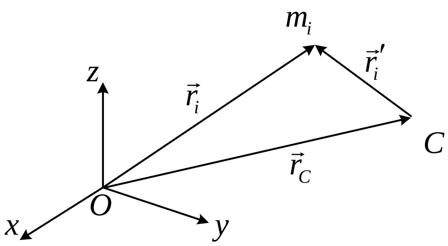
3 Dynamics of a System of Particles

In studying a system or collection of many free particles, we shall be mainly interested in the general features of the motion of such a system. Consider a system consists of *n* particles of masses $m_1,\ m_2, \ldots.\ m_n$, whose position vectors are $\vec{r}_1,\ \vec{r}_2, \ldots.\ \vec{r}_n$, and velocities are $\vec{v}_1,\ \vec{v}_2, \ldots.\ \vec{v}_n$ respectively.

Introduce the centre of mass of system of particles:
The geometrical point relative to the statical momentum of a system of particles is zero is called the center of mass of the system:



$$\vec{r}_i = \vec{r}_C + \vec{r}_i\,' \rightarrow \vec{r}_i\,' = \vec{r}_i - \vec{r}_C$$

$$\sum_{i=1}^{n} m_i \vec{r}_i\,' = 0$$

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i - \vec{r}_C \right) = 0$$

$$\sum_{i=1}^{n} m_i \vec{r}_i = \vec{r}_C \sum_{i=1}^{n} m_i$$

Introduce the total mass of the system: *M*

$$M = \sum_{i=1}^{n} m_i$$

Therefore, the position vector of the center of mass:

$$\vec{r}_C = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M}$$

To obtain the expression for the centre of mass of a continuous body we begin subdividing the body into *n* small elements of mass Δm_i located at the point $\vec{r}_i$, then:

$$\vec{r}_C \approx \frac{\sum_{i=1}^{n} \Delta m_i \vec{r}_i}{M}$$

Let the number of elements tend to infinity while their mass approaches zero:

$$\vec{r}_C = \frac{1}{M} \lim_{\substack{\Delta m_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \Delta m_i \vec{r}_i$$

If the density distribution of the body is given, then $\Delta m_i = \rho\left(\vec{r}_i\right) \Delta V_i$, and

$$\vec{r}_C = \frac{1}{M} \cdot \lim_{\substack{\Delta V_i \to 0 \\ n \to \infty}} \vec{r}_i\ \rho\left(\vec{r}_i\right) \Delta V_i$$

By definition it is an integral with respect to volume over the whole volume of the body:

$$\vec{r}_C = \frac{1}{M}\int_V \vec{r}\rho dV$$

Suppose that there are external forces $\vec{F}_1, \vec{F}_2 ... \vec{F}_n$ acting on the respective particles. In addition, there may be internal forces of interaction between any two particles of the system. Denote these internal forces by $\vec{F}_{ik}$ meaning the force exerted on particle *i* by particle *k*. The equation of motion of particle *i* is:

$$\dot{\vec{p}}_i = \vec{F}_i + \sum_{\substack{k=1 \\ k\neq i}}^{n} \vec{F}_{ik}$$

The $k = i$ is excluded from the summation. Adding this equation for the *n* particles we have:

$$\sum_{i=1}^{n} \dot{\vec{p}}_i = \sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n}\sum_{k=1}^{n} \vec{F}_i$$

From Newton's third law, or the law of action and reaction:

$$\vec{F}_{ik} = -\vec{F}_{ki}$$

Consequently, in the double summation the internal forces cancel in pairs, and the value of this sum vanishes.
Introduce the total momentum of the system, and the total external force by the next definitions:

$$\vec{p} = \sum_{i=1}^{n} \vec{p}_i \text{ ,and } \vec{F} = \sum_{i=1}^{n} \vec{F}_i$$

We can therefore write:

$$\dot{\vec{p}} = \vec{F}$$

The time rate of change of the total momentum of a system is equal to the total external force. The internal forces do not change the momentum of the system.

If the system of particles is isolated that is the external force $\vec{F} = 0$, then the total momentum of the system remains constant.

$$\vec{F} = 0 \quad \Rightarrow \quad \dot{\vec{p}} = 0 \quad \Rightarrow \quad \vec{p} = \text{constant}$$

Taking the time derivative of the he centre of mass:

$$\vec{r}_C = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M}$$

$$\vec{v}_C = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{M} = \frac{\sum_{i=1}^{n} \vec{p}_i}{M}$$

$$M\vec{v}_C = \sum_{i=1}^{n} \vec{p}_i = \vec{p}$$

The result of the next time derivative:

$$M\vec{a}_C = \dot{\vec{p}}$$

$$\vec{F} = \dot{\vec{p}}$$

$$\vec{F} = M\vec{a}_C$$

The centre of mass of a system of particles moves as if it was a particle of mass equal to the total mass of the system and subject to the external force applied to the system.
In an isolated system the centre of mass has no acceleration, it is at rest or it moves along a straight line with constant velocity.
We may attach a frame of reference to the centre of mass whose velocity is $\vec{v}_C$ relative to the inertial system but it does not rotate relative to the inertial system. This frame of reference is called the centre-of-mass frame of reference.
In case of isolated system the centre-of-mass frame of reference is an inertial system.
Consider again the definition of the centre of mass, and take the time derivative:

$$\sum_{i=1}^{n} m_i \vec{r}_i\,' = 0$$

$$\sum_{i=1}^{n} m_i \vec{v}_i\,' = \sum_{i=1}^{n} \vec{p}_i\,' = 0$$

The total momentum of a system of particles referred to the *C*-frame of reference is zero.

3.1 Angular momentum of a System

As we know the angular momentum of a particle relative to the point *A* is given by:

$$\vec{L}_A = \left(\vec{r} - \vec{r}_A\right) \times \vec{p}$$

If *A* is the origin of the reference system then:

$$\vec{L} = \vec{r} \times \vec{p}$$

The relation between the angular momentum and torque is:

$$\dot{\vec{L}}_A = \vec{M}_A - \vec{v}_A \times m\vec{v}$$

If *A* is the origin:

$$\dot{\vec{L}} = \vec{M}$$

Consider now a system of particles. The angular momentum $\vec{L}$ of a system of particles about *O* is defined as the vector sum of the individual angular momentum, namely:

$$\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i$$

Let’s calculate the time derivative of the angular momentum:

$$\dot{\vec{L}} = \sum_{i=1}^{n} \dot{\vec{r}}_i \times m_i \vec{v}_i + \sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{v}}_i$$

Use that $\vec{r}_i = \vec{v}_i$, and $\vec{v}_i \times m_i \vec{v}_i = 0$ due to parallel vectors, and $\dot{\vec{v}}_i = \vec{a}_i$:

$$\dot{\vec{L}} = \sum_{i=1}^{n} \dot{\vec{r}}_i \times m_i \vec{a}_i = \sum \vec{r}_i \times \left(\vec{F}_i + \sum_{k=1}^{n} \vec{F}_{ik} \right)$$

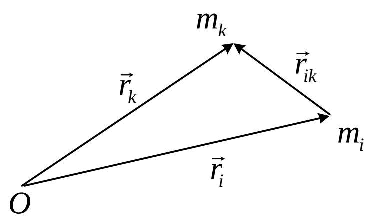
$$\dot{\vec{L}} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ i \neq k}}^{n} \vec{r}_i \times \vec{F}_{ik}\,.$$

The double summation on the right consists of pairs of term line these:

$$\vec{r}_i \times \vec{F}_{ik} + \vec{r}_k \times \vec{F}_{ki} = \vec{r}_i \times \vec{F}_{ik} - \vec{r}_k \times \vec{F}_{ik} = -\left(\vec{r}_k - \vec{r}_i\right) \times \vec{F}_{ik} = \vec{r}_{ik} \times \vec{F}_{ik}$$

We have used that $\vec{F}_{ik} = -\vec{F}_{ki}$ and we denote the position of particle *k* relative to particle *i* by:

$$\vec{r}_{ik} = \vec{r}_k - \vec{r}_i$$



If the internal forces are central, these terms vanish. They act along the lines connecting the pairs of particles. Hence the double sum vanishes.

The sum $\sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i$ is the total moment of all the external forces acting on the system. If we denote it by $\vec{M}$ them:

$$\dot{\vec{L}} = \vec{M}$$

That is the time rate of change of the angular momentum of a system is equal to the total moment (torque) of all the external forces acting on the system.

If the system is isolated, $\vec{M} = 0$, and the angular momentum remains constant, both in magnitude and direction. This statement is the principle of conservation of angular momentum.

3.2 Kinetic energy of a System of Particles

The total kinetic energy *T* of a system in the inertial system is:

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$$

As we have seen:

$$\vec{r}_i = \vec{r}_C + \vec{r}_i{}'$$

Where $\vec{r}_i{}'$ is the position of particle *i* relative to the centre of mass.

Taking the derivative with respect to *t* we have:

$$\vec{v}_i = \vec{v}_C + \vec{v}_i{}'$$

$$v_i^2 = v_C^2 + 2\vec{v}_i{}' \cdot \vec{v}_C + \vec{v}_i^2 ,$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 + \frac{1}{2}\sum_{i=1}^{n} m_i \cdot 2\vec{v}_i{}' \cdot \vec{v}_C + \frac{1}{2}\sum_{i=1}^{n} m_i \vec{v}_i'^2$$

$$T = \frac{1}{2} v_C^2 \sum_{i=1}^{n} m_i + \vec{v}_C \sum_{i=1}^{n} m_i \vec{v}_i{}' + \frac{1}{2}\sum_{i=1}^{n} m_i \vec{v}\,'^2_i$$

Apply the next formulas:

$$M = \sum_{i=1}^{n} m_i , \quad \sum_{i=1}^{n} m_i \vec{v}_i{}' = 0 , \quad T' = \frac{1}{2}\sum_{i=1}^{n} m_i \vec{v}\,'^2_i$$

$$T = \frac{1}{2} M v_C^2 + T'$$

Thus the total kinetic energy of a system of particles is given by the sum of the kinetic energy of translation of the center of mass plus the kinetic energy of motion of the individual particles relative to the center of mass.