

## 2 Particle dynamics

### 2.1 Basic concepts

The aim of dynamics is to find the cause of motion, and knowing the cause to give description of motion. The cause is always some interaction.

There are two types of interaction:

- body-body interaction, they are in contact,
- body-field interaction

The particle is called free body if it is no in interaction with any other body or field.

Newton's first law:

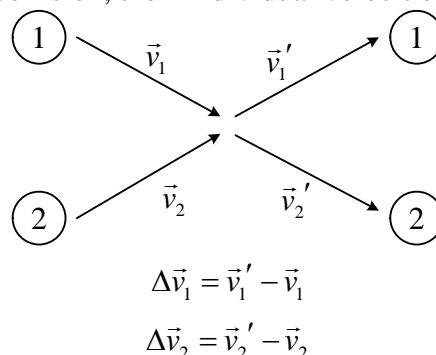
It is possible to find a reference system in which the free body or particle does not experience acceleration or its speed is constant or zero. This reference system is called inertial system.

Due to experiences the reference system fixed to very distant stars is always inertial system. In a lot of simple cases the reference system fixed to the Earth is also can consider as an inertial system. If we have one inertial system, we have infinite number, because all systems move with a uniform straight line motion relative to it is also an inertial system.

### 2.2 Mass and momentum of a body

It comes from Newton's first law, every body resists changing its state of motion. This property of the body is called inertia. The inertia is characterized quantitatively by a physical quantity called the mass of a body.

Consider now an ideal situation. Instead of observing an isolated (free) particle, we observe two interacting particles otherwise isolated from the rest of the world. As a result of their interaction, for example in a collision, their individual velocities change with time:



Experimentally we find that the two changes in velocities have opposite direction and the ratio of the magnitudes of the velocity changes is always the same. Therefore:

$$\Delta \vec{v}_1 = -K \Delta \vec{v}_2,$$

where  $K$  is the same for a pair of particle, independent of how they are moving. Suppose now that we choose a certain particle as our standard body and designate it by (0). Next we let particles (1), (2), (3) ... each interact with the standard particle. In each case we determine the constant  $K$  and for each of these pairs designate by  $m_1, m_2, \dots$

$$\Delta \vec{v}_0 = -m_1 \Delta \vec{v}_1, \quad \Delta \vec{v}_0 = -m_2 \Delta \vec{v}_2,$$

and we call the coefficients  $m_1, m_2 \dots$  etc. the inertial mass of particles when the mass of the standard particle is set at unity  $m_0 = 1$ .

When the experiment is carried out with particles (1) and (2) the constant  $K$  is equal  $\frac{m_2}{m_1}$ , that is:

$$\Delta \vec{v}_1 = -\frac{m_2}{m_1} \Delta \vec{v}_2 \rightarrow m_1 \Delta \vec{v}_1 = -m_2 \Delta \vec{v}_2$$

The linear momentum of a particle is defined as the product of its mass and its velocity, designated by:  $\vec{p}$

$$\vec{p} = m\vec{v}$$

The total momentum of system of two particles is given by:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2$$

In case of an interaction the change of momentum, since  $m_1$  is a constant, can be written:

$$\Delta \vec{p}_1 = \Delta(m_1 \vec{v}_1) = m_1 \Delta \vec{v}_1,$$

so in a pair interaction:

$$\Delta \vec{p}_1 = -\Delta \vec{p}_2 \rightarrow \Delta \vec{p}_1 + \Delta \vec{p}_2 = 0$$

$$\Delta(\vec{p}_1 + \vec{p}_2) = 0$$

$$\vec{p}_1 + \vec{p}_2 = \text{constant}$$

The above equation is the mathematical form of the conservation of momentum. The total momentum of two particles in a pair interaction remains constant. If the change of the momentum is faster then we say, that the interaction is stronger.

To characterise the strength of the interaction we introduce the concept of force.

$$\frac{d\vec{p}}{dt} = \vec{F} \quad \text{or} \quad \dot{\vec{p}} = \vec{F}$$

So we consider force as a quantitative expression of interaction.

$$\frac{d}{dt}(m\vec{v}) = m \frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

If  $m = \text{constant}$

$$m\ddot{\vec{r}} = \vec{F}$$

This equation is often called Newton II. law.

Newton's II law states, that if the mass is constant the force is equal to mass times acceleration.

$$\vec{F} = m\vec{a}$$

To determine the  $\vec{r} = \vec{r}(t)$  function from this equation we have to know the so called force laws. How the force does depend on the position vector, the velocity, and the time.

Some examples as force laws:

gravitational force:

$$\vec{F} = -\gamma \frac{mM}{r^2} \vec{e}_r$$

elastic force

$$F_x = -Dx$$

electrostatic force (Coulomb's law)

$$\vec{F} = k \frac{Q_1 Q_2}{r^2} \vec{e}_r$$

resistance in a liquid

$$\vec{F} = -\kappa \vec{v}$$

air resistance

$$\vec{F} = -\beta v^2 \vec{v}$$

Lorentz' force

$$\vec{F} = Q\vec{v} \times \vec{B}$$

The Newton's II. law in vector form:

$$\vec{F} = m \ddot{\vec{r}}$$

In rectangular coordinate system:

$$F_x = m \ddot{x}$$

$$F_y = m \ddot{y}$$

$$F_z = m \ddot{z}$$

To solve this three second order differential equation for:

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t),$$

we need the force laws in the problem in question, and 6 initial conditions:

$$\text{if } t = 0, \text{ then } \vec{v} = \vec{v}_0, \quad (v_{0x}, v_{0y}, v_{0z}), \text{ and}$$

$$\text{if } t = 0, \text{ then } \vec{r} = \vec{r}_0, \quad (x_0, y_0, z_0).$$

Consider again a pair interaction:

$$\dot{\vec{p}}_1 = \vec{F}_{21},$$

$\vec{F}_{21}$  is the force acting on the first body due to the second.

$$\dot{\vec{p}}_2 = \vec{F}_{12}$$

$\vec{F}_{12}$  is the force acting on the second body due to the first.

Add the two equations:

$$\begin{aligned} \dot{\vec{p}}_1 + \dot{\vec{p}}_2 &= \vec{F}_{21} + \vec{F}_{12} \\ (\vec{p}_1 + \vec{p}_2)' &= \vec{F}_{21} + \vec{F}_{12} \end{aligned}$$

In a pair interaction the momentum of the system remains constant:

$$\vec{p}_1 + \vec{p}_2 = \text{constant},$$

and the derivative of a constant is zero:

$$(\vec{p}_1 + \vec{p}_2)' = 0$$

Finally:

$$\vec{F}_{21} = -\vec{F}_{12}$$

The above equation is the action-reaction theorem, or Newton's III. Law.

When two particles interact, the force on one particle is equal and opposite of the force on the other.

Consider now an object interacting with several other objects. The experience shows that the time rate of change of the momentum of the particle is equal to the resultant force acting on the object, that is

$$\dot{\vec{p}} = \vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \dots,$$

where  $\vec{F}_1, \vec{F}_2, \dots$  are the forces due to the pair interactions. This axiom is called superposition axiom or independent action.

### 2.3 Instantaneous power, kinetic energy, and their connection

The instantaneous power delivered to a body by a force  $\vec{F}$  that acts on it is defined as the scalar product of the force and the instantaneous velocity:

$$P = \vec{F} \cdot \vec{v}$$

The unit of power:  $[P] = 1W = 1watt$

The kinetic energy of a particle is defined as:

$$T = \frac{1}{2}mv^2, \text{ or } T = \frac{p^2}{2m}$$

Suppose that  $m = \text{constant}$ . Take Newton's II law:

$$\vec{F} = m\vec{a} \quad / \cdot \vec{v}$$

$$\vec{F}\vec{v} = m\dot{\vec{v}} \cdot \vec{v}$$

$$P = m \left( \frac{\vec{v}^2}{2} \right) \dot{\quad}$$

$$P = \frac{d}{dt} \left( \frac{1}{2}m\vec{v}^2 \right)$$

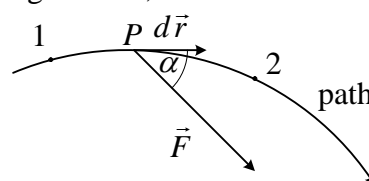
In short form:

$$P = \frac{dT}{dt}, \quad P = \dot{T}.$$

This equation states that the time derivative of the kinetic energy is equal to the power delivered by the force. This is the so called power law.

#### 2.3.1 Work done by a force:

Consider a particle A moving along a curve, under the action of the force  $\vec{F}$ .



If its displacement is  $d\vec{r}$ , the elementary work done on the particle by the force is defined as:

$$\delta W = \vec{F} \cdot d\vec{r}$$

$$\delta W = F dr \cos \alpha$$

This is only an elementary work done due to the infinitesimal displacement. The total work done on the particle when moving from point 1 to 2 is the sum of all the infinitesimal works done, that is:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$$

The work is the line integral of the force along the curve from point 1. to 2.  
Unit of work, and kinetic energy:

$$[W] = [T] = 1J = 1Joule$$

As we know:

$$\vec{v} = \frac{d\vec{r}}{dt}, \rightarrow d\vec{r} = \vec{v} dt$$

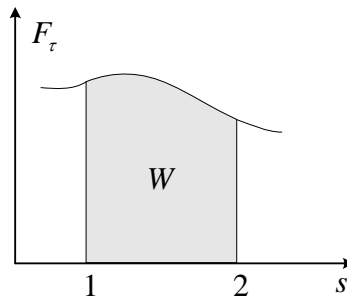
$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt = \int_{t_1}^{t_2} P dt$$

The work is the time integral of the instantaneous power between the two instants  $t_1$  to  $t_2$ .

$$d\vec{r} = \vec{v} dt = v \vec{\tau} dt = ds \vec{\tau}$$

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 \vec{F} \cdot \vec{\tau} ds = \int_1^2 F_{\tau} ds$$

Only the tangential component of the force does any work. The work done by the normal component of the force is zero. Sometimes it is convenient to represent  $F_{\tau}$  graphically. If we know the  $F_{\tau}$  versus distance graph, the work done is equal to the area under this graph.



Consider now the power law and integrate it for a time interval  $t_1$  to  $t_2$ :

$$P = \frac{dT}{dt}, \rightarrow \int_{t_1}^{t_2} P dt = \int_{t_1}^{t_2} \frac{dT}{dt} dt$$

$$W_{12} = T_2 - T_1.$$

This equation is called work-energy theorem. This equation states that the work done by the resultant force acting on a body is equal to the change in the kinetic energy of the body.

$T$  is the kinetic energy:  $T = \frac{1}{2} mv^2$ .

## 2.4 Force field

The force field is a region in which a body experiences a force as a result of the presence of some other body or bodies. So the field concept is a method of representing the way in which bodies not in contact can influence each other.

In general a field may be any physical quantity which can be specified simultaneously for all points within a given region of interest. A field variable may be vector or scalar. The force field is vector field. In the force field, the force acting on a particle is continuous and differentiable function of time and position.

The field is called stationary if the field variable does not depend on time  $\frac{\partial \vec{F}}{\partial t} = 0$ .

The field is called homogeneous if the field variable does not depend on space at any time. In the following we consider only stationary force fields.

### 2.4.1 Conservative forces

The stationary force field in which the elementary work done by a force is total differential, that is exists a scalar function of the position vector whose negative differential is the elementary work, is called conservative force field or conservative field. The scalar function  $V(\vec{r})$  is called potential energy:

$$\delta W = -dV$$

$$\vec{F} \cdot d\vec{r} = -dV$$

In rectangular coordinate system:

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k},$$

$$d\vec{r} = dx \vec{i} + dy \vec{j} + dz \vec{k},$$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz,$$

due to definition:

$$F_x dx + F_y dy + F_z dz = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz,$$

$$F_x = -\frac{\partial V}{\partial x}, F_y = -\frac{\partial V}{\partial y}, F_z = -\frac{\partial V}{\partial z},$$

In a vector equation:

$$\vec{F} = -\nabla V = -\text{grad } V$$

The  $\nabla$  vector is the so called Hamiltonian operator or del operator:

$$\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

In conservative field the force is the negative gradient of the potential energy.

A question arises: how can we decide that a force field is conservative or not? If it is conservative, there is potential energy function and for example:

$$F_x = -\frac{\partial V}{\partial x}, F_y = -\frac{\partial V}{\partial y}$$

Take the second order mixed partial derivatives:

$$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial x \partial y}, \quad \frac{\partial F_y}{\partial x} = -\frac{\partial^2 V}{\partial y \partial x}$$

Due to a mathematical theorem the mixed second order partial derivatives are equal:

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0.$$

In the same way we can obtain the next two equations:

$$\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = 0, \quad \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0$$

Instead of these three scalar equations we can write one vector equation:

$$\begin{aligned} \text{rot} \vec{F} &= \nabla \times \vec{F} = \vec{0} \\ \text{rot} \vec{F} = \nabla \times \vec{F} &= \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \vec{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \vec{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \vec{k} = 0 \end{aligned}$$

In some books curl is used instead of rot. Thus the criterion for a force field to be conservative is that:

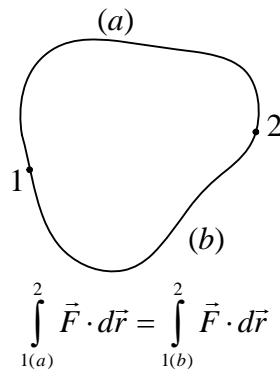
$$\nabla \times \vec{F} = 0$$

What is the connection between the work done and the potential energy in conservative field?

By definition:

$$\begin{aligned} \delta W &= -dV \\ W_{12} &= \int_{12} \vec{F} \cdot d\vec{r} = - \int_1^2 dV = -[V]_1^2 = -[V_2 - V_1] = V_1 - V_2, \end{aligned}$$

so the work done by the force on the particle between the two points is equal to the negative change of its potential energy, that is initial minus final potential energy. It means that the work done by a conservative force on a particle that moves between two points is independent of the path taken, depends only on these points.



(a) and (b) are two different curves between point 1 and 2.

If the path is closed the work done is zero:

$$\oint \vec{F} \cdot d\vec{r} = V_1 - V_1 = 0$$

The circle on the integral sign indicates that the path is closed.

The potential energy is always defined within an arbitrary constant. We have to choose a reference point  $\vec{r}_0$ , where the potential energy is chosen to be zero  $V(\vec{r}_0) = 0$ . So the potential energy at a point  $\vec{r}_1$  is:

$$\begin{aligned} -dV &= \vec{F} \cdot d\vec{r} \\ -\int_{\vec{r}_1}^{\vec{r}_0} dV &= -\int_{\vec{r}_1}^{\vec{r}_0} \vec{F} \cdot d\vec{r} \\ V(\vec{r}_1) - V(\vec{r}_0) &= \int_{\vec{r}_1}^{\vec{r}_0} \vec{F} \cdot d\vec{r} \end{aligned}$$

$$V(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_0} \vec{F} \cdot d\vec{r}$$

The potential energy of the particle at point  $\vec{r}_1$  is equal to the work done by the conservative force as it moves from point  $\vec{r}_1$  to the reference point  $\vec{r}_0$ .

Consider now the work-energy theorem in conservative field:

$$W_{12} = T_2 - T_1,$$

$$W_{12} = V_1 - V_2,$$

$$V_1 - V_2 = T_2 - T_1,$$

$$T_1 + V_1 = T_2 + V_2$$

Introduce the mechanical energy, by the next definition:

$$E = T + V,$$

$$E_1 = E_2$$

When the forces are conservative the mechanical energy of the particle remains constant.

In the nature there are several non-conservative forces like friction. A particle in many cases may be subject at the same time to conservative and non-conservative forces. In such cases:

$$W_{12}^C + W_{12}^{NC} = T_2 - T_1$$

$$W_{12}^C = V_1 - V_2$$

$W_{12}^C$  is the work by conservative forces,  $W_{12}^{NC}$  is the work done by non-conservative ones.

$$W_{12}^{NC} = T_2 + V_2 - T_1 - V_1$$

$$W_{12}^{NC} = E_2 - E_1$$

The change of the mechanical energy is equal to the work done by the non-conservative forces. Of course if there are no non-conservative forces the mechanical energy remains constant.