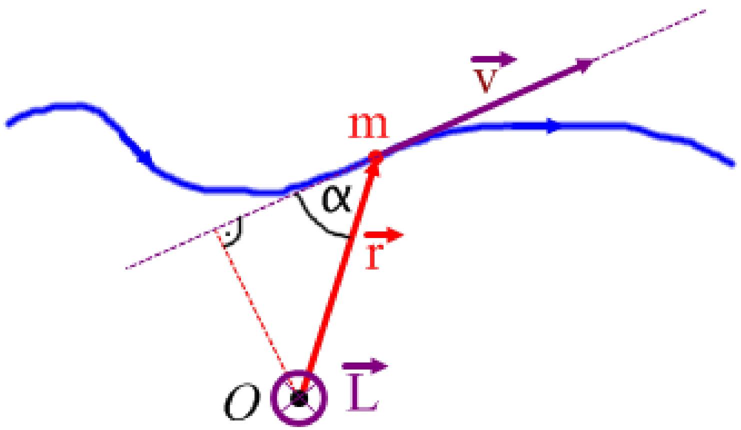
Angular momentum

Generally the **angular momentum** of a point-like body is: $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$
(similar to the definition of torque, replacing force with momentum)



If the position and velocity vectors are perpendicular, as for uniform circular motion:

$$L = rmv = mrv = mr\omega r = mr^2\omega$$

The angular momentum vector changes under the influence of torque:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(m\vec{r} \times \vec{v}) = m\left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) = m(\vec{v} \times \vec{v} + \vec{r} \times \vec{a}) =$$

$$= \vec{r} \times (m\vec{a}) = \vec{r} \times \vec{F}_{net} = \vec{\tau}_{net}$$

Torque-angular momentum law: $$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

Moment of inertia

<u>Special case:</u> point mass moves at a constant distance around a **fixed axis** (circular motion)

$$L(t) = mr^2\omega(t)$$

Derivative of the angular momentum: $\dfrac{dL}{dt} = mr^2\dfrac{d\omega(t)}{dt} = mr^2\beta(t)$

β is the angular acceleration, and the term mr^2 is the **moment of inertia** of the point mass.

The moment of inertia for the center of mass is therefore: $I = mr^2$,
where r is the distance from the axis.

Using the torque-angular momentum law: $\dfrac{dL}{dt} = mr^2\dfrac{d\omega(t)}{dt} = I\beta(t) = \tau$

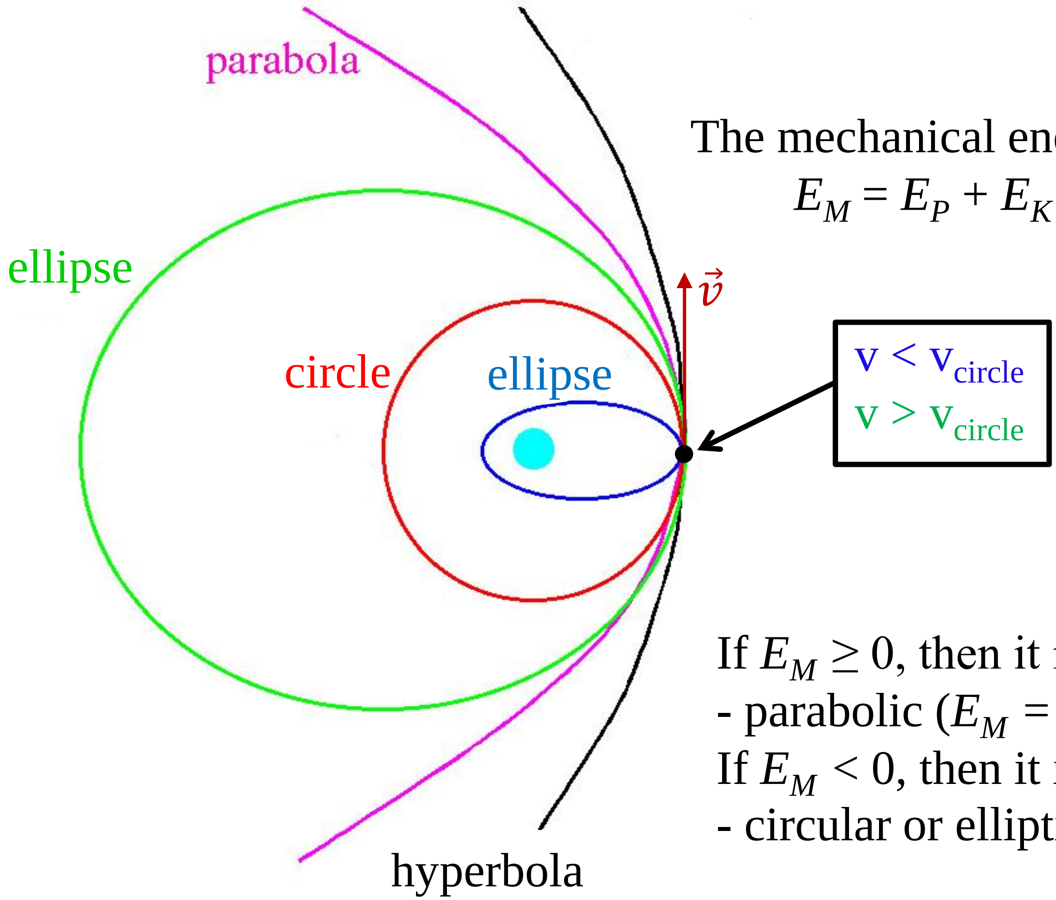
We get the fundamental equation of rotary motion: $\tau = I\beta$

The **kinetic energy** of the point mass: $E_K = \dfrac{1}{2}mv^2 = \dfrac{1}{2}mr^2\omega^2 = \dfrac{1}{2}I\omega^2$

Movement of planets and moons

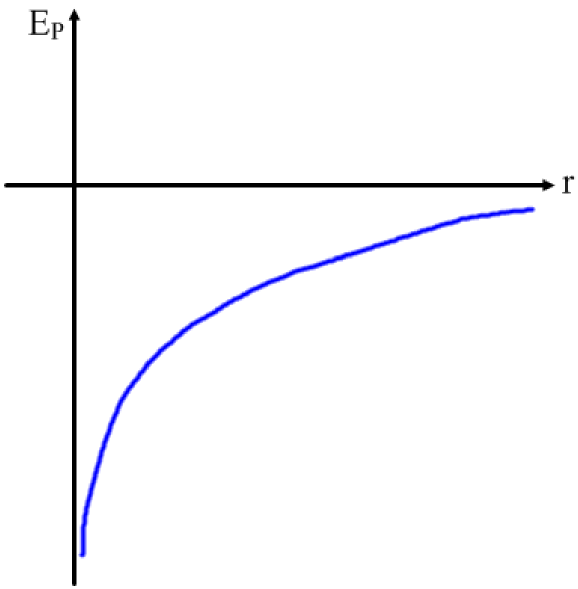
Suppose a body of mass m moves in the gravitational field of a body of much larger mass (M). Since M is much larger than m, it can be considered stationary. e.g. Sun and Earth.

A body of mass m has E_K kinetic energy and E_P potential energy.
We take the zero point at infinity: $E_P = 0$, if $r \rightarrow \infty$



The mechanical energy:

$$E_M = E_P + E_K$$



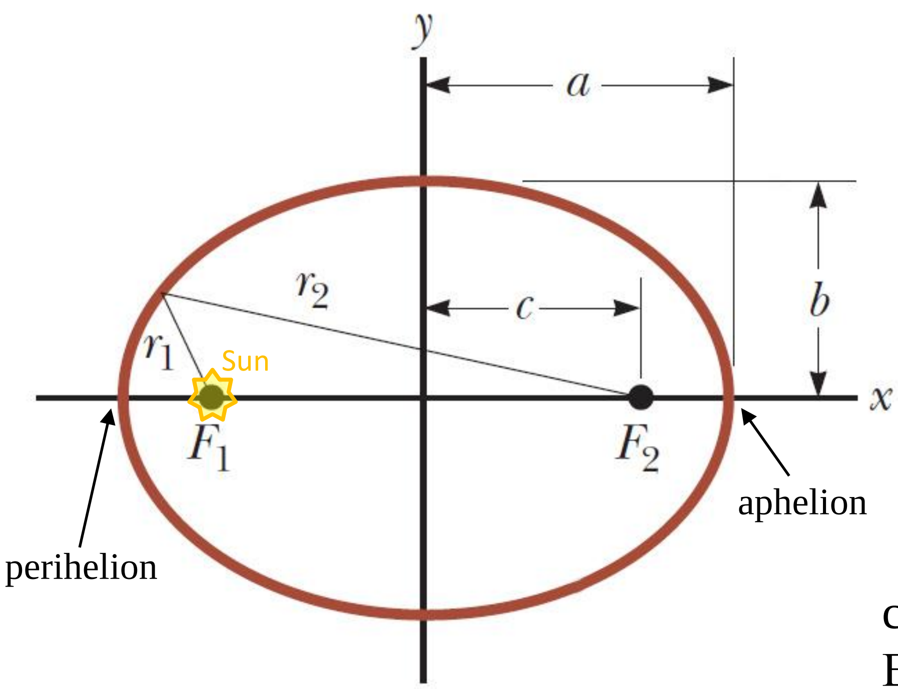
If $E_M \geq 0$, then it is free (can move away to infinity)
- parabolic ($E_M = 0$) or hyperbolic trajectory

If $E_M < 0$, then it is bound (remains at a finite distance)
- circular or elliptical orbit

Kepler's 1st law

For bodies moving in a bound state in the gravitational field of a massive body (e.g. planets).

I. The orbits of the planets are ellipses, and the Sun is at one of its foci.



ellipse: points for which $r_1 + r_2 = \text{constant}$

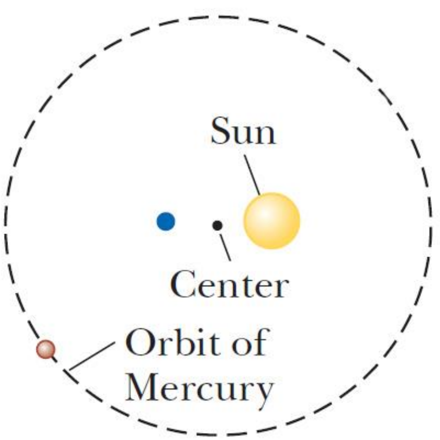
$$a^2 = b^2 + c^2$$

a: semi-major axis
b: semi-minor axis

eccentricity: $e = c/a$

$0 < e < 1$

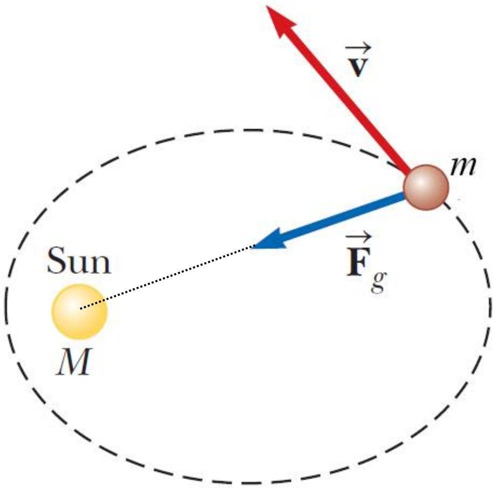
circle: $e = 0$
Earth's orbit: $e = 0.017$
Mercury's orbit: $e = 0.21$
Comet Halley: $e = 0.97$



For moons around planets: perigee and apogee

Kepler's 2nd law

II. (Equal area law) The lines between the Sun and the planets sweep out equal areas in equal times. Planets move faster near the perihelion. It follows from the conservation of angular momentum: there is no torque in a central force field.



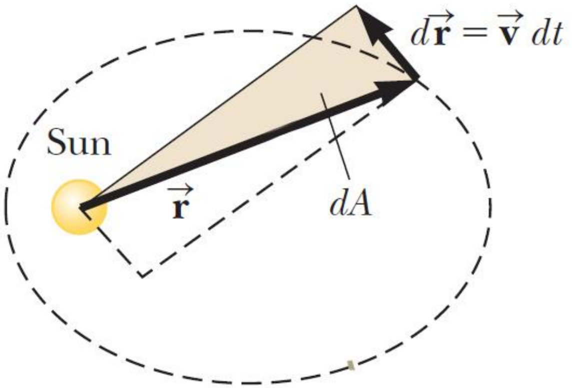
$$\vec{L} = \text{constant}$$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$$

For its magnitude: $L = m|\vec{r} \times \vec{v}|$

In time dt the area dA is swept out:

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}dt| = \frac{1}{2}|\vec{r} \times \vec{v}|dt$$



$$dA = \frac{L}{2m}dt$$

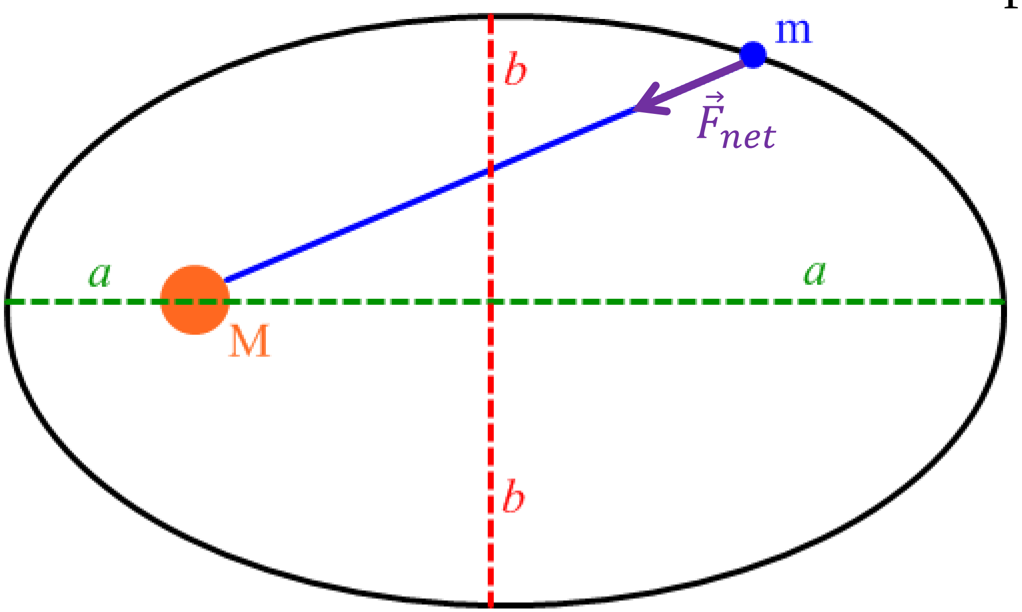
$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

Kepler's 3rd Law

III. The cubes of the semi-major axes (a) of elliptical orbits are proportional to the squares of the orbital periods (T) of the planets moving in the given orbits.

So for every planet (and any body) orbiting the Sun: $\frac{a^3}{T^2} = \text{constant}$

All three laws can be derived from Newton's laws and the Newtonian force law of gravity.



Proof for a circular orbit: $a = b = R$

$$F_{net} = ma$$

$$\gamma \frac{Mm}{R^2} = m\omega^2 R$$

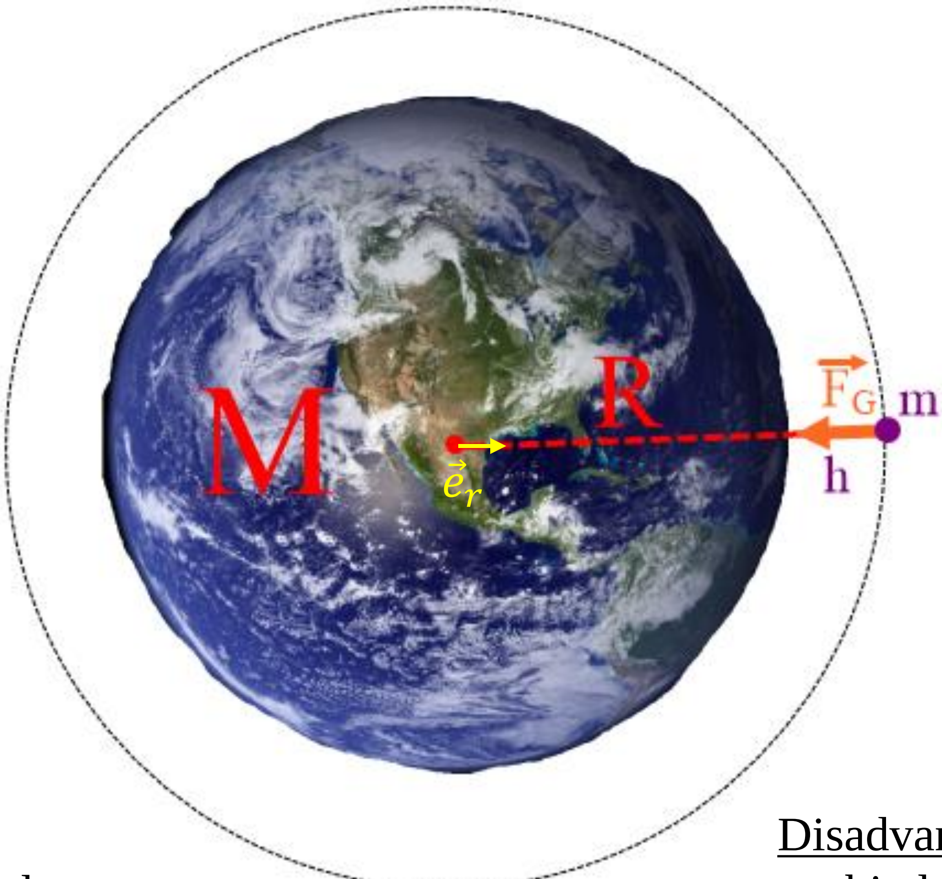
$$\gamma \frac{M}{R^2} = \frac{4\pi^2}{T^2} R$$

$$\frac{\gamma M}{4\pi^2} = \frac{R^3}{T^2} = \text{constant}$$

Orbits of satellites

Low Earth orbit (LEO)

Circular orbits at altitudes between 160km and 2000km. For the ISS it is 408km.



The only force acting is gravity:

$$\vec{F}_{net} = m\vec{a} = -ma_{cp}\vec{e}_r$$

$$\vec{F}_{net} = \vec{F}_G = -G\frac{Mm}{(R+h)^2}\vec{e}_r$$

$$G\frac{M}{(R+h)^2} = a_{cp} = \frac{v^2}{R+h}$$

$$\frac{GM}{(R+h)^2} = \omega^2(R+h) = \frac{4\pi^2}{T^2}(R+h)$$

$$T^2 = \frac{4\pi^2}{GM}(R+h)^3$$

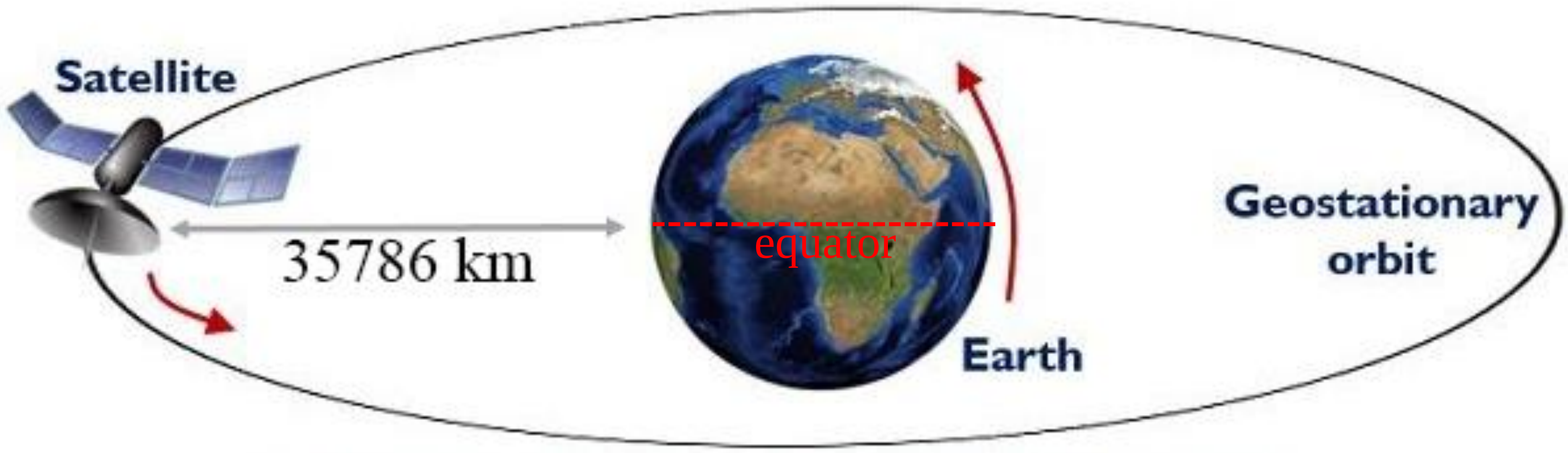
$$T = \frac{2\pi}{\sqrt{GM}}(R+h)^{3/2}$$

<u>Advantages:</u>
- quick communication
- low launch cost
- high-resolution imaging

<u>Disadvantages:</u>
- orbital decay
- space debris
- atomic oxygen
- small viewing area

Geostationary orbit (GEO)

Circular orbit above the equator with exactly one day period.



$$T = \frac{2\pi}{\sqrt{GM}}(R+h)^{3/2}$$

<u>Advantages:</u>
- large viewing area
- always above same spot

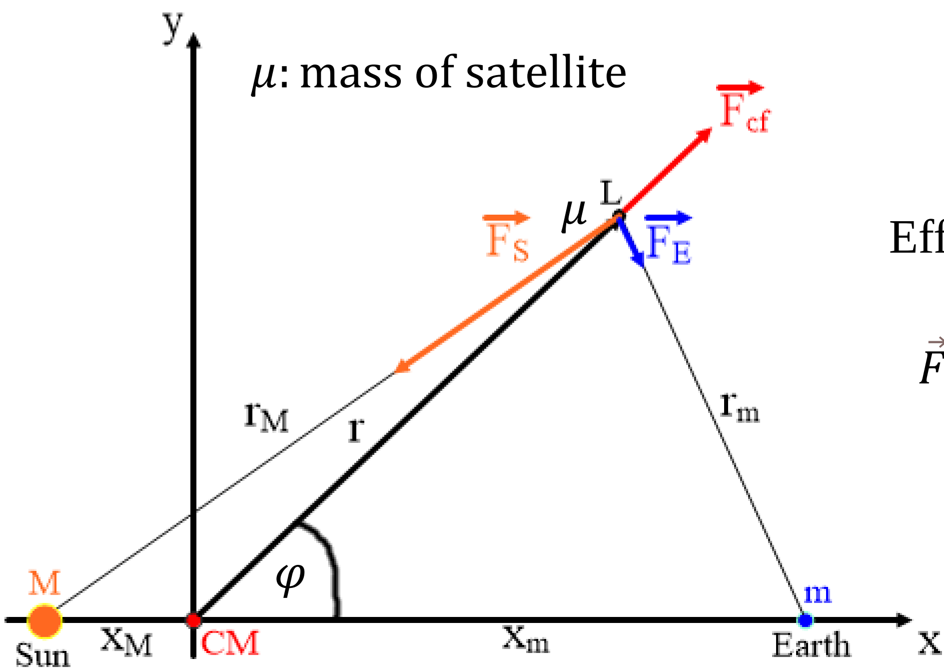
<u>Disadvantages:</u>
- lag because of distance
- crowded area

Lagrange points

Equilibrium points in the Sun-Earth rotating reference frame.
Rotation center is the center of mass.

$$T = 365.256 \text{ days}$$

$$R = x_M + x_m = 149.6 \cdot 10^6 \text{ km}$$



$$\vec{F}_{cf} = \mu a_{cp} \vec{e}_r = \mu \omega^2 r \vec{e}_r \qquad \omega = \frac{2\pi}{T}$$

Effective potential energy of $\vec{F}_{cf}$, such that:

$$\vec{F}_{cf} = -\nabla E_{Pcf} = -\vec{e}_r \frac{\partial E_{Pcf}}{\partial r} - \underbrace{\vec{e}_\varphi \frac{1}{r} \frac{\partial E_{Pcf}}{\partial \varphi}}_{0}$$

$$E_{Pcf} = -\frac{1}{2} \mu \omega^2 r^2$$

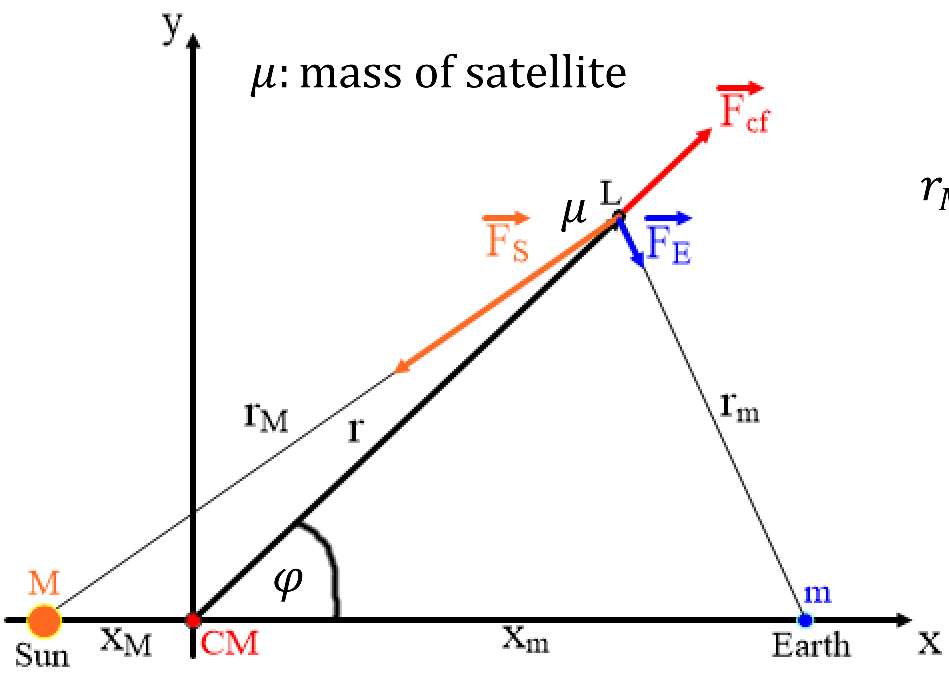
$$\frac{x_M}{x_m} = \frac{m}{M} \quad \rightarrow \quad R = x_M + x_m$$

$$R = x_M + x_M \frac{M}{m} = x_M \left(1 + \frac{M}{m}\right)$$

$$E_{PS} = -\frac{GM\mu}{r_M} \qquad E_{PE} = -\frac{Gm\mu}{r_m}$$

$$x_m = x_M \frac{M}{m} = \frac{MmR}{m^2 + mM} = \frac{MR}{m + M} \quad \leftarrow \quad x_M = \frac{R}{1 + \frac{M}{m}} = \frac{mR}{m + M}$$

Potential energy



$$r_M{}^2 = r^2 + x_M{}^2 - 2rx_M\cos(180° - \varphi) =$$
$$= r^2 + x_M{}^2 + 2rx_M\cos\varphi$$
$$r_M = \sqrt{r^2 + x_M{}^2 + 2rx_M\cos\varphi}$$

$$r_m{}^2 = r^2 + x_m{}^2 - 2rx_m\cos\varphi =$$
$$r_m = \sqrt{r^2 + x_m{}^2 - 2rx_m\cos\varphi}$$

Total potential energy: $E_P = -\frac{GM\mu}{r_M} - \frac{Gm\mu}{r_m} - \frac{1}{2}\mu\omega^2 r^2$

Equilibrium points where $\nabla E_P = 0$

Using polar coordinates: $\nabla E_P = \vec{e}_r\frac{\partial E_P}{\partial r} + \vec{e}_\varphi\frac{1}{r}\frac{\partial E_P}{\partial \varphi}$

$$\frac{\partial E_P}{\partial r} = 0 \quad \text{and} \quad \frac{1}{r}\frac{\partial E_P}{\partial \varphi} = 0$$

Equations

$$r_M = \sqrt{r^2 + {x_M}^2 + 2r x_M \cos\varphi} \qquad r_m = \sqrt{r^2 + {x_m}^2 - 2r x_m \cos\varphi}$$

$$E_P = -\frac{GM\mu}{r_M} - \frac{Gm\mu}{r_m} - \frac{1}{2}\mu\omega^2 r^2$$

$$0 = \frac{\partial E_P}{\partial r} = \frac{GM}{{r_M}^2}\frac{1}{2r_M}(2r + 2x_M\cos\varphi) + \frac{Gm}{{r_m}^2}\frac{1}{2r_m}(2r - 2x_m\cos\varphi) - \omega^2 r$$

$$0 = \frac{GM}{{r_M}^3}(r + x_M\cos\varphi) + \frac{Gm}{{r_m}^3}(r - x_m\cos\varphi) - \omega^2 r \qquad (1)$$

$$0 = \frac{1}{r}\frac{\partial E_P}{\partial\varphi} = \frac{1}{r}\left[\frac{GM}{{r_M}^2}\frac{1}{2r_M}(-2r x_M\sin\varphi) + \frac{Gm}{{r_m}^2}\frac{1}{2r_m}(2r x_m\sin\varphi)\right]$$

$$0 = \frac{m}{{r_m}^3}(x_m\sin\varphi) - \frac{M}{{r_M}^3}(x_M\sin\varphi) \qquad (2)$$

In equation (2) solution when $\sin\varphi = 0 \;\rightarrow\; \varphi = 0$ and $\varphi = \pi$

These are points on the line connecting the Sun and Earth.
In equation (2), if $\sin\varphi \neq 0$, it can be simplified:

$$0 = \frac{m}{{r_m}^3}x_m - \frac{M}{{r_M}^3}x_M \;\rightarrow r_m = r_M \qquad \text{because} \qquad \frac{x_M}{x_m} = \frac{m}{M}$$

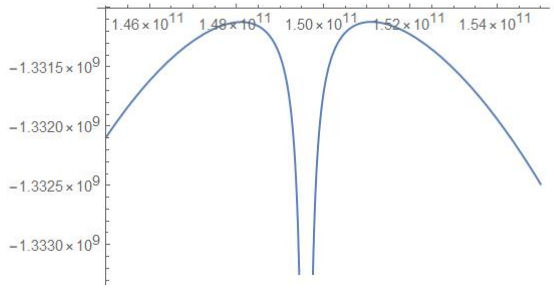
Solutions for L1, L2, L3

In[21]:= $E_P = -\frac{G\,M}{r_S} - \frac{G\,m}{r_E} - \frac{1}{2}\omega^2 r^2$

Out[21]= $-1.98202\times10^{-14}\, r^2 - \frac{3.98602\times10^{14}}{\sqrt{2.238\times10^{22} - 2.99199\times10^{11}\, r + r^2}} - \frac{1.32717\times10^{20}}{\sqrt{2.01877\times10^{11} + 898615.\, r + r^2}}$

In[29]:= Plot[E_P, {r, 1.45×10^{11}, 1.55×10^{11}}]

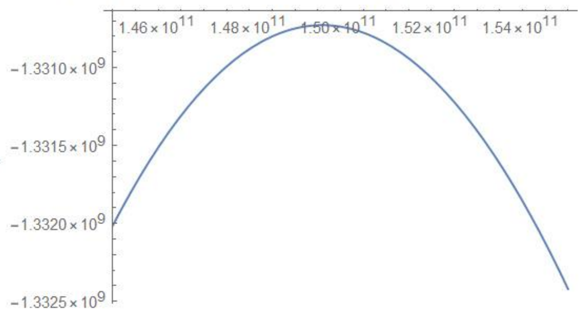
Out[29]=



$\varphi = 0$

In[44]:= Plot[E_P, {r, 1.45×10^{11}, 1.55×10^{11}}]

Out[44]=



$\varphi = \pi$

In[31]:= $dr = \partial_r E_P$

Out[31]= $-3.96404\times10^{-14}\, r + \frac{1.99301\times10^{14}\,(-2.99199\times10^{11} + 2\,r)}{(2.238\times10^{22} - 2.99199\times10^{11}\, r + r^2)^{3/2}} + \frac{6.63584\times10^{19}\,(898615. + 2\,r)}{(2.01877\times10^{11} + 898615.\, r + r^2)^{3/2}}$

In[32]:= Solve[dr == 0, r]

Out[32]= $\{\{r \to -1.496\times10^{11}\}, \{r \to -1.496\times10^{11}\}, \{r \to 1.48108\times10^{11}\}, \{r \to 1.51101\times10^{11}\}\}$

$\varphi = \pi$ $\qquad\qquad$ $\varphi = 0$

Solutions for L4, L5

In[15]:= $\phi = \pi / 3$

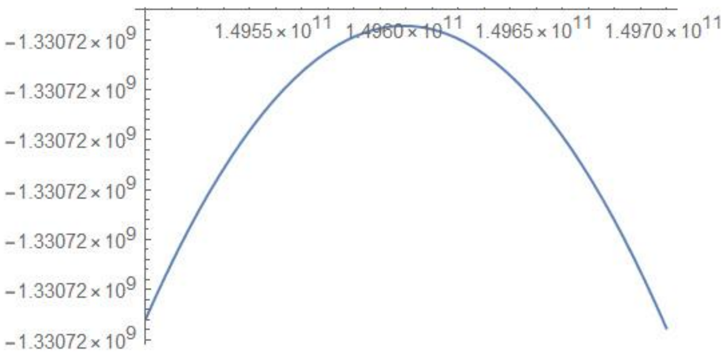
Out[15]= $\frac{\pi}{3}$

In[16]:= $E_P = -\frac{G\,M}{r_S} - \frac{G\,m}{r_E} - \frac{1}{2}\omega^2 r^2$

Out[16]= $-1.98202\times10^{-14}\, r^2 - \frac{3.98602\times10^{14}}{\sqrt{2.238\times10^{22} - 1.496\times10^{11}\, r + r^2}} - \frac{1.32717\times10^{20}}{\sqrt{2.01877\times10^{11} + 449\,308.\, r + r^2}}$

In[25]:= Plot[E_P, {r, 1.495×10^{11}, 1.497×10^{11}}]

Out[25]=



$\varphi = \pi/3$ or $-\pi/3$

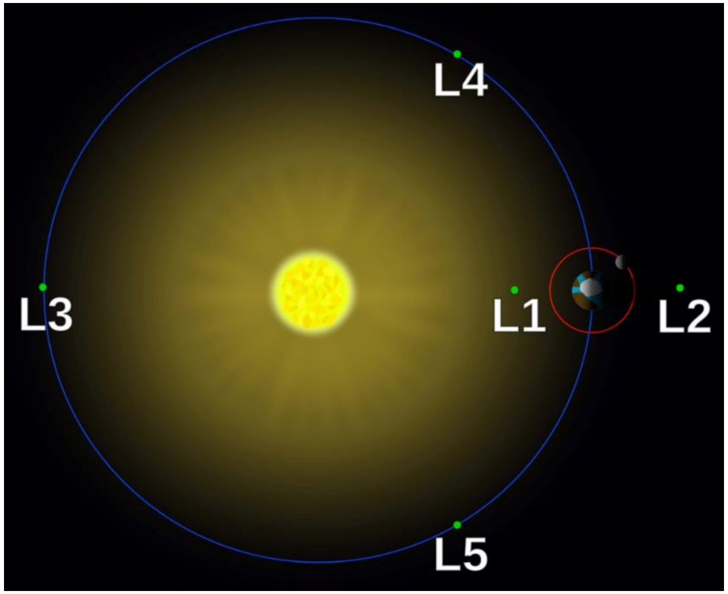
In[26]:= $dr = \partial_r E_P$

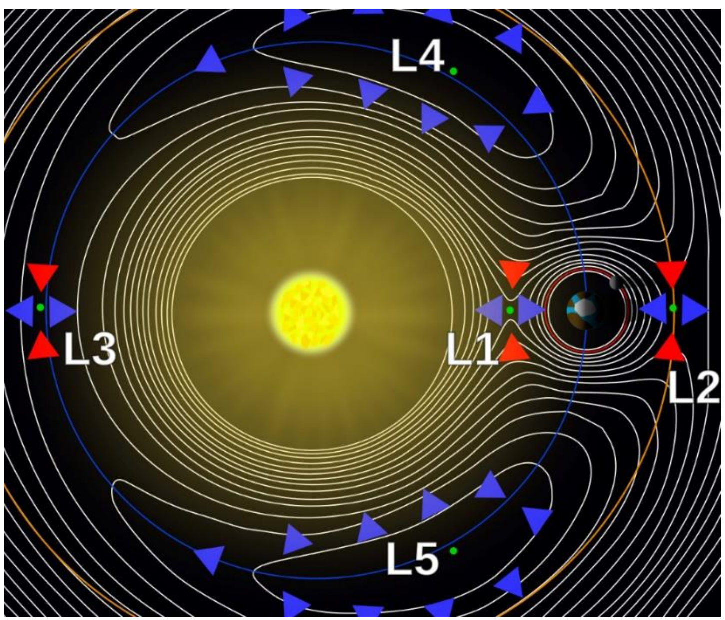
Out[26]= $-3.96404\times10^{-14}\, r + \frac{1.99301\times10^{14}\,(-1.496\times10^{11} + 2\,r)}{(2.238\times10^{22} - 1.496\times10^{11}\, r + r^2)^{3/2}} + \frac{6.63584\times10^{19}\,(449\,308. + 2\,r)}{(2.01877\times10^{11} + 449\,308.\, r + r^2)^{3/2}}$

In[27]:= Solve[dr == 0, r]

Out[27]= $\{\{r \to -1.49599\times10^{11}\}, \{r \to -1.49599\times10^{11}\}, \{r \to -224\,654.\}, \{r \to 7.48033\times10^{10} - 1.29547\times10^{11}\, i\}, \{r \to 7.48033\times10^{10} + 1.29547\times10^{11}\, i\}, \{r \to 1.49599\times10^{11}\}, \{r \to 1.49599\times10^{11}\}\}$

Locations





<u>L1 and L2:</u>
1.5 million km from Earth
L1 for solar observation
L2 for deep space infrared (JWST)

<u>L3:</u>
150 million km from center of mass
Sun blocks direct communication

Unstable points, station keeping is needed.

<u>L4 and L5:</u>
Forming equilateral triangle.
L4 is 60º ahead, L5 is 60º behind.
Stable for Sun-Earth pair.

$$\frac{M}{m} > 24.96$$