Uniformly changing motion along a straight line

If the initial velocity vector and the acceleration vector fall in the same line, then the body will move along that line (one coordinate is enough, e.g. z).

Thus, the body moves in the direction of the *z* axis with constant a_z acceleration and a time-dependent velocity v_z (these components can also be negative!). The other components (*x*, *y*) are zero. The velocity at time t_1 is:

$$v_z(t_1) = \int_{t_0}^{t_1} a_z(t)\,dt + v_z(t_0) = a_z(t_1 - t_0) + v_z(t_0)$$

let $t_0 = 0$

$v_z(t_0) = v_{z0}$

With these: $\boxed{v_z(t) = a_z t + v_{z0}}$

$(\Delta v_z = a_z \Delta t\,)$

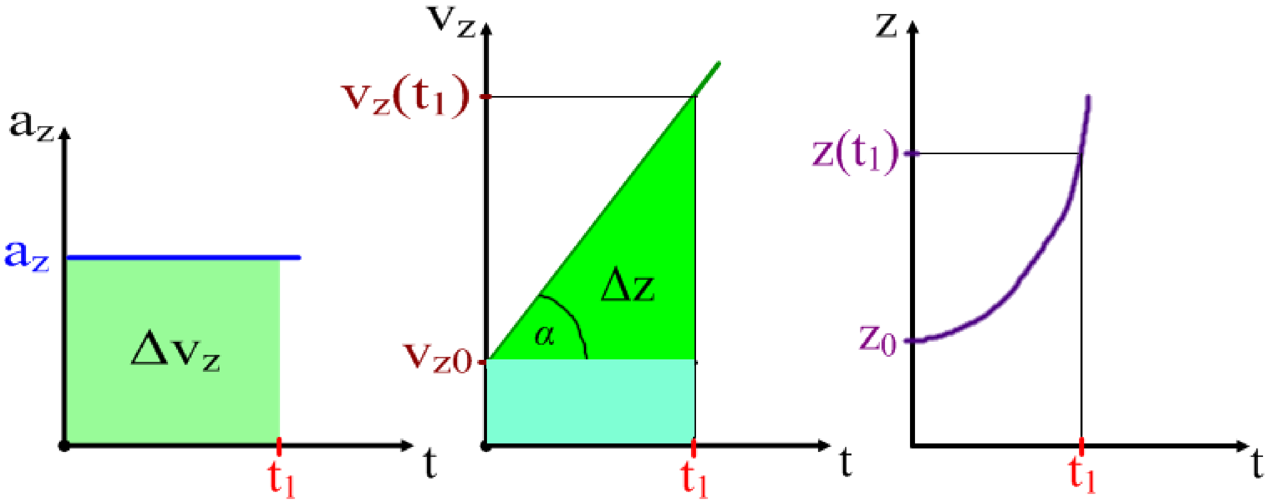
The position:

$$z(t_1) = \int_{t_0}^{t_1} v_z(t)\,dt + z(t_0) = \int_{t_0}^{t_1} (a_z t + v_{z0})\,dt + z(t_0)$$

let $z(t_0) = z_0$

Thus if $t_0 = 0$:

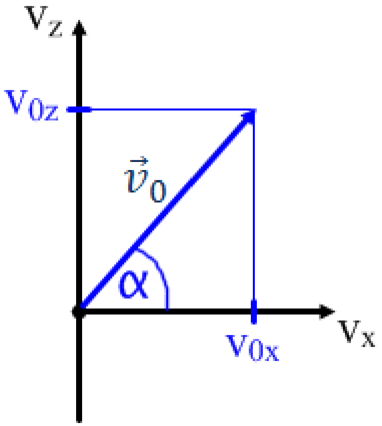
$$\boxed{z(t) = \frac{1}{2} a_z t^2 + v_{z0} t + z_0}$$



Projectile motion

The acceleration is constant (g), but is not parallel with the initial velocity vector:
Projecting the initial velocity (2D – *x* and *z*):

$$v_{0x} = v_0 \cos\alpha \qquad v_{0z} = v_0 \sin\alpha$$



The acceleration: $\vec{a} = -g\vec{k}$

The velocity-time function: $\vec{v}(t) = v_{0x}\vec{i} + 0\vec{j} + (-gt + v_{0z})\vec{k}$

The position vector: $\vec{r}(t) = v_{0x}t\vec{i} + 0\vec{j} + \left(-\frac{g}{2}t^2 + v_{0z}t\right)\vec{k}$

CLICK HERE FOR ANIMATION!

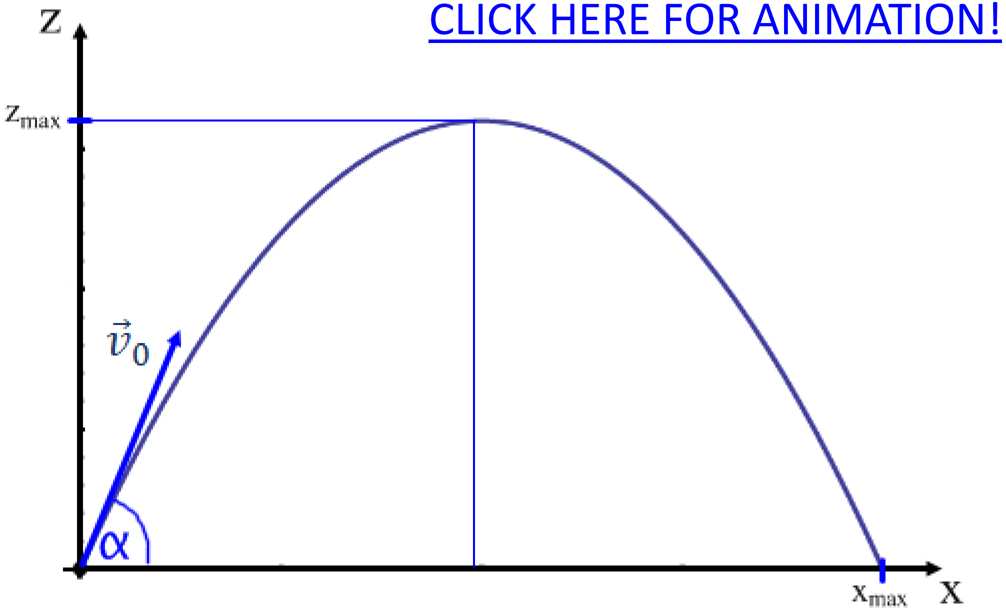
The body hits the ground when *z* = 0:

$$-\frac{g}{2}t^2 + v_0 \sin\alpha\, t = 0$$

Solving for time: $t = \dfrac{2v_0 \sin\alpha}{g}$

Substituting into the *x* coordinate
we get the range of the throw:

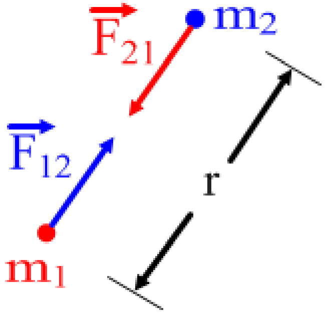
$$x_{max} = \frac{2v^2{}_0 \sin\alpha \cos\alpha}{g}$$



Newtonian gravitational force

The force between two point masses is proportional to the product of the two masses and inversely proportional to the square of their distance.

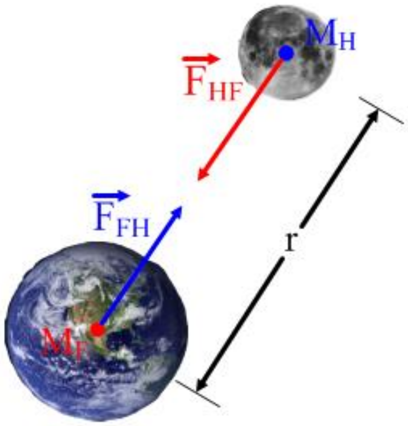
$$F = \frac{\gamma m_1 m_2}{r^2}$$



The interaction is always attractive.

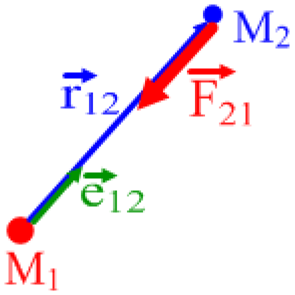
The proportionality factor is the universal gravitational constant: $\gamma = 6{,}67 \cdot 10^{-11} \frac{\mathrm{Nm}^2}{\mathrm{kg}^2}$

The simple form of the force law also applies to extended bodies, as long as they are spherically symmetric. The distance is measured between the centers.



The vector form also gives the direction of the force:

$$\vec{F}_{21} = -\frac{\gamma M_1 M_2}{r^2{}_{12}} \frac{\vec{r}_{12}}{r_{12}} = -\frac{\gamma M_1 M_2}{r^2} \vec{e}_{12}$$



Weight force

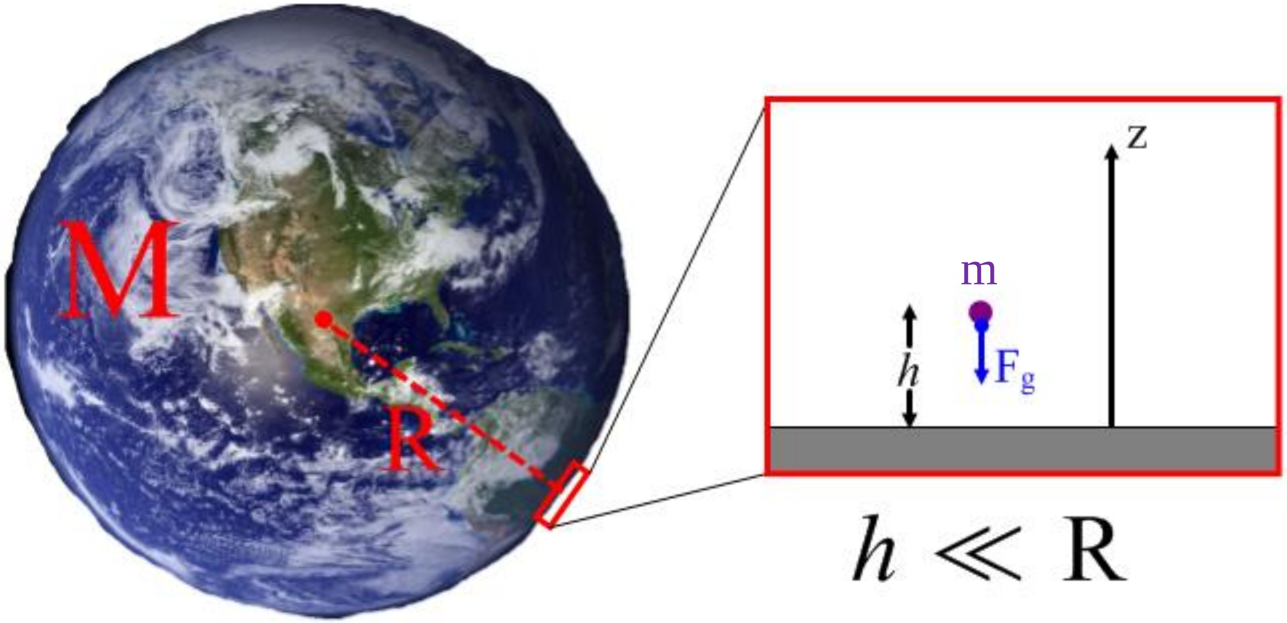
When the displacement of the body is negligible compared to the radius of the planet or moon, the gravitational force can be considered homogeneous (independent of location).

E. g. for movements occurring near the surface of the Earth from the general Newtonian force law we get:

$$\vec{F}_g = -\frac{\gamma M m}{r^2}\vec{e}_r = -\frac{\gamma M m}{(R+h)^2}\vec{e}_r \approx -\frac{\gamma M m}{R^2}\vec{e}_r = -mg\vec{k}$$

$$g = \frac{\gamma M}{R^2} = 9{,}8\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

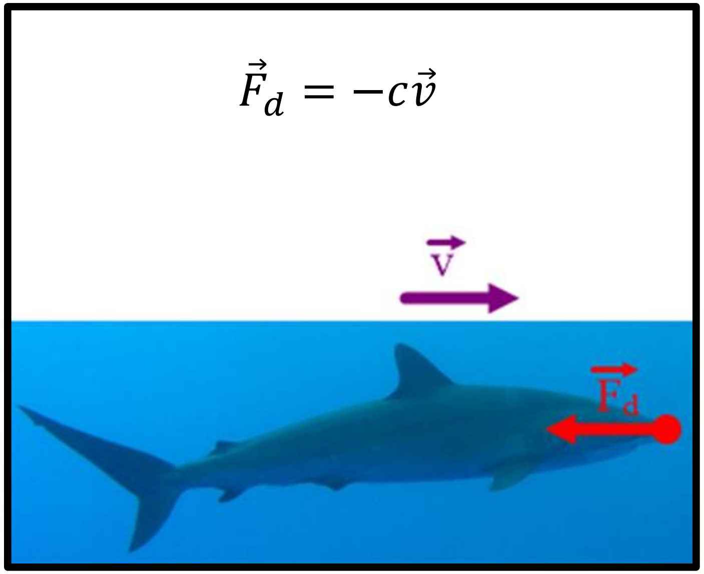
gravitational acceleration
on the surface of the Earth



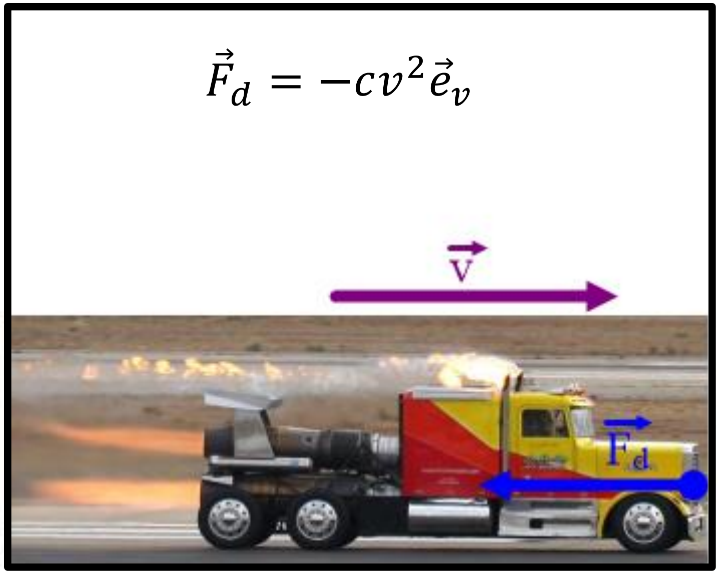
Drag forces

It is proportional to the velocity of the body, or the square of the velocity, and in the opposite direction.

$$\vec{F}_d = -c\vec{v}$$



$$\vec{F}_d = -cv^2\vec{e}_v$$



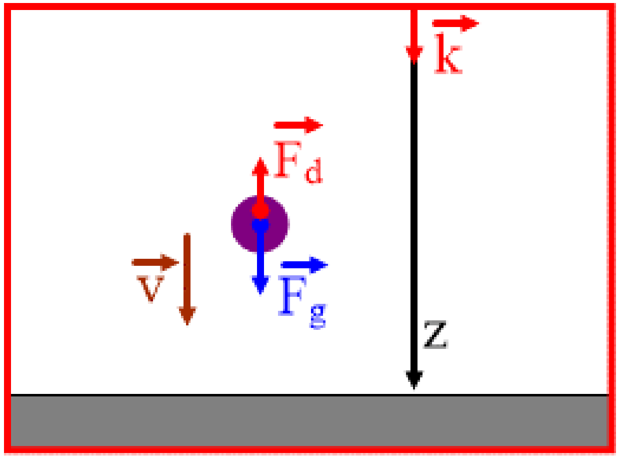
The coefficient *c* depends on:

- the size of the body's surface perpendicular to the movement
- the shape of the body (how streamlined it is) – drag coefficient C_d
- the density of the medium

$$c = \frac{1}{2}\rho A C_d$$

Free fall with drag

In addition to its weight, the object is also affected by the air resistance (drag). Starting with relatively large v_0 speed downward.



$$m\vec{a} = \vec{F}_{net} = mg\vec{k} - cv^2\vec{k} \qquad c = \frac{1}{2}\rho A C_d$$

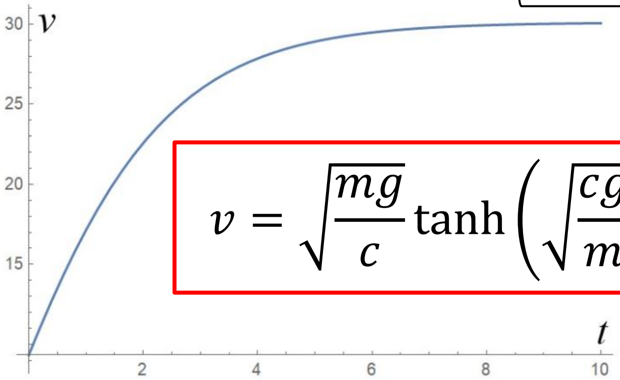
Motion along z axis:

$$m\frac{dv}{dt} = mg - cv^2 \quad \rightarrow \quad \frac{dv}{dt} = g - \frac{c}{m}v^2$$

$$\frac{dv}{g - \frac{c}{m}v^2} = dt \quad \rightarrow \quad \int_{v_0}^{v}\frac{dv'}{1 - \frac{c}{mg}v'^2} = \int_0^t g\,dt'$$

$$\left[\sqrt{\frac{mg}{c}}\tanh^{-1}\left(\sqrt{\frac{c}{mg}}\,v'\right)\right]_{v_0}^{v} = gt \;\rightarrow\; \tanh^{-1}\left(\sqrt{\frac{c}{mg}}\,v\right) \underbrace{- \tanh^{-1}\left(\sqrt{\frac{c}{mg}}\,v_0\right)}_{A} = \sqrt{\frac{c}{mg}}\,gt$$

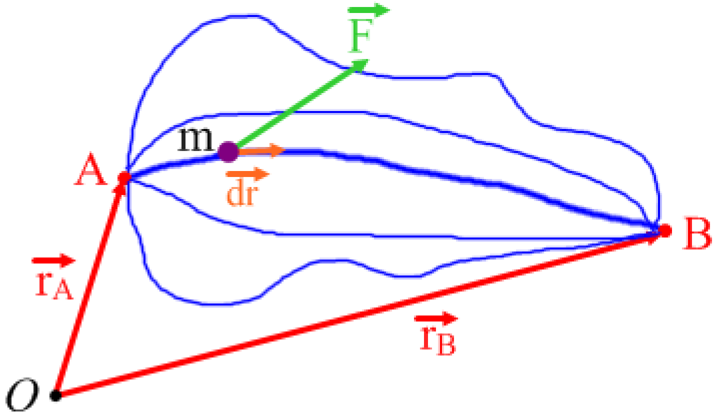
$$\sqrt{\frac{c}{mg}}\,v = \tanh\left(\sqrt{\frac{cg}{m}}\,v + A\right) \quad \rightarrow \quad \boxed{v = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,v + A\right)}$$



$C_d = 0.2$
$r = 0.11$
$\rho = 1.225$
$c = 0.004655$
$m = 0.43$
$v_0 = 10$
(soccer ball)

Conservative force fields

Conservative force field: A time-independent force field in which the work done by the force field between two points is independent of the path (this is equivalent to the work being zero for any closed curve).

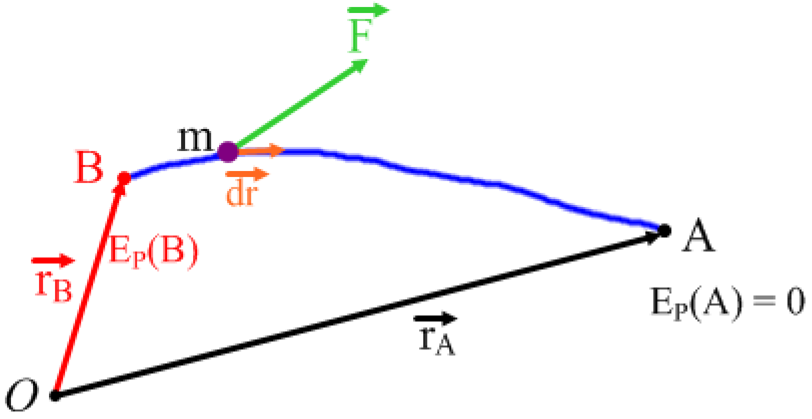


Then we can characterize the points (e.g. B) by the work that the field does while the body moves from there to a selected zero point (e.g. A).

Potential energy:

The potential energy at a point (B) is equal to the work done by the force field as the body moves from there to the zero point (A).

$$E_p(B) = W_{BA} = \int_B^A \vec{F} \cdot d\vec{r}$$



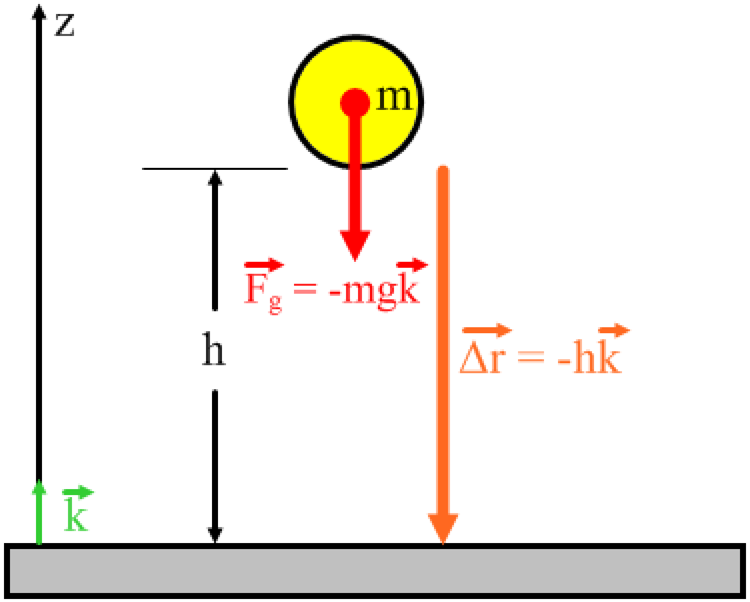
Potential energy of the weight force

Let the floor level be the zero point of the potential energy and drop a body with a weight F_g = 20N from a height of 80m.

Then the work of the weight force (i.e. the potential energy at a height of 80m) is:

$$E_p(h) = W_{h0} = \int_{h\vec{k}}^{0} \vec{F} \cdot d\vec{r} = \int_{h\vec{k}}^{0} (-mg\vec{k}) \cdot (dz\vec{k}) =$$

$$= \int_{h}^{0} -mg\,dz = -mg[z]_h^0 = mgh$$

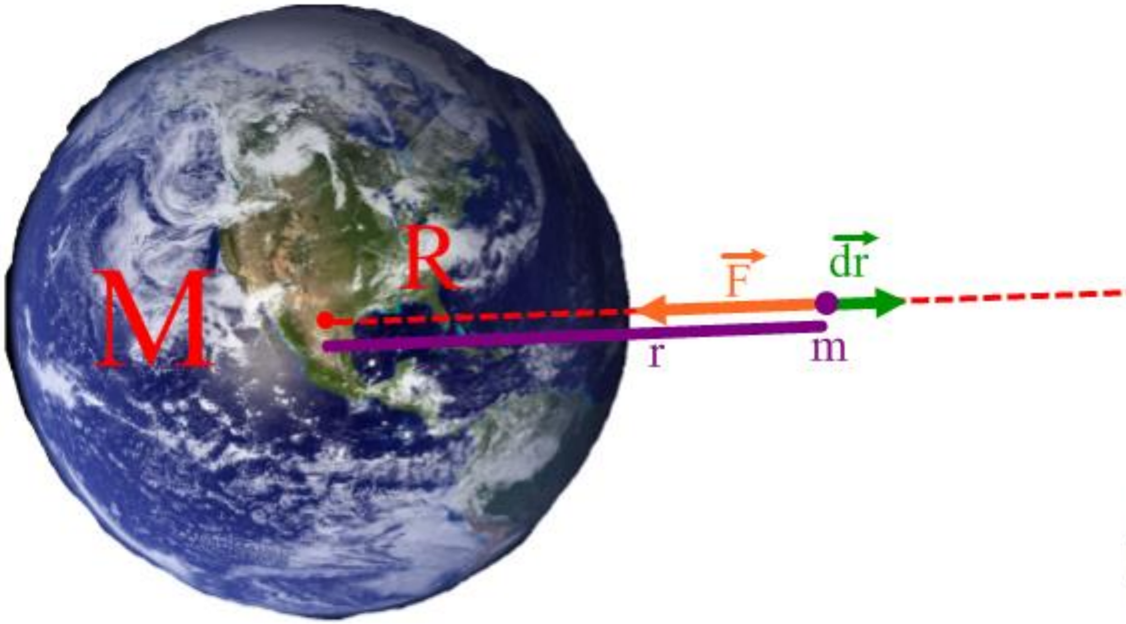


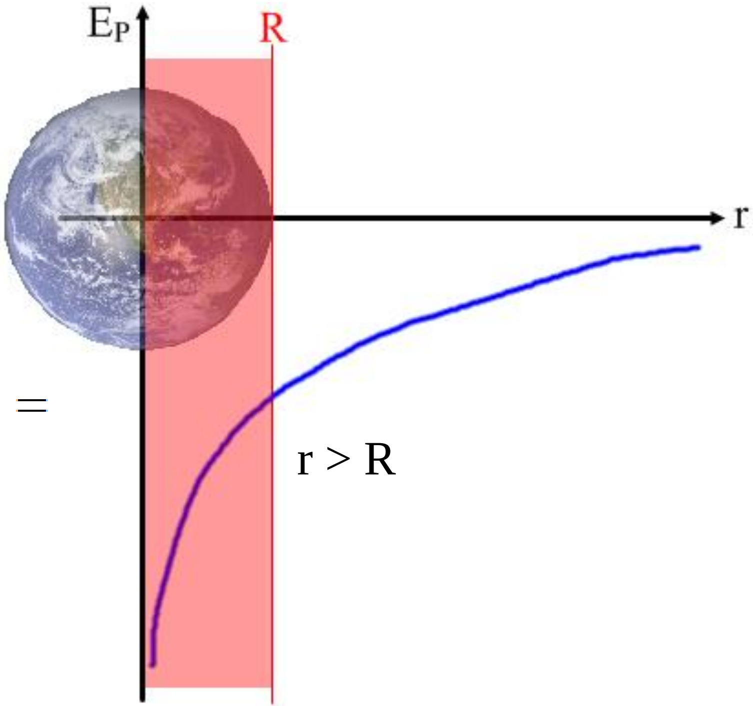
Of course, in this simple case the formula $W = Fs$ can also be used, so the $W = (mg)h = mgh$ is immediately obvious.

That is, in our case: W = (20N)(80m) = 1600J

Potential energy in a Newtonian gravitational field

Let the body of mass *M* be fixed, and at a distance *r* from it we calculate the potential energy of the body of mass *m*. The force is radial, so it is advisable to take a radial path. It is advisable to take the zero point at infinity because *r* = 0 is problematic.





$$E_P(\vec{r}) = W_{\vec{r},\infty} = \int_{\vec{r}}^{\infty} \vec{F} \cdot d\vec{r} = \int_{\vec{r}}^{\infty} \left(-\frac{\gamma Mm}{r^2}\vec{e}_r\right) dr\vec{e}_r =$$

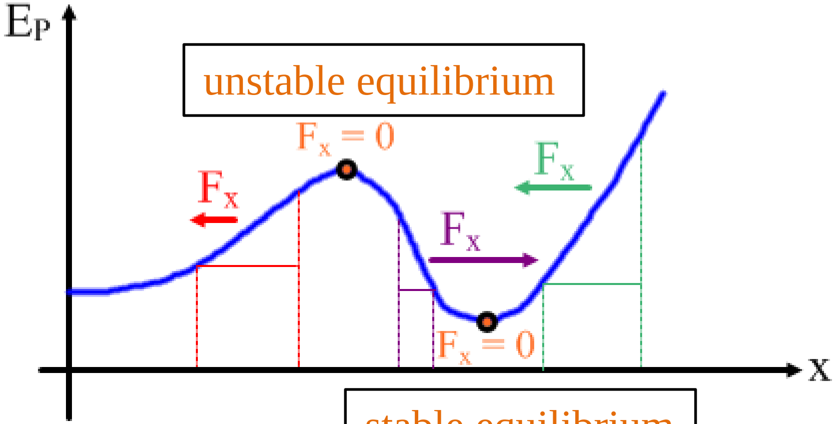
$$= \int_{r}^{\infty} -\frac{\gamma Mm}{r^2}\,dr = \gamma Mm\left[\frac{1}{r}\right]_r^{\infty} = -\frac{\gamma Mm}{r}$$

Principle of minimum energy

A greater force means a greater potential energy difference between the same two points. Reversed: The greater the rate at which potential energy changes with changing location, the greater the force exerted **by the force field**.

In general, at an arbitrary point in one dimension: $F_x = -\dfrac{\partial E_P}{\partial x}$

Principle of minimum energy: The force acts in the direction of decreasing potential energy (negative sign).



unstable equilibrium

stable equilibrium

In three dimensions:

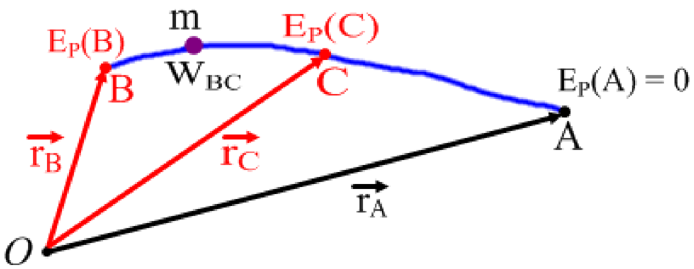
$$\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} =$$

$$= -\frac{\partial E_P}{\partial x}\vec{\imath} - \frac{\partial E_P}{\partial y}\vec{\jmath} - \frac{\partial E_P}{\partial z}\vec{k} =$$

$$= -\nabla E_P = -\mathrm{grad}E_P$$

Mechanical energy

Let's take the special case where **only conservative forces act,** while the body moves from point B to point C.



Then for any points B and C: $E_P(B) - E_P(C) = W_{BC} = E_K(C) - E_K(B)$

$$-\Delta E_P = W_{BC} = \Delta E_K$$

Rearranging the original equation: $E_P(B) + E_K(B) = E_P(C) + E_K(C)$
The sum of potential and kinetic energy is the **same at every point**.

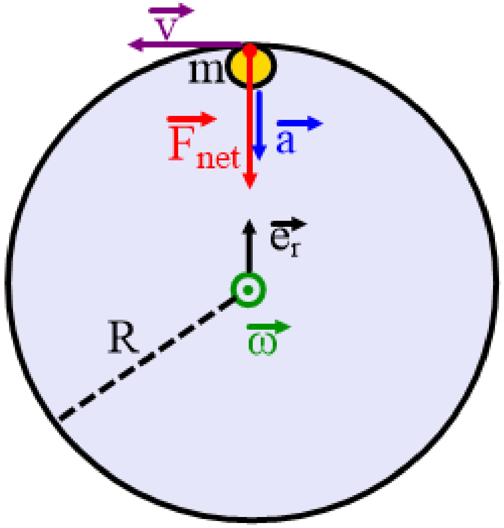
Let's introduce the **mechanical energy**, which is the sum of kinetic and potential energies:

$$E_M = E_P + E_K$$

This mechanical energy is conserved in a conservative force field: $E_M(B) = E_M(C)$

Dynamics of uniform circular motion

Uniform circular motion: During the motion, the magnitude of the velocity is constant, but its direction is constantly changing. So there is acceleration that points towards the center (**centripetal**).
The condition for this is that the net force also points in that direction (centripetal force).



WE DISCUSS IN AN INERTIAL REFERENCE FRAME

Fundamental equation of dynamics: $m\vec{a} = \vec{F}_{net}$

Acceleration has only a centripetal (radial) component.

The magnitude of the net force:

$$F_{net} = ma = ma_{cp} = m\frac{v^2}{R} = m\omega^2 R$$

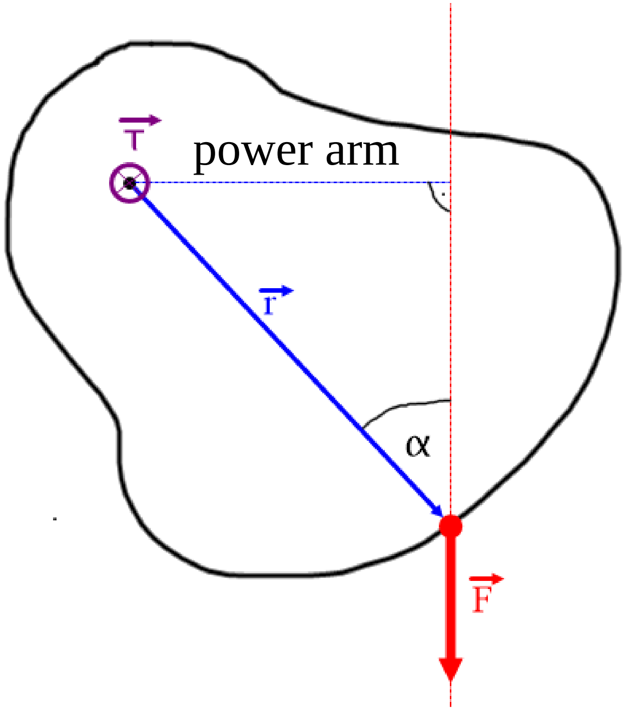
This net force can be provided by many different interactions: e. g. gravitational force, Coulomb force, rope force, normal force, Lorentz force, etc.
Then $\vec{F}_{net} \perp \vec{v}$ the **work done is zero**. The centripetal force does no work.

The direction of the angular velocity vector can be determined using the right-hand rule.
E. g. in the figure outwards.

Variable circular motion - torque

The **torque** of a force about the origin (axis of rotation): $\vec{\tau} = \vec{r} \times \vec{F}$

Power arm: the distance of the line of action of the force from the axis of rotation



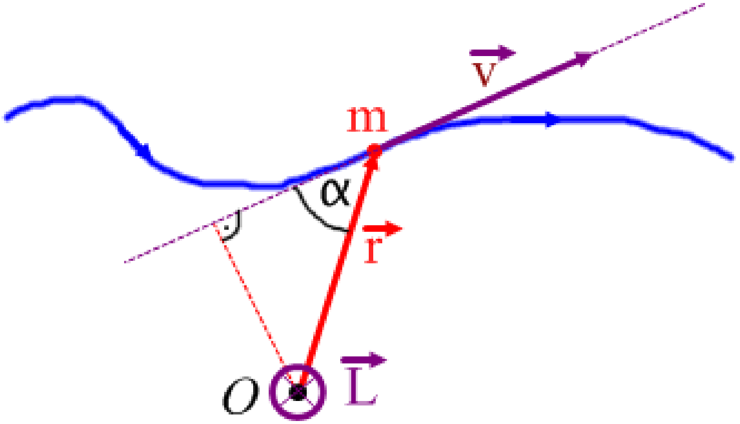
its magnitude: force × power arm
i.e. $\tau = Fr_{\perp} = Fr\sin\alpha$

The torque is zero if the line of action of the force passes through the axis of rotation, and is maximum if it is perpendicular to the $\vec{r}$ position vector.

its direction: based on the cross product (right-hand rule)
perpendicular to the plane defined by the
force and position vectors.

Angular momentum

Generally the **angular momentum** of a point-like body is: $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$
(similar to the definition of torque, replacing force with momentum)



If the position and velocity vectors are perpendicular, as for uniform circular motion:

$$L = rmv = mrv = mr\omega r = mr^2\omega$$

The angular momentum vector changes under the influence of torque:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(m\vec{r} \times \vec{v}) = m\left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) = m(\vec{v} \times \vec{v} + \vec{r} \times \vec{a}) =$$

$$= \vec{r} \times (m\vec{a}) = \vec{r} \times \vec{F}_{net} = \vec{\tau}_{net}$$

<u>Torque-angular momentum law:</u> $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_e}$

Moment of inertia

Special case: point mass moves at a constant distance around a **fixed axis** (circular motion)

$$L(t) = mr^2\omega(t)$$

Derivative of the angular momentum: $\frac{dL}{dt} = mr^2\frac{d\omega(t)}{dt} = mr^2\beta(t)$

β is the angular acceleration, and the term mr^2 is the **moment of inertia** of the point mass.

The moment of inertia for the center of mass is therefore: $I = mr^2$,
where r is the distance from the axis.

Using the torque-angular momentum law: $\frac{dL}{dt} = mr^2\frac{d\omega(t)}{dt} = I\beta(t) = \tau$

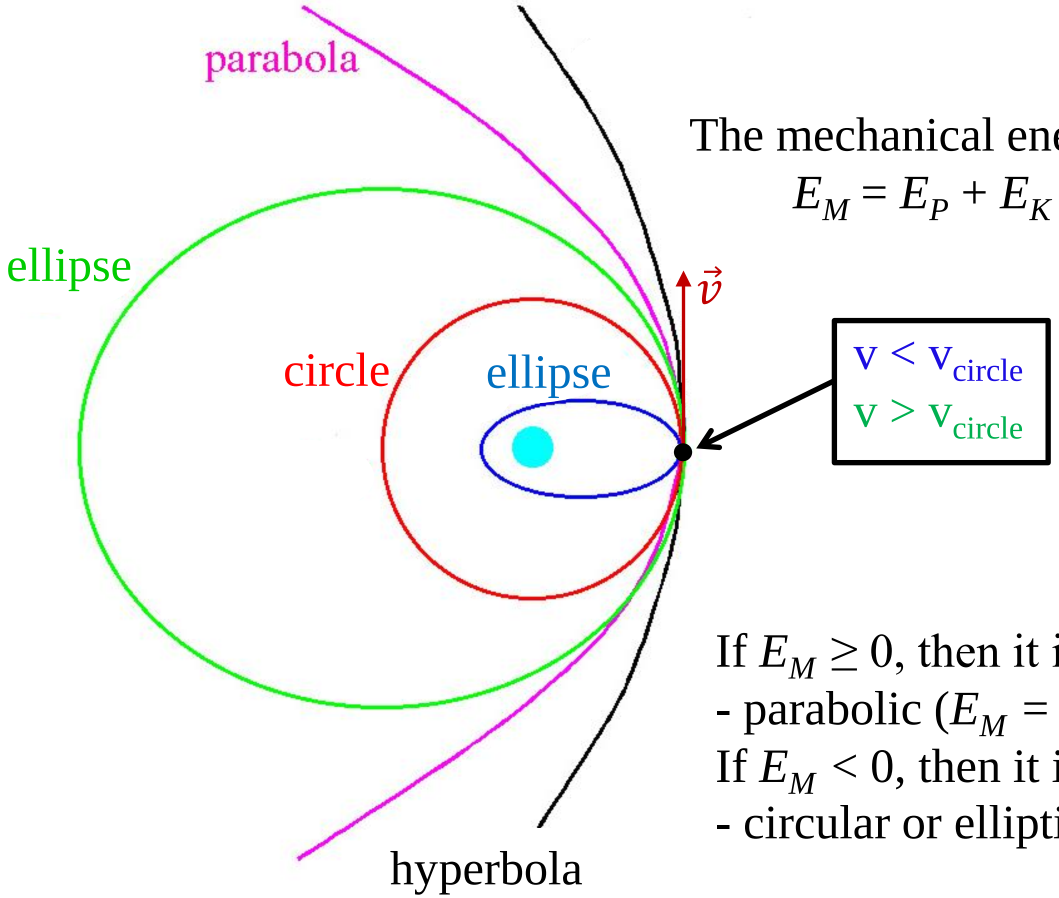
We get the fundamental equation of rotary motion: $\tau = I\beta$

The **kinetic energy** of the point mass: $E_K = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$

Movement of planets and moons

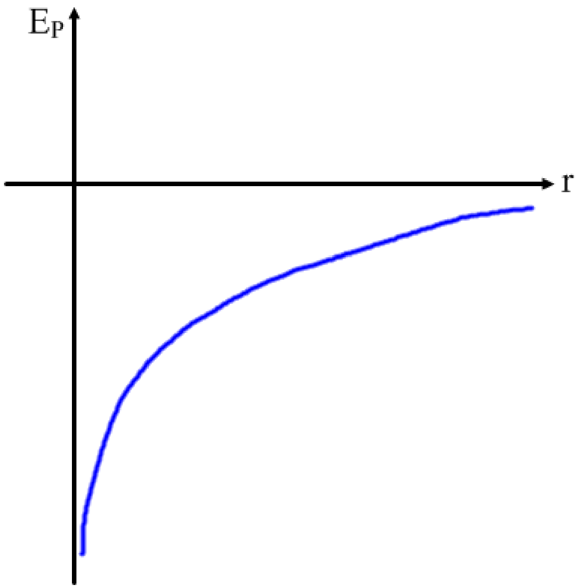
Suppose a body of mass *m* moves in the gravitational field of a body of much larger mass (*M*). Since *M* is much larger than *m*, it can be considered stationary. e.g. Sun and Earth.

A body of mass m has E_K kinetic energy and E_P potential energy.
We take the zero point at infinity: $E_P = 0$, if $r \rightarrow \infty$



The mechanical energy:

$$E_M = E_P + E_K$$



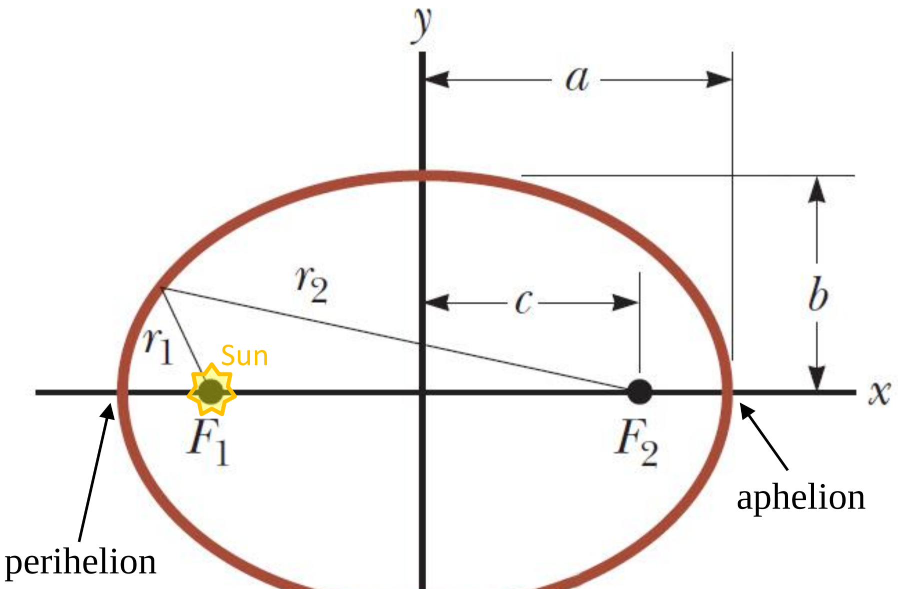
If $E_M \geq 0$, then it is free (can move away to infinity)
- parabolic ($E_M = 0$) or hyperbolic trajectory

If $E_M < 0$, then it is bound (remains at a finite distance)
- circular or elliptical orbit

Kepler’s 1st law

For bodies moving in a bound state in the gravitational field of a massive body (e.g. planets).

I. The orbits of the planets are ellipses, and the Sun is at one of its foci.



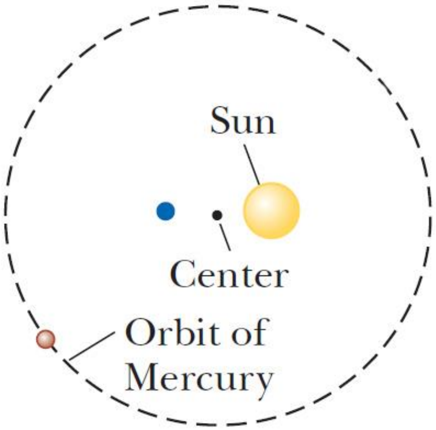
<u>ellipse:</u> points for which $r_1 + r_2 = \text{constant}$

$$a^2 = b^2 + c^2$$

a: semi-major axis
b: semi-minor axis

eccentricity: $e = c/a$
$0 < e < 1$

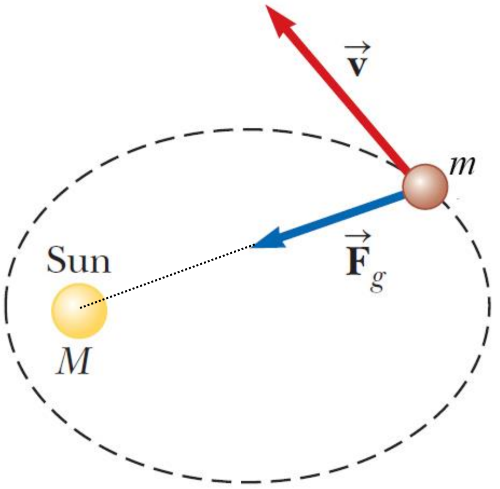
circle: $e = 0$
Earth’s orbit: $e = 0.017$
Mercury’s orbit: $e = 0.21$
Comet Halley: $e = 0.97$



For moons around planets: perigee and apogee

Kepler's 2nd law

II. (Equal area law) The lines between the Sun and the planets sweep out equal areas in equal times. Planets move faster near the perihelion. It follows from the conservation of angular momentum: there is no torque in a central force field.



$$\vec{L} = \text{constant}$$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$$

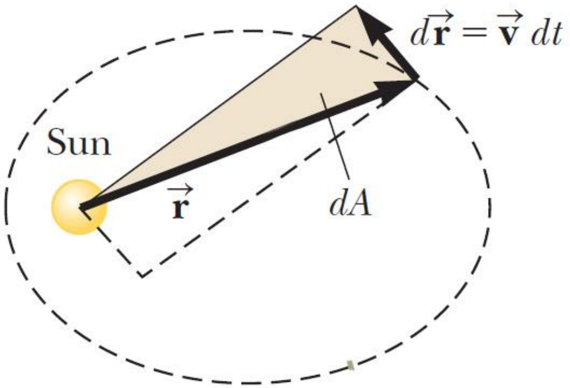
For its magnitude: $L = m|\vec{r} \times \vec{v}|$

In time dt the area dA is swept out:

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}dt| = \frac{1}{2}|\vec{r} \times \vec{v}|dt$$

$$dA = \frac{L}{2m}dt$$

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

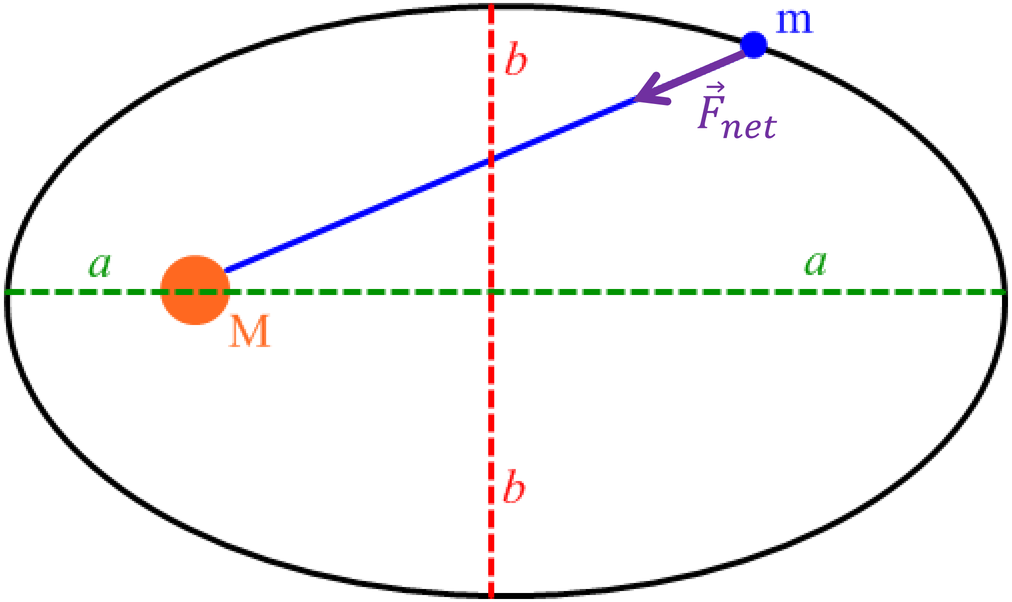


Kepler's 3rd Law

III. The cubes of the semi-major axes (*a*) of elliptical orbits are proportional to the squares of the orbital periods (*T*) of the planets moving in the given orbits.

So for every planet (and any body) orbiting the Sun: $\frac{a^3}{T^2} = \text{constant}$

All three laws can be derived from Newton's laws and the Newtonian force law of gravity.



Proof for a circular orbit: *a* = *b* = *R*

$$F_{net} = ma$$

$$\gamma \frac{Mm}{R^2} = m\omega^2 R$$

$$\gamma \frac{M}{R^2} = \frac{4\pi^2}{T^2} R$$

$$\frac{\gamma M}{4\pi^2} = \frac{R^3}{T^2} = \text{constant}$$