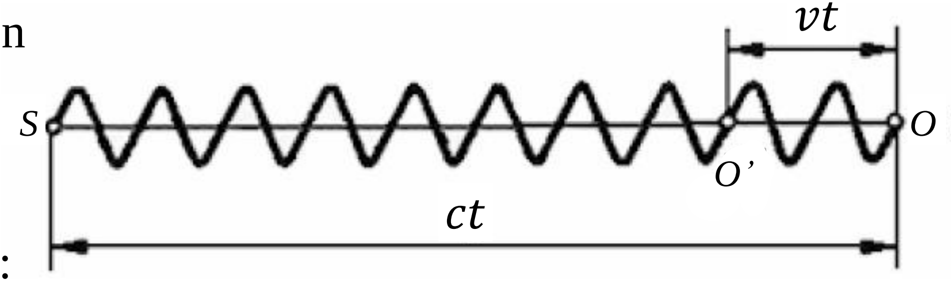
If the source of the wave and the observer are moving relative to each other, the emitted frequency will not match the detected frequency.

E.g. sound of approaching and receding siren

If **observer** *O* **approaches** with speed *v*:

Detects not only a wave train of length *ct*, but length *ct* + *vt*.

Number of vibrations detected per unit time:

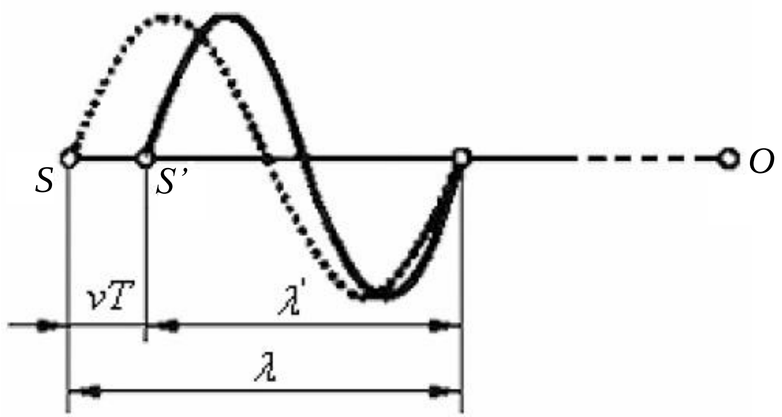


$$f' = \frac{ct \pm vt}{\lambda t} = f\left(1 \pm \frac{v}{c}\right)$$

(-*v*/*c* if receding)

If **source** *S* **approaches** with speed *v*:

The wavelength appears to shorten with *vT*:



$$\lambda' = \lambda - vT$$

$$f' = \frac{c}{\lambda'} = \frac{c}{\lambda \mp vT} = \frac{c}{\lambda}\frac{1}{1 \mp \frac{vT}{\lambda}} = f\frac{1}{1 \mp \frac{v}{c}}$$

(+*v*/*c* if receding)

If both are moving **relative to the medium**: $f' = f\,\dfrac{1 \pm \frac{v_O}{c}}{1 \mp \frac{v_S}{c}}$