1 Kinematics

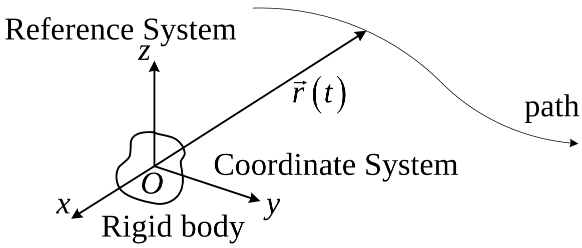
1.1 Basic concepts of kinematics

Kinematics is the pure mathematical and geometrical description of motion. When a body changes its position relative to other bodies it is said to be in mechanical motion.

We consider only mass points. A mass point or particle is a body whose dimension or size can be neglected in the given problem. The characteristic measurement is relatively small.

The set of locations of a body in time is called the path or trajectory.

The reference body is a rigid body which is fixed for a given problem, and with respect to which the motion of a particle is described.



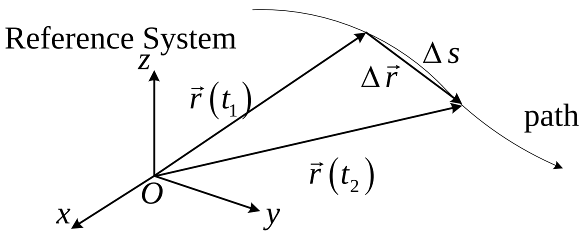
The reference system is the reference body and a coordinate system fixed to this body. It should be:

- Rectangular or Cartesian coordinate system,
- Plane polar coordinate system,
- Cylindrical coordinate system, or
- Spherical coordinate system.

The position vector or radius vector is a vector drawn from the origin of the Reference System to the momentary position of the mass-point, denoted by: $\vec{r}$.
In case of motion the position vector is a function of time:

$$\vec{r} = \vec{r}(t).$$

Assume that a particle travelled along a certain trajectory form point 1. to point 2. The displacement vector is a vector drawn between the tow positions directed towards the later one.

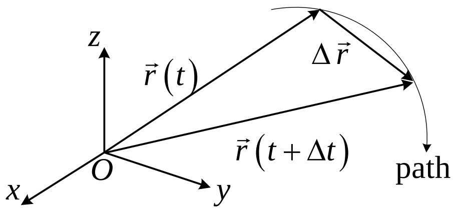


$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1), \quad t_2 > t_1$$

The path measured along the trajectory is called distance travelled by the particle:

$$\Delta s \geq |\Delta\vec{r}|$$

Consider the position vector $\vec{r}(t)$ at time *t*, and denote the elapsed time by Δt :



The average velocity is defined as:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t}$$

The instantaneous velocity (velocity) is the time derivative of the position vector:

$$\vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

As the displacement $d\vec{r}$ coincides with an infinitely small element of the path consequently the vector $\vec{v}$ is directed along a tangent to the trajectory.

If the position vector is written as the function of the distance measured along the path:

$$\vec{r} = \vec{r}(s)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

The tangent unit vector is defined as:

$$\vec{\tau} = \frac{d\vec{r}}{ds}, \text{ and } |\vec{\tau}| = 1$$

The absolute value, or magnitude of the velocity is called speed:

$$|\vec{v}| = v = \frac{ds}{dt} \geq 0$$

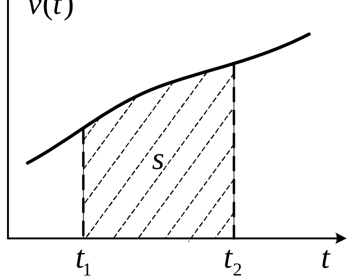
And the velocity vector is:

$$\vec{v} = v\,\vec{\tau}$$

The velocity is a vector which is tangent to the path, while the speed is scalar quantity. As the speed is the derivative of the distance with respect to time, the travelled distance is the integral of the speed for the time interval in question.

$$v = \frac{ds}{dt} = \dot{s}$$

$$s = \int_{t_1}^{t_2} v\,dt$$



The travelled distance is the area under the speed time graph.

The velocity $\vec{v}$ of a particle can change with time both in magnitude and in direction.

The acceleration of a particle is the rate of change of the velocity. The average acceleration is:

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t}$$

The instantaneous acceleration (acceleration) is the first time derivative of the velocity or the second derivative of the position vector.

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$$

1.1.1 Tangential and normal component of the acceleration:

$$\vec{v} = \frac{d\vec{r}}{dt} = v\,\vec{\tau}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v\,\vec{\tau}\right) = \dot{v}\,\vec{\tau} + v\frac{d\vec{\tau}}{dt} = \dot{v}\,\vec{\tau} + v\frac{d\vec{\tau}}{ds}\frac{ds}{dt} = \dot{v}\,\vec{\tau} + v^2\frac{d\vec{\tau}}{ds}$$

It can be proved that:

$$\frac{d\vec{\tau}}{ds} = \kappa\,\vec{n}\ ,$$

where $\vec{n}$ is the normal unit vector, and κ is the curvature.
The components of the acceleration:

$$\vec{a} = \dot{v}\,\vec{\tau} + v^2\kappa\vec{n}$$

Introducing the radius of curvature ρ:

$$\kappa = \frac{1}{\rho}$$

$$\vec{a} = \dot{v}\,\vec{\tau} + \frac{v^2}{\rho}\vec{n}$$

So the acceleration has a tangential and a normal or centripetal component:

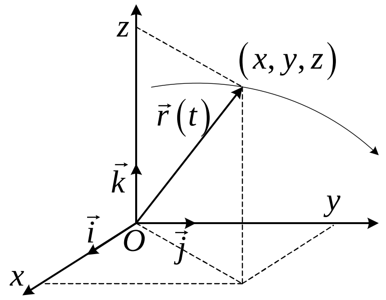
$$a_t = \dot{v}$$

$$a_n = \frac{v^2}{\rho}$$

1.1.2 The description of motion in different coordinate system

Rectangular coordinate system consists of three mutually perpendicular lines are drawn through a point O called origin. These lines are the three coordinate axis: *x*, *y*, *z*. The unit vectors are $\left\{\vec{i}\,,\vec{j},\vec{k}\right\}$ and in this order form a right hand system.

We know the motion of a particle if we know the position vector at any moment $\vec{r} = \vec{r}\left(t\right)$.



The projections of the position vector $\vec{r}$ onto the coordinate axes are the rectangular coordinates x, y, z.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

We know the motion if we know the coordinates as a function of time:

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

this is the parametric equation system of path.
The velocity by definition:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

The unit vectors do not follow the motion, so they are constant in direction and in magnitude and their time derivatives are zero:

$$\dot{\vec{i}} = \dot{\vec{j}} = \dot{\vec{k}} = 0$$

$$\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} ,$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

The rectangular coordinates of the velocity are:

$$v_x = \dot{x}$$

$$v_y = \dot{y}$$

$$v_z = \dot{z} .$$

The magnitude of the velocity:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The acceleration is the time derivative of the velocity:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}\right)$$

$$\vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} ,$$

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

The three rectangular components of the acceleration:

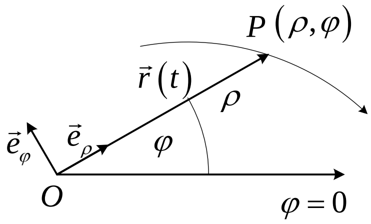
$$a_x = \ddot{x}$$

$$a_y = \ddot{y}$$

$$a_z = \ddot{z}$$

2. Plane polar coordinate system

The position of any point *P* lying in a plane can be completely determined by its distance ρ from the origin of the reference system and with the angle φ that the radius vector makes with the $\varphi = 0$ reference line passing through the origin *O*.



The length of the position vector:

$$|\vec{r}| = \rho$$

φ is positive measured counter-clock wise. The polar coordinates of point *P* are ρ are φ. In case of motion these are functions of time, and the parametric equation system of path is:

$$\rho = \rho(t)$$

$$\varphi = \varphi(t)$$

The unit vectors are $\vec{e}_\rho$ and $\vec{e}_\varphi$.

$$|\vec{e}_\rho| = |\vec{e}_\varphi| = 1$$

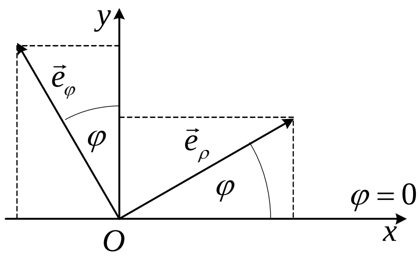
The position vector is:

$$\vec{r} = \rho\vec{e}_\rho$$

The velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\rho\vec{e}_\rho) = \dot{\rho}\vec{e}_\rho + \rho\dot{\vec{e}}_\rho$$

These unit vectors follow the motion so their time derivatives are no longer zero.



$$\vec{e}_\rho = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$$

$$\vec{e}_\varphi = \sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$$

$$\dot{\vec{e}}_\rho = -\dot{\varphi}\sin\varphi\,\vec{i} + \dot{\varphi}\cos\varphi\,\vec{j} = \dot{\varphi}\vec{e}_\varphi$$

$$\dot{\vec{e}}_\varphi = -\dot{\varphi}\cos\varphi\,\vec{i} + \dot{\varphi}\sin\varphi\,\vec{j} = -\dot{\varphi}\vec{e}_\rho$$

So the velocity:

$$\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi$$

The two components of the velocity:

$$v_\rho = \dot{\rho}$$

$$v_\varphi = \rho\dot{\varphi}$$

The acceleration by definition:

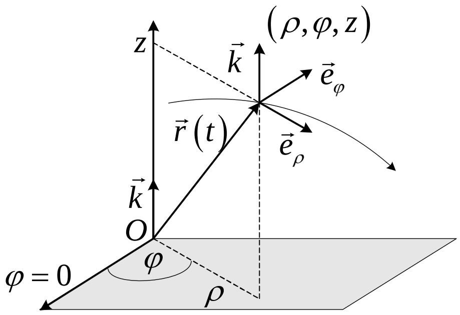
$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}\left(\dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi\right) = \ddot{\rho}\vec{e}_\rho + \dot{\rho}\dot{\vec{e}}_\rho + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi + \rho\dot{\varphi}\dot{\vec{e}}_\varphi$$

$$\vec{a} = \ddot{\rho}\vec{e}_\rho + 2\dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi - \rho\dot{\varphi}^2\vec{e}_\rho$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\varphi}^2\right)\vec{e}_\rho + \left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\vec{e}_\varphi$$

3. Cylindrical coordinate system

The cylindrical coordinate system is a combination of a plane polar coordinate system with a perpendicular *z* axis.



The cylindrical coordinates are:

$$\rho,\ \varphi,\ z.$$

The unit vectors are:

$$\left\{\vec{e}_\rho, \vec{e}_\varphi, \vec{k}\right\}$$

Position vector:

$$\vec{r} = \rho\vec{e}_\rho + z\vec{k}$$

Velocity vector:

$$\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{k}$$

Acceleration vector:

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\varphi}^2\right)\vec{e}_\rho + \left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\vec{e}_\varphi + \ddot{z}\vec{k}$$

1.2 Special motions

1.2.1 Rectilinear uniform motion, uniform straight line motion

Let's consider a particle travelling with constant velocity denoted by $\vec{v}_0$. There is no acceleration, $\vec{a} = 0$, $\vec{v}_0 = \text{constant}$.

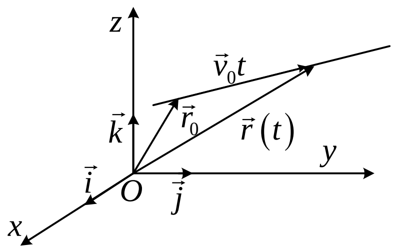
$$\vec{v}_0 = \frac{d\vec{r}}{dt}, \quad d\vec{r} = \vec{v}_0 dt$$

$$\vec{r} = \int \vec{v}_0 dt = \vec{v}_0 t + \vec{c}$$

To determine the constant of the integration we need initial conditions:

$$\text{if } t = 0 \quad \vec{r} = \vec{r}_0 \Rightarrow \vec{c} = \vec{r}_0$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t$$



The components of this vector equation:

$$x = x_0 + v_{0x} t \ ,$$

$$y = y_0 + v_{0y} t \ ,$$

$$z = z_0 + v_{0z} t \ .$$

For simplicity one coordinate axis is often chosen to be the straight line of the motion. In this case one component equation is enough to describe the motion:

$$x = x_0 + v_{0x} t$$

The $x - t$ function is a linear function.

1.2.2 Uniformly accelerated straight line motion

Consider a motion in which the particle has constant acceleration. Suppose that $\vec{a}$ and the initial velocity $\vec{v}_0$ is parallel.

$$\vec{a} = \text{constant, and } \vec{v}_0 \parallel \vec{a}$$

$$\vec{a} = \frac{d\vec{v}}{dt}, \rightarrow \vec{v} = \int \vec{a} dt = \vec{a}t + \vec{c}_1$$

$\vec{c}_1$ is the first integration constant.

If $t = 0$ then $\vec{v} = \vec{v}_0$ (initial condition):

$$\vec{v}_0 = \vec{c}_1 \text{ and } \vec{v}(t) = \vec{v}_0 + \vec{a}t$$

$$\vec{v} = \frac{d\vec{r}}{dt} \rightarrow \vec{r} = \int \vec{v} dt = \int (\vec{v}_0 + \vec{a}t) dt$$

$$\vec{r} = \vec{v}_0 t + \frac{\vec{a}}{2} t^2 + \vec{c}_2$$

If the initial position is $\vec{r}_o$, then:

$$t = 0, \text{ when } \vec{r} = \vec{r}_0 \Rightarrow \vec{c}_2 = \vec{r}_0$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{\vec{a}}{2} t^2$$

If the motion takes place along the x-axis then:

$$x = x_0 + v_{0x}t + \frac{a}{2}t^2$$

1.2.3 Motion in a plane with constant acceleration, projectile motion

Consider the motion, when $\vec{a} = \text{constant}$, but the acceleration $\vec{a}$ is not parallel with the initial velocity $\vec{v}_0$. In this case the trajectory is a curve.

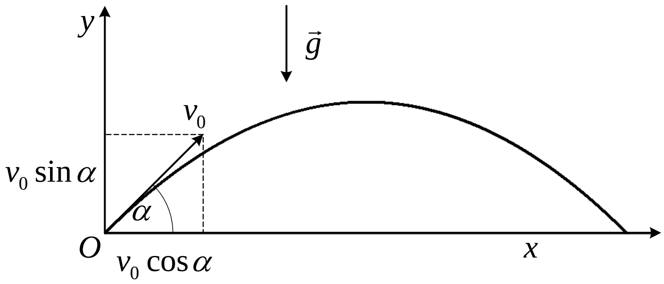
$$\vec{a} = \text{constant}$$
$$\vec{v} = \vec{v}_0 + \vec{a}t$$
$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{\vec{a}}{2}t^2$$

This expression gives the position of the particle at any time.

An important example for this motion is the motion of a projectile. In this case $\vec{a}$ is just the acceleration due to gravity:

$$\vec{a} = \vec{g}\ .$$

Choose an $x - y$ plane defined by $\vec{v}_0$ and $\vec{g}$, the y-axis directed upward so $\vec{g} = -g\vec{j}$, and suppose that the initial position $\vec{r}_0 = 0$ (The motion starts from the origin.)



The components of the initial velocity:

$$v_{0x} = v_0 \cos\alpha$$
$$v_{0y} = v_0 \sin\alpha$$

The *x*-component of the velocity remains constant, the y-component is linear function:

$$v_x = v_0 \cos\alpha, \quad v_y = v_0 \sin\alpha - gt$$

The coordinates of the particle as a function of time:

$$x = v_0 \cos\alpha\ t$$
$$y = v_0 \sin\alpha\ t - \frac{g}{2}t^2$$

The equation of the path can be obtained by eliminating time *t* from the system, and get a parabola:

$$y = -\frac{g}{2v_0^2 \cos^2\alpha}x^2 + x\,tg\alpha$$

The horizontal range:

$$R = \frac{v_0^2}{g}\sin 2\alpha\ ,\ (y = 0)$$

The time of rising:

$$t_r = \frac{v_0 \sin\alpha}{g}, \left(v_y = 0\right)$$

The maximum height:

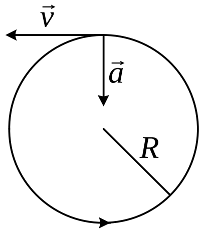
$$h = \frac{v_0^2 \sin^2\alpha}{2g}, \left(h = y\left(t_r\right)\right)$$

1.2.4 Uniform circular motion

Take that motion as the speed is constant, the magnitude of the acceleration is constant, and the direction of the acceleration is always perpendicular to the velocity.

$$\vec{v} = v\vec{\tau}$$

$$\vec{a} = \dot{v}\vec{\tau} + \frac{v^2}{\rho}\vec{n}$$



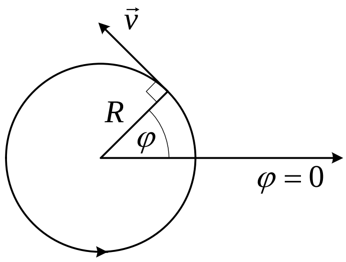
$$v = \text{constant}, \ \dot{v} = 0$$

In case of circular motion the radius of curvature is just the radius of the circle.

$$\vec{a} = \frac{v^2}{R}\vec{n}$$

This acceleration is called centripetal acceleration.

If the centre of the circular motion and the origin of a plane polar coordinate system coincides:



$$\vec{v} = v\vec{\tau}$$

$$\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi$$

$$\rho = R = \text{constant}$$

The derivative of the radius is:

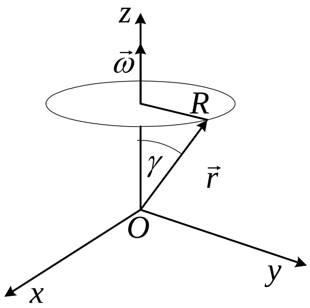
$$\dot{\rho} = 0 \Rightarrow \vec{v} = R\dot{\varphi}\vec{e}_\varphi$$

$$\left|\vec{v}\right| = v = R\dot{\varphi}$$

Introduce the angular speed as the time rate of change of the angle φ:

$$\omega = \frac{d\varphi}{dt} = \dot{\varphi}$$
$$v = R\omega$$

The angular velocity may be expressed as a vector quantity, whose direction is perpendicular to the plane of motion with a sense given by the thumb of the right hand when the fingers point in the sense of the motion:



$$v = R\omega$$
$$R = r\sin\gamma$$
$$v = r\omega\sin\gamma\ ,$$

and the following vector relation is valid:

$$\vec{v} = \vec{\omega}\times\vec{r}$$

1.2.5 Uniformly accelerated circular motion

If the angular speed of a particle changes with time, the angular acceleration is defined as:

$$\beta = \frac{d\omega}{dt}$$

Consider now the motion of a particle whose angular acceleration is constant, and the radius of the motion is *R*.

$$\beta = \text{constant}\ ,$$
$$\beta = \frac{d\omega}{dt} \rightarrow \ \omega = \int \beta dt = \beta t + \omega_0$$
$$\omega = \omega_0 + \beta t$$
$$v = R\omega$$
$$v = R\left(\omega_0 + \beta t\right)$$

The velocity vector:

$$\vec{v} = v\vec{\tau} = R\left(\omega_0 + \beta t\right)\vec{\tau}$$

The acceleration:

$$\vec{a} = \dot{v}\,\vec{\tau} + \frac{v^2}{R}\vec{n}$$
$$\vec{a} = R\beta\vec{\tau} + R\omega^2\vec{n}$$

The tangential acceleration:

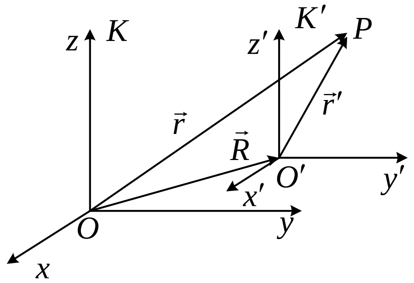
$$a_t = R\beta$$

The centripetal or normal acceleration:

$$a_n = a_{cp} = \frac{v^2}{R} = R\omega^2$$

1.3 Relative velocity and acceleration

Motion is a relative concept and it must always be referred to a particular frame of reference chosen by the observer. Consider now two observers who move relative to each other with translational uniform motion. The *K'* reference system moves with constant velocity with respect to *K*. They do not rotate.



The two observers of the two reference systems are watching the motion of the same mass point *P*. The relation between the position rectors:

$$\vec{r} = \vec{R} + \vec{r}\,',$$

where $\vec{R}$ is the vector between O and O'. If the time flows in the same way in both reference systems then the time derivatives are:

$$\vec{v} = \frac{d\vec{R}}{dt} + \vec{v}\,'$$

Let's denote the velocity of *K'* with respect to *K* by $\vec{V}$:

$$\vec{V} = \frac{d\vec{R}}{dt},$$

$$\vec{v} = \vec{V} + \vec{v}\,'$$

$\vec{V} = \text{constant}$, $\dot{\vec{V}} = 0$ so the accelerations:

$$\vec{a} = \vec{a}\,'$$

That is, if two observers moving with constant velocity with respect to each other, observe the same acceleration of a point in motion.

[We have assumed that time has an absolute meaning independent of any particular frame of reference. Einstein was who discovered that it is not true and he developed the modified theory called special theory of relativity.]