2.5 Oscillatory motion

2.5.1 Simple harmonic oscillation

One of the most frequent motions in nature is oscillatory or vibrational motion. A particle is oscillating when it moves periodically about an equilibrium position. The oscillatory motion is the result of the so called linear restoring force. This is a force whose magnitude is proportional to the displacement of a particle from some equilibrium position, and the direction is opposite to that of the displacement. Such a force is exerted by an elastic cord or by a spring obeying Hooke's law:

$$F_x = -Dx, \quad D > 0$$

The proportionality factor *D* is called stiffness, or force constant of the spring.
Suppose that the straight line of the motion is the *x*-axis. Apply Newton's II law. The equation of motion:

$$m\ddot{x} = -Dx$$

Solve this second order differential equation for the *x(t)* function:

$$\ddot{x} = -\frac{D}{m}x\,,$$

Introduce the next symbol:

$$\omega_0^2 = \frac{D}{m}$$

The differential equation:

$$\ddot{x} = -\omega_0^2 x\,,$$

$$\ddot{x} + \omega_0^2 x = 0$$

It is a second order homogeneous linear differential equation with constant coefficients. The general solution is the linear combination of the two independent partial solutions like:

$$x_1 = \sin \omega_0 t\,,$$

$$x_2 = \cos \omega_0 t\,.$$

The linear combination:

$$x(t) = C_1 \sin \omega_0 t + C_2 \cos \omega_0 t \text{ , or}$$

$$x(t) = A \sin\left(\omega_0 t + \delta\right).$$

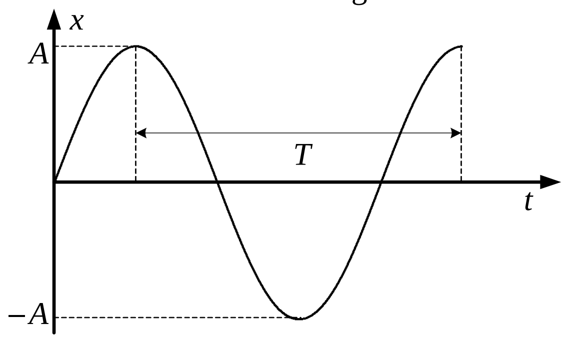
In the first general solution the two integration constants are C_1, and C_2. In the second solution the two integration constants are *A* and δ. The displacement *x* is a sinusoidal function of the time *t*. The coefficient ω_0 is called angular frequency. The maximum value of the displacement is called amplitude of the oscillation *A*. As the function repeats itself after a time $\frac{2\pi}{\omega_0}$ so this is the period of the motion *T*.

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{D}}$$

The time of period is determined completely by the mass of the oscillating particle and the stiffness of the spring.
The frequency of the oscillation is the number of cycles in unit time:

$$f_0 = \frac{1}{T}$$

The quantity $(\omega_0 t + \delta)$ is called phase of the oscillation and this δ is the initial phase. The amplitude A and the initial phase can be obtained from the initial condition. The plot of the displacement versus time function is shown on the figure:



displacement time function:

$$x = A\sin(\omega_0 t + \delta)$$

velocity time function:

$$\dot{x} = A\omega_0 \cos(\omega_0 t + \delta)$$

acceleration time function:

$$\ddot{x} = -A\omega_0^2 \sin(\omega_0 t + \delta).$$

The linear restoring force is a conservative force. To obtain the potential energy we apply:

$$\vec{F} = -\nabla V$$

In one dimension:

$$F_x = -\frac{\partial V}{\partial x}, \text{ but the linear restoring force } F_x = -Dx$$

$$-Dx = -\frac{\partial V}{\partial x}$$

$$V = \frac{1}{2}Dx^2 + C$$

Choosing the zero of the potential energy at the equilibrium position, we get: $C = 0$

$$V(x) = \frac{1}{2}Dx^2$$

In case of conservative force the mechanical energy remains constant:

$$E = T + V = \text{constant}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Dx^2 = \frac{1}{2}mA^2\omega_0^2 \cos^2(\omega_0 t + \delta) + \frac{1}{2}DA^2 \sin^2(\omega_0 t + \delta) =$$

$$= \frac{1}{2}DA^2 \cos^2(\omega_0 t + \delta) + \frac{1}{2}DA^2 \sin^2(\omega_0 t + \delta) =$$

$$= \frac{1}{2}DA^2\left[\sin^2(\omega_0 t + \delta) + \cos^2(\omega_0 t + \delta)\right] = \frac{1}{2}DA^2$$

During an oscillation, there is a continuous exchange of kinetic and potential energies.

2.5.2 Damped oscillations

In simple harmonic motion the oscillations have constant amplitude. If there are friction or air resistance the amplitude gradually decreases, that is the oscillatory motion is damped. To explain the damping dynamically we assume a force that is proportional to the velocity but oppositely directed. The forces acting on the particle are the linear restoring force, and the damping force:

$$F_x = -Dx$$

$$F_x^d = -\kappa \dot{x}$$

The equation of motion:

$$m\ddot{x} = -Dx - \kappa \dot{x}$$

$$m\ddot{x} + \kappa \dot{x} + Dx = 0$$

$$\ddot{x} + \frac{\kappa}{m}\dot{x} + \frac{D}{m}x = 0$$

Introducing:

$$\omega_0 = \frac{D}{m}, \quad 2\alpha = \frac{\kappa}{m}$$

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0$$

ω_0 is the natural angular frequency without damping.
This is a homogeneous linear second order differential equation with constant coefficients. Use the next exponential function as a trial solution:

$$x_p = e^{\lambda t}$$

$$\lambda^2 e^{\lambda t} + 2\alpha\lambda e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0$$

$$e^{\lambda t}\left(\lambda^2 + 2\alpha\lambda + \omega_0^2\right) = 0, \qquad e^{\lambda t} \neq 0$$

The characteristic equation:

$$\lambda^2 + 2\alpha\lambda + \omega_0^2 = 0$$

The roots of this equation:

$$\lambda_{12} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

If the roots do not coincide $\lambda_1 \neq \lambda_2$, then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are linearly independent and the general solution is the linear combination:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

If $\lambda_1 = \lambda_2$, it can be shown that the general solution is:

$$x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}.$$

Due to the roots there are three different cases:
If $\alpha > \omega_0$, it is called over damping, the motion is not oscillatory motion.
If $\alpha = \omega_0$, it is called critical damping, the motion is not oscillatory motion.
If $\alpha < \omega_0$, it is called under damping, or small damping, the two roots are conjugate complex numbers.
Introduce:

$$\gamma = \sqrt{\omega_0^2 - \alpha^2}$$

$$\lambda_{12} = -\alpha \pm i\gamma$$

And the general solution:

$$x(t) = C_1 e^{(-\alpha+i\gamma)t} + C_2 e^{(-\alpha-i\gamma)t} = e^{-\alpha t}\left(C_1 e^{i\gamma t} + C_2 e^{-i\gamma t}\right)$$

Use the Euler-relation:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$x(t) = e^{-\alpha t}\left[C_1\left(\cos\gamma t + i\sin\gamma t\right) + C_2\left(\cos\gamma t - i\sin\gamma t\right)\right]$$

$$x(t) = e^{-\alpha t}\left[\left(C_1 + C_2\right)\cos\gamma t + i\left(C_1 - C_2\right)\sin\gamma t\right]$$

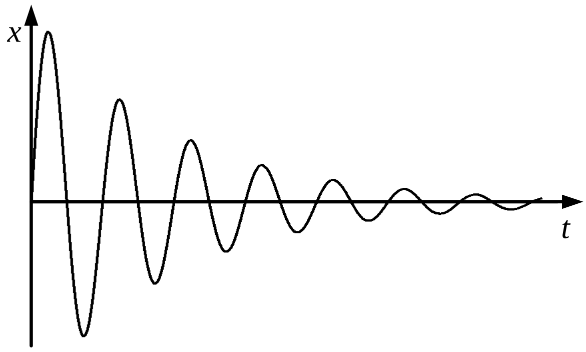
The general solution of the differential equation:

$$x(t) = e^{-\alpha t}\left[A\cos\gamma t + B\sin\gamma t\right],$$

the other form:

$$x(t) = Ce^{-\alpha t}\sin\left(\gamma t + \delta\right).$$

The real form of the solution shows that the motion is oscillatory, and that the amplitude $Ce^{-\alpha t}$ decays exponentially with time. The angular frequency of oscillation γ is less than that of the undamped oscillation ω_0.



This is the *x*-versus time function in case of under damping.
In the first real solution the two integration constants are *A* and *B*, in the second one the two constants are *C* and δ. They can be determined by using the initial conditions. It is needed to know the initial position x_0, and the initial velocity of the particle v_0.

2.5.3 Forced harmonic oscillations, Resonance

In this section we shall study the motion of a damped harmonic oscillator that is driven by an external harmonic force that varies sinusoidally with time. The forces are:

$$F_x = -Dx \text{, linear restoring force}$$

$$F_x^d = -\kappa \dot{x} \text{, damping force}$$

$$F_x^{driving} = F_0 \cos\omega t \text{, driving force}$$

The equation of motion:

$$m\ddot{x} = -Dx - \kappa\dot{x} + F_0\cos\omega t$$

Here F_0 is the amplitude of the force, and ω is the cyclic frequency of this harmonic driving force.

$$m\ddot{x} + \kappa\dot{x} + Dx = F_0\cos\omega t$$

$$\ddot{x}+\frac{\kappa}{m}\dot{x}+\frac{D}{m}x=\frac{F_0}{m}\cos\omega t$$

Introducing:

$$\omega_0=\frac{D}{m},\quad 2\alpha=\frac{\kappa}{m}\text{, and }\; f_0=\frac{F_0}{m}$$

$$\ddot{x}+2\alpha\,\dot{x}+\omega_0^2x=f_0\cos\omega t$$

This is a non-homogeneous second order linear differential equation with constant coefficients. The general solution of this equation equals the sum of the general solution of the corresponding homogeneous equation and a partial solution of the non-homogeneous equation:

$$x_{\text{gen. inh.}}(t)=x_{\text{gen. hom.}}(t)+x_{\text{part. inh.}}(t)$$

The homogeneous equation:

$$\ddot{x}+2\alpha\,\dot{x}+\omega_0^2x=0,$$

and as we have seen, the solution of this equation contains a term $e^{-\alpha t}$ which tends to zero. This is called transient term. Therefore after sufficient time this solution may be disregarded

$$x_{\text{gen. hom.}}(t)\to 0.$$

Thus the steady-state function or stationary solution is the partial solution of the non-homogeneous equation. Find this partial solution of the inhomogeneous equation:

$$\ddot{x}+2\alpha\,\dot{x}+\omega_0^2x=f_0\cos\omega t$$

Set up a complex helping equation, it has no physical meaning, i is the complex unit:

$$i\,\ddot{y}+i2\alpha\,\dot{y}+i\omega_0^2y=if_0\sin\omega t$$

Add the two equations:

$$\ddot{x}+i\,\ddot{y}+2\alpha\left(\dot{x}+i\,\dot{y}\right)+\omega_0^2\left(x+iy\right)=f_0\left(\cos\omega t+i\sin\omega t\right)$$

Introduce a new complex variable:

$$z=x+iy$$

The two derivatives:

$$\dot{z}=\dot{x}+i\,\dot{y}$$

$$\ddot{z}=\ddot{x}+i\,\ddot{y}$$

Apply the Euler-relation:

$$\ddot{z}+2\alpha\,\dot{z}+\omega_0^2z=f_0e^{i\omega t}$$

This is a complex equation; let's look for the solution in the next form:

$$z=Ae^{i(\omega t-\delta)}$$

$$-A\omega^2e^{i(\omega t-\delta)}+2\alpha Ai\omega\,e^{i(\omega t-\delta)}+\omega_0^2A\,e^{i(\omega t-\delta)}=f_0e^{i\omega t}$$

Simplify the equation by the term $e^{i\omega t}$,

$$Ae^{-i\delta}\left[-\omega^2+2\alpha i\omega+\omega_0^2\right]=f_0,$$

$$A\left(\cos\delta-i\sin\delta\right)\left[\left(\omega_0^2-\omega^2\right)+2\alpha\omega i\right]=f_0.$$

The above equation is a complex algebraic equation, and it can be satisfied by two real equations as:

$$A\left[\cos\delta\left(\omega_0^2-\omega^2\right)+2\alpha\omega\sin\delta\right]=f_0$$
$$A\left[\sin\delta\left(\omega_0^2-\omega^2\right)+2\alpha\omega\cos\delta\right]=0\,.$$

The unknown quantities are A and δ, and the solutions:

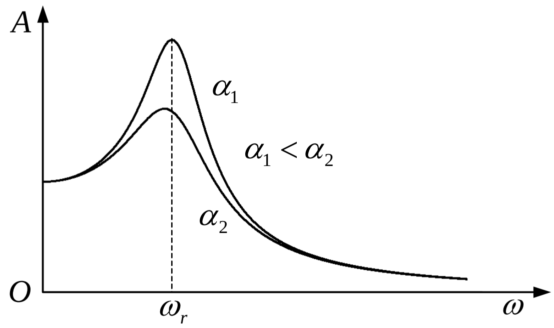
$$\tan\delta=\frac{2\alpha\omega}{\omega_0^2-\omega^2}\,,$$
$$A=\frac{f_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\alpha^2\omega^2}}$$

The real part of the solution is:

$$x(t)=A\cos(\omega t-\delta)$$

So the stationary solution is a simple harmonic motion with constant amplitude and with the same angular frequency as the driving force.

The amplitude A depends on the amplitude of the driving force and on the cyclic frequency of the driving force. The next graph shows that A assumes a maximum value at a certain frequency ω_r, called resonant frequency.



The separate curves correspond to different values of the parameter α. When the frequency of the driving force is the resonance frequency, the oscillating system is in the state of resonance. As the damping decreases the height of the resonance curve increases and becomes sharper. To determine the resonance frequency consider the amplitude:

$$A=\frac{f_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\alpha^2\omega^2}}$$

The amplitude has maximum if:

$$f(\omega)=\left(\omega_0^2-\omega^2\right)^2+4\alpha^2\omega^2$$

has minimum.

To get the minimum, take the derivative of the $f(\omega)$ function:

$$\frac{df}{d\omega}=2\left(\omega_0^2-\omega^2\right)(-2\omega)+4\alpha^2 2\omega=4\omega\left(2\alpha^2-\omega_0^2+\omega^2\right).$$

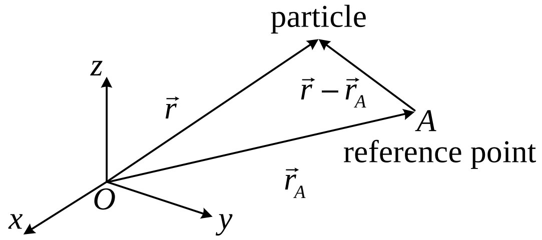
For the minimum:

$$2\alpha^2-\omega_o^2+\omega_r^2=0$$
$$\omega_r=\sqrt{\omega_0^2-2\alpha^2}=\sqrt{\gamma^2-\alpha^2}$$

Resonance occurs only in under damping system. In case of resonance the frequency of the driving force is less than the natural frequency.

2.6 First order momenta

Consider a particle in a reference system with a position vector $\vec{r}$ and consider a reference point *A* with position vector $\vec{r}_A$.



The statical momentum or the momentum of mass relative to point *A* is defined as:

$$\vec{S}_A = (\vec{r}-\vec{r}_A)m$$

The angular momentum or the momentum of the linear momentum of a particle relative to the point *A* is given by:

$$\vec{L}_A = (\vec{r}-\vec{r}_A)\times\vec{p}$$

The torque about a given point *A*, it is also called the momentum of the force is defined as:

$$\vec{M}_A = (\vec{r}-\vec{r}_A)\times\vec{F}$$

2.6.1 Relation between angular momentum and torque

Consider the angular momentum of a particle about a point *A:*

$$\vec{L}_A = (\vec{r}-\vec{r}_A)\times\vec{p}$$

Take the first time derivative of the equation:

$$\dot{\vec{L}}_A = (\vec{r}-\vec{r}_A)^{\bullet}\times\vec{p} + (\vec{r}-\vec{r}_A)\times\dot{\vec{p}} = \left(\dot{\vec{r}}-\dot{\vec{r}}_A\right)\times\vec{p} + (\vec{r}-\vec{r}_A)\times\vec{F} =$$

$$= (\vec{v}-\vec{v}_A)\times\vec{p} + \vec{M}_A = -\vec{v}_A\times\vec{p} + \vec{M}_A$$

We have obtained:

$$\dot{\vec{L}}_A = \vec{M}_A - \vec{v}_A\times m\vec{v}$$

In the calculation we have used that the vector product of two parallel vectors is zero:

$$\vec{v}\times\vec{p} = \vec{v}\times m\vec{v} = 0$$

If the reference point is at rest in the reference system, $\vec{v}_A = 0$, and:

$$\dot{\vec{L}}_A = \vec{M}_A$$

The time rate of change of the angular momentum of a particle is equal to the torque of the force applied to it with respect to the same point *A*. (the point *A* is at rest.) We call this equation as theorem for angular momentum.

2.7 Central forces

A force whose line of action passes through a fixed point or centre of force is called a central force. Central forces are of fundamental importance in physics, they include such forces as gravity, electrostatic forces, and others.

Choose the fixed point to be the origin of the reference system. In this case $\vec{r}$ and $\vec{F}$ are parallel vectors, and:

$$\vec{M}_O = \vec{r} \times \vec{F} = 0 .$$

In central field the torque on a particle about the origin is zero.
Due to the theorem for angular momentum:

$$\dot{\vec{L}}_O = \vec{M}_O ,$$

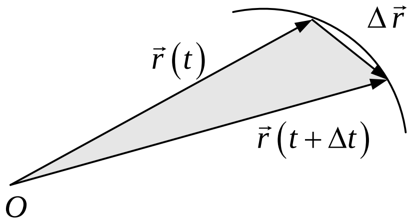
but the momentum of the force is zero:

$$\vec{M}_O = 0$$

therefore:

$$\vec{L}_O = \text{constant vector}$$

In case of central force, the angular momentum relative to the centre of force is a constant of motion.
As $\vec{L}_O = \vec{r} \times m\vec{v} = \text{constant vector}$, in central field the trajectory of the particle is always in the same plane.
The angular momentum of a particle is related to the rate at which the position vector sweeps out area. Consider the motion of a particle in central field during Δt:



The area ΔA of the shaded triangular segment is:

$$\Delta A = \frac{1}{2}\left|\vec{r} \times \Delta\vec{r}\right| .$$

Divided by Δt and taking the limiting value as Δt tends to zero we get:

$$\Lambda = \frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2}\left|\vec{r} \times \vec{v}\right|$$

The swept area in unit time is called areal velocity vector:

$$\vec{\Lambda} = \frac{1}{2}\left(\vec{r} \times \vec{v}\right)$$

$\vec{\Lambda}$ is a vector, whose magnitude is given above and perpendicular to the plane of the motion.
Apply the definition of the angular momentum:

$$\vec{L} = \vec{r} \times m\vec{v}$$

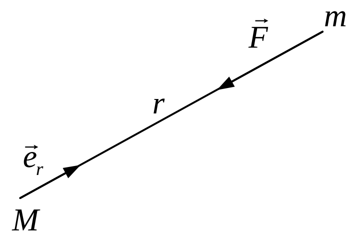
The connection between angular momentum and areal velocity:

$$\vec{\Lambda} = \frac{1}{2}\vec{r} \times \vec{v} = \frac{\vec{L}}{2m}$$

In central field $\vec{L} = \text{constant vector}$ so $\vec{\Lambda}$ the areal velocity is also constant vector.

2.8 The Law of Universal Gravitation

All bodies in nature mutually attract one another. The law which this attraction obeys was established by Newton (1687). The gravitational interaction between two point particles can be expressed by an attractive central force proportional to the gravitational masses of the particles and inversely proportional to the square of the distance between them.



$$\vec{F} = -\gamma \frac{Mm}{r^2}\vec{e}_r$$

γ is called the universal gravitational constant measured by Cavendish (1798) with a very sensitive torsion balance.

$$\gamma = 6,67 \cdot 10^{-11} \frac{Nm^2}{kg^2}.$$

$\vec{e}_r$ is a unit vector:

$$\vec{e}_r \cdot \vec{e}_r = 1$$

In the law of universal gravitation we have used the so called gravitational mass of a particle. The gravitational mass characterizes the participation in the gravitational interaction, and can be measured by spring. The inertial mass characterizes the inertial properties of bodies. The next question arises: we ought to distinguish the inertial and gravitational mass? In 1909 the Hungarian physicist L. Eötvös was the first who measured that the gravitational mass is proportional to the inertial mass, and for simplicity the proportionality factor is chosen to be one, so:

$$m_{gr} = m_{in} = m$$

The gravitational interaction is carried out through a gravitational field. The gravitational field is conservative field. Consider the elementary work done:

$$\vec{F} \cdot d\vec{r} = -\gamma \frac{Mm}{r^2}\vec{e}_r \cdot d\vec{r} = -\gamma \frac{Mm}{r^2}\vec{e}_r \left(dr\,\vec{e}_r + r\,d\vec{e}_r\right)$$

As $\vec{e}_r \cdot \vec{e}_r = 1$, and $\vec{e}_r \cdot d\vec{e}_r = 0$, we get:

$$\vec{F} \cdot d\vec{r} = -\gamma \frac{Mm}{r^2} dr$$

The right side can be written as a differential:

$$\vec{F} \cdot d\vec{r} = -d\left(-\gamma \frac{Mm}{r} + \text{constant}\right)$$

$$\vec{F} \cdot d\vec{r} = -dV$$

Therefore the potential energy:

$$V = -\gamma \frac{Mm}{r} + \text{constant}$$

The arbitrary constant is chosen so that the potential energy in the infinity is zero:

if $r = \infty$, and $V = 0$, then $\text{constant} = 0$

The gravitational potential energy:

$$V = -\gamma \frac{Mm}{r}.$$

The gravitational force on a test particle is proportional to its mass *m*, so the force can be written:

$$\vec{F} = m\vec{f}$$

The second term $\vec{f}$ characterizes the gravitational field which is produced by a mass distribution. $\vec{f}$ is called gravitational field strength, or gravitational field intensity. The gravitational field strength due to the mass point *M* at a distance *r* is:

$$\vec{f} = \frac{\vec{F}}{m} = -\gamma \frac{M}{r^2}\vec{e}_r$$

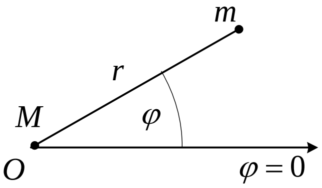
We can introduce the gravitational potential as the gravitational potential energy of unit mass:

$$U = \frac{V}{m} = -\gamma \frac{M}{r} .$$

Close to the surface of the Earth the gravitational potential energy is $V = mgh$, if the reference point is on the Earth and there the potential energy is zero.

2.9 The Motion of Planets in Gravitational Field

The gravitational field is central and conservative field. Describe the motion of a planet around the Sun. Denote the mass of the planet by *m* and the mass of the Sun by *M*. We assume that *M* greatly exceeds the mass of the planet. So we suppose that the Sun remains at rest at the origin of the plane polar coordinate system.



In central field the torque on a particle about the origin is zero:

$$\vec{M}_O = 0$$

Due to the theorem for angular momentum we have:

$$\dot{\vec{L}}_O = 0 \text{, so } \vec{L}_O = \text{constant vector}$$

The angular momentum of the particle about the Sun is constant. As we know the sector or areal velocity is in a connection with the angular momentum, and it must be also constant:

$$\vec{\Lambda} = \frac{\vec{L}_O}{2m} ,\ \vec{\Lambda} = \text{constant} .$$

In central field the sector or areal velocity is constant. That is in the gravitational field of the Sun the planet moves in a plane which passes through the centre of force (Sun) and the position vector sweeps out equal areas in equal time. So we have proved Kepler’s II. Law.
Kepler’s II. Law states:
The position vector of any planet relative to the Sun sweeps out equal areas in equal times (law of areas).

The gravitational field is central field:

$$\vec{\Lambda} = \text{constant} .$$

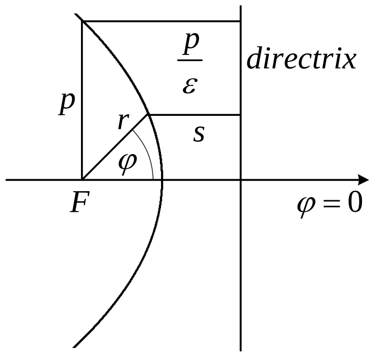
The gravitational field is conservative field, so the mechanical energy remains constant too.

$$T + V = \frac{1}{2}mv^2 - \gamma \frac{Mm}{r} = \text{constant}$$

From this two conservation laws we can deduce the equation of the orbit in plane polar coordinate system.
Without calculation:

$$r(\varphi) = \frac{p}{1 + \varepsilon \cos\varphi},$$

where $p = \frac{4\Lambda^2}{\lambda M}$.
It is just the equation of a conic section (ellipse, parabola, or hyperbola) with the origin at the focus. So it is the general equation of a second order curve, which is the locus of points P whose distance from a given point (focus) and a given straight line (directrix) have a constant quotient. The distance *p* is called the parameter of the curve, ε is called the excentricity of the curve.



$$\frac{r}{s} = \varepsilon = \text{constant}$$

From the figure:

$$r\cos\varphi + s = \frac{p}{\varepsilon}, \text{ and } s = \frac{p}{\varepsilon} - r\cos\varphi$$

Instead of *s* we can use the definition:

$$s = \frac{r}{\varepsilon}$$

$$\frac{r}{\varepsilon} = \frac{p}{\varepsilon} - r\cos\varphi, \text{ so } r = p - \varepsilon r\cos\varphi, \text{ and } r(1 + \varepsilon\cos\varphi) = p$$

Therefore:

$$r = \frac{p}{1 + \varepsilon\cos\varphi}$$

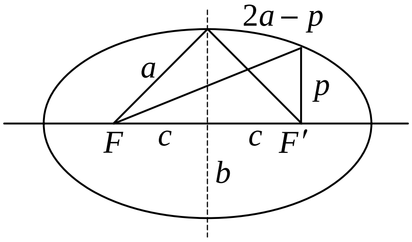
The excentricity ε is the quotient of two distances that is $\varepsilon \ge 0$.
if $\varepsilon < 1$, the curve is ellipse
(if $\varepsilon = 0$, the curve is circle, special ellipse)
if $\varepsilon = 1$, the curve is parabola
if $\varepsilon > 1$, the curve is hyperbola.

When the curve is circle or ellipse, we call the moving particle as a planet. So we have got Kepler's I. Law.
Kepler's I. Law states:
All planets move in elliptical orbits having the Sun as one focus (law of orbits).

To obtain Kepler's III. law write the area of an ellipse:

$$A = ab\pi \ ,$$

where *a* and *b* are the semi-major and semi-minor axes respectively.



$$a^2 = b^2 + c^2 \text{, and}$$

$$(2a - p)^2 = p^2 + (2c)^2$$

$$4a^2 + p^2 - 4ap = p^2 + 4c^2$$

$$4\left(a^2 - c^2\right) = 4ap$$

$$b^2 = ap \ .$$

Calculate the area of the ellipse:

$$A = ab\pi = a\sqrt{ap}\pi = a^{\frac{3}{2}} p^{\frac{1}{2}}\pi$$

If we multiply the sector velocity with the time T required for the particle to complete one orbital path we obtain also the area of the ellipse.

$$A = \Lambda T \ .$$

Take the square of the equation:

$$A^2 = \Lambda^2 T^2 \text{, or } a^3 p\pi^2 = \Lambda^2 T^2$$

It is known that:

$$p = \frac{4\Lambda^2}{\gamma M} \text{, and } \frac{\gamma M p}{4} = \Lambda^2$$

We can obtain:

$$a^3 p \pi^2 = \frac{\gamma M p}{4} T^2 \ ,$$

Finally:

$$T^2 = \frac{4\pi^2}{\gamma M} a^3$$

This equation is just Kepler's III. law.
Kepler's III. Law states:
The square of the period of any planet is proportional to the cube of the semi-major axis of its orbit, and the proportionality factor is the same for all planets around the Sun (law of periods).