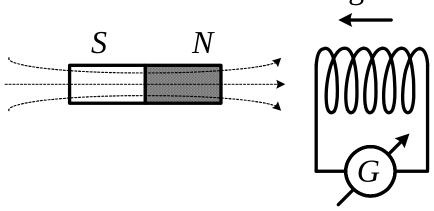
4 Time varying electromagnetic field

4.1 Electromagnetic Induction

4.1.1 Induction due to motion of conductor

Consider the Faraday's experiment. The figure shows a coil of wire connected to a current measuring galvanometer. Near the coil there is a bar magnet. When the coil is hold stationary the galvanometer does not show current flow, but if we move the coil either toward or away from magnet the meter shows current in opposite direction respectively.



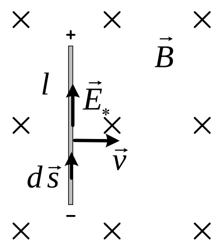
We call this an induced current, and the corresponding emf that has to be present to cause this current is called induced emf. As we move the conductor in magnetic field the charge carriers move with the conductor and experience a force (Lorentz Force).
This force is an extraneous force, it has not electrostatic origin:

$$\vec{F_*} = q\left(\vec{v}\times\vec{B}\right) \rightarrow \vec{E_*} = \frac{\vec{F_*}}{q} = \vec{v}\times\vec{B}$$

$\vec{E}^*$ is the strength of the extraneous field. The induced electromotive force (emf) along the conductor is:

$$\mathcal{E}_{AB} = \int_{AB} \vec{E_*}\cdot d\vec{s} = \int_{AB}\left(\vec{v}\times\vec{B}\right)\cdot d\vec{s}$$

If the moving conductor forms a loop, due to the induced emf a current flows through it.
Consider a straight conductor moving in homogeneous magnetic field, perpendicular to the plane directed into the page.



The vectors $\vec{v}, \vec{B}$ *and* $d\vec{s}$ mutually perpendicular to each other. Due to the induced electric field the emf along the conductor:

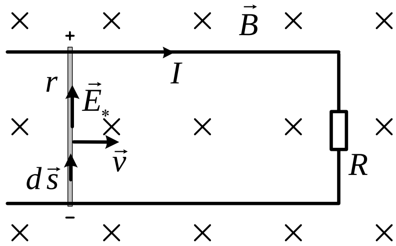
$$\mathcal{E} = \int_{\widehat{AB}} \vec{E^*}\cdot d\vec{s} = \int_{-}^{+}\left(\vec{v}\times\vec{B}\right)\cdot d\vec{s} = \int_{-}^{+} vBds = Blv$$

$$\mathcal{E} = B\,l\,v\,.$$

We used that $\vec{v}$ and $\vec{B}$ are perpendicular, both are constant over the length of the conductor, and their vector product is parallel to $d\vec{s}$.

Suppose the moving conductor slides along a stationary U shaped conductor as in figure. The moving conductor has become a source of emf, charges moves within it from lower to higher potential and in the remainder of the circuit charge moves from higher to lower potential. If the resistance of the sliding bar is *r* and the external resistance is *R* then:

$$\mathcal{E} = B\,l\,v\,.$$



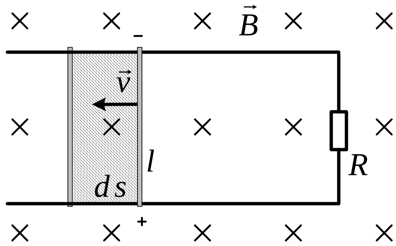
$$I = \frac{\mathcal{E}}{R+r} \Rightarrow U = IR = \mathcal{E}\frac{R}{R+r} < \mathcal{E}$$

The device above is called linear generator. The mechanical power needed to move the rod is converted into electrical power.

If a directed closed conductor loop moves in magnetic field the induce emf is:

$$\overset{\circ}{\mathcal{E}} = \oint\left(\vec{v}\times\vec{B}\right)\cdot d\vec{s}\,,\qquad \text{(Neumann' Law)}$$

Consider the linear generator again, when the conductor moves toward the right a distance *ds*, the area enclosed by the circuit is increased by $d\,A = l\,ds$ and the change in magnetic flux through the circuit is $d\Phi = B\,dA = B\,l\,ds$.



The time rate of change of flux is therefore:

$$\frac{d\Phi}{dt} = Bl\frac{ds}{dt} = Blv$$

Faraday's Law of induction states that the induced emf in a circuit is equal to the negative rate of change of the magnetic flux through it. This is called flux-rule.

$$\overset{0}{\mathcal{E}} = -\frac{d\Phi}{dt}$$

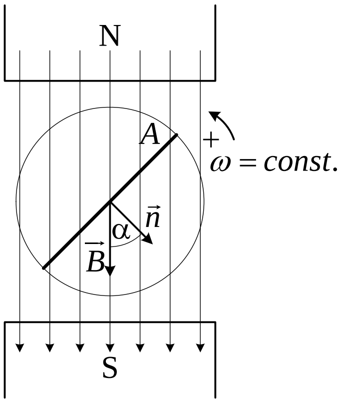
This is the alternative form of the equation for the emf in a moving conductor. It is often easies to apply the flux-rule instead of Neumann's Law to obtain the induce emf. The negative sign is due to Lenz's Law: The direction of the induced current is such that its effect would oppose the change in magnetic flux, which gave rise to the current.

Remark: If we have a coil of *N* turns and the flux varies the same rate through each turn, the induced emf's are in series and must be added:

$$\overset{0}{\mathcal{E}} = -N\frac{d\Phi}{dt}\,.$$

4.1.2 The emf induced in a rotating coil, AC generator

Consider a conducting coil of *N* turns each of area *A* being made to rotate with angular speed ω in a uniform magnetic field. *B*.



At $t = 0$ be $\vec{n} \uparrow\uparrow \vec{B}$, where $\vec{n}$ is the normal of the coil. As $\omega = \text{const.}$, the angle $\alpha = \omega t$. The flux through each turn is $\Phi = BA\cos\alpha = BA\cos\omega t$.
Apply the Faraday's Law. The induced emf in the coil is:

$$\overset{0}{\mathcal{E}} = -N\frac{d\Phi}{dt} = -NBA\omega\left(-\sin\omega t\right).$$

$$\overset{0}{\mathcal{E}} = NBA\omega\sin\omega t\ .$$

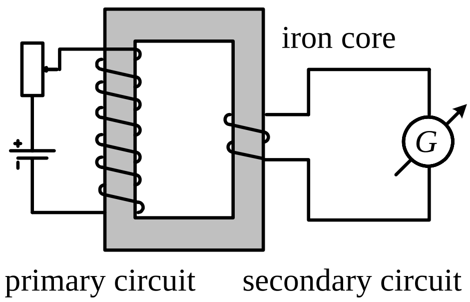
Introduce $\mathcal{E}_o = NBA\omega$, it is called maximum value of the emf,

$$\overset{0}{\mathcal{E}} = \mathcal{E}_o \sin\omega t$$

The output from a simple generator is a sinusoidally varying emf.

4.1.3 Induction due to the change of flux-linkages

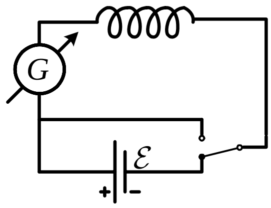
a) mutual induction



As the current in the primary circuit is varied with the rheostat, the magnetic field due to the current is varied also. A part of the magnetic flux through the secondary circuit is also varied. It is found experimentally that an emf and induced current appears in the secondary circuit. The secondary circuit is not moving in a magnetic field so no "motional" emf is induced in it. In such a situation no one portion of circuit can be considered the source of emf, the entire circuit constitutes the source.
The setting up of an induced current signifies that the changes in the magnetic field produce extraneous forces in the loop. The electric field due to the extraneous forces causes the current carriers in a conductor to start moving and an induced current is set up. In case of mutual

induction one circuit acted as a source of magnetic field and emf was induced in a separate independent circuit linking some of the flux.

b) self induction



Due to experiences if we disconnect the source from the coil and make short-circuit, the ammeter shows a decreasing current. Whenever a current is present in any circuit, this current sets up magnetic field that links with the same circuit and varies when the current varies. So there is an induced emf in it resulting from the variation in its own magnetic field. Such an emf is called self induced emf.

In the two previous cases, we must conclude that the induced current in the loop is caused by an induced electric field which is associated with the changing magnetic field. This field is called non-electrostatic field and denote it by $\vec{E}_n$. So the induced emf in this case is the line integral of $\vec{E}_n$ around the loop, and according to experiences the Faraday's Law of induction in case of changing flux:

$$\overset{o}{U} = -\frac{d\Phi}{dt}$$

Faraday's Law, integrated form:

$$\oint_g \overrightarrow{E_n} \cdot d\vec{s} = -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}$$

In a varying magnetic field an emf is induced in any closed circuit and is equal to the negative of the time rate of change of the magnetic flux through the circuit.
If an electrostatic field, produced by electric charges is also present in the same region, it is always conservative and so its line integral around any closed path is always zero. Hence in the next equation $\vec{E}$ is the total electric field inducting both electrostatic and non electrostatic contributions:

$$\oint_g \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}$$

To obtain the differential form of Faraday's Law of induction, let's transform the left hand side in accordance with Stokes's theorem:

$$\oint_g \vec{E} \cdot d\vec{s} = \int_A \left(\nabla \times \vec{E}\right) \cdot d\vec{A}$$

Since the loop and the surface are stationary the operation of time differentiation and integration over the surface can have their places exchanged; or:

$$-\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

The Faraday's Law:

$$\int_A \left(\nabla \times \vec{E}\right) \cdot d\vec{A} = -\int_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

$$\int_A \left(\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{A} = 0 ,$$

Due to the arbitrary chosen surface the integrand must be zero:

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 ,$$

The differential form of Faraday's Law of induction:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The negative time rate of change of magnetic field at a point equals to the curl of the field $\vec{E}$. The curl of the vector $\vec{E}$ is not zero, consequently the induced electric field is non conservative field. It is important to remark that the time-varying magnetic field causes electric field to appear in space regardless of whether or not there is a wire loop in this space. The presence of a loop only makes it possible to detect the existence of an electric field at the corresponding points of space as a result of a current being induced in the loop.

Electric field is set up not only by charges but time-varying magnetic field as well. The field set up by the charges have sources so the field lines begins and terminate at charges, and if they are at rest or in stationary motion this field is conservative. The field set up by time-varying magnetic field have no sources, the field lines are closed curves, and non conservative.

4.1.4 Self Inductance of a Long Solenoid

The magnetic field and induction inside a long solenoid is:

$$H = \frac{NI}{l}, \; B = \mu \frac{NI}{l}$$

Denote the flux of one turn by Φ_m. The flux through each turn:

$$\Phi_m = \int_A \vec{B} \cdot d\vec{A} = \mu \frac{NI}{l} A$$

The flux of the coil is:

$$\Phi = N\Phi_m = \mu \frac{N^2 A}{l} I$$

The flux of a long solenoid is proportional to the current. The proportionality factor is called self inductance, denoted by *L*:

$$\Phi = LI ,$$

$$L = \frac{\mu N^2 A}{l}$$

L is numerically equal to the flux linkage of a circuit when unit current flows through it. The coefficient *L* depends only on the shape of the conductor, so the geometry and medium inside it.

Unit of *L* is:

$$[\mathrm{L}] = 1\frac{\mathrm{Vs}}{\mathrm{A}} = 1\mathrm{henry} = 1\mathrm{H}$$

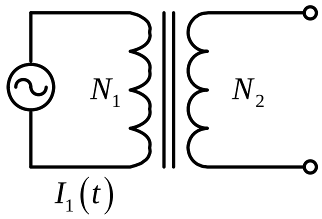
If the current varying with time then $\Phi(t) = LI(t)$ and so the self induced emf is:

$$U_i = -\frac{d\Phi}{dt} = -L\frac{dI}{dt} = -L\dot{I}$$

In case of a solenoid generally the number of turns is a great number and as L is proportional to N^2 the self inductance is so great that we can consider it as the self inductance of the whole circuit.

4.1.5 Mutual inductance of two closely wound coils

Consider the next figure:



The current I_1 in coil 1 sets up magnetic field as:

$$B_1 = \mu\frac{N_1 I_1(t)}{l}$$

The flux through one turn:

$$\Phi_1 = \mu\frac{N_1 A}{l} I_1(t)$$

In case of closely wound coils this is also the flux of one turn through the second coil, so the whole flux of coil 2 is:

$$\Phi_{12} = N_2\Phi_1 = \mu\frac{N_1 N_2 A}{l} I_1(t) .$$

The total flux through the second coil is proportional to the current of the first coil. The proportionality factor is called the mutual inductance of the two coils and denoted by L_{12} or M:

$$\Phi_{12} = L_{12} I_1$$

And the mutual inductance:

$$L_{12} = M = \mu\frac{N_1 N_2 A}{l}$$

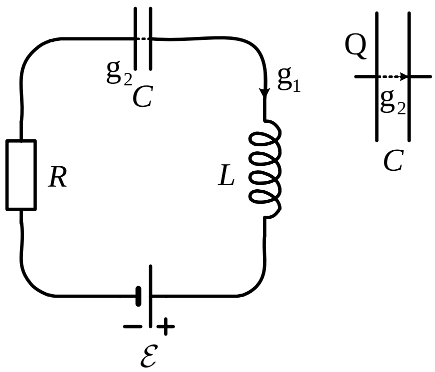
The induced emf in the coil 2:

$$U_{12} = -\frac{d\Phi_{12}}{dt} = -L_{12}\frac{dI_1}{dt} = -M\frac{dI_1}{dt}$$

The idea of mutual inductance is used in the transformer.

4.2 The generalization of the loop theorem for a single loop

Consider the next single loop. Denote the resistance of the whole circuit by R, the capacitance by C, the inductance of the coil by L, (at the same time this is the inductance of the whole circuit), and the applied emf by $\mathcal{E}$.



Set up a closed curve *g* along the circuit shown in the figure, and apply the Faraday's Law of induction:

$$\overset{0}{U} = -\frac{d\Phi}{dt}$$

$$\oint_g \vec{E}\cdot d\vec{s} = -\frac{d\Phi}{dt}$$

$$\int_{g_1} \vec{E}\cdot d\vec{s} + \int_{g_2} \vec{E}\cdot d\vec{s} = -L\frac{dI}{dt}$$

The next term is the potential difference across the capacitor:

$$\int_{g_2} \vec{E}\cdot d\vec{s} = \int E\,ds = \frac{Q}{C}$$

Due to the differential form of Ohm's Law:

$$\rho\vec{j} = \vec{E} + \vec{E}^* \Rightarrow \vec{E} = \rho\vec{j} - \vec{E}^*,$$

$$\int_{g_1} \rho\vec{j}\cdot d\vec{s} - \int_{g_1} \vec{E}^*\cdot d\vec{s} + \frac{Q}{C} = -L\frac{dI}{dt}$$

The next term is just the emf of the circuit by definition:

$$\int_{g_1} \vec{E}^*\cdot d\vec{s} = \mathcal{E}$$

In case of thin wires: $\vec{j} \uparrow\uparrow d\vec{s}$ and $j = \frac{I}{A}$:

$$\int_{g_1} \rho\vec{j}\cdot d\vec{s} = I\int_{g_1} \rho\frac{ds}{A} = IR,$$

R is the resistance of the whole circuit.

$$IR - \mathcal{E} + \frac{Q}{C} = -L\frac{dI}{dt}$$

The generalization of the loop theorem:

$$L\dot{I} + RI + \frac{Q}{C} = \mathcal{E}$$

The charge of the capacitor is varying due to the current. The transferred charge in time *dt* is:

$dQ = Idt$, that is $I = \frac{dQ}{dt} = \dot{Q}$, and $\dot{I} = \ddot{Q}$

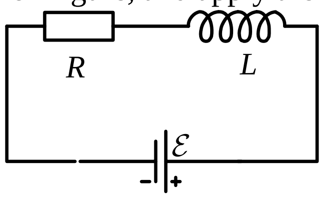
The differential equation for the charge:

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \mathcal{E}$$

This is a non-homogeneous linear differential equation of the second order with constant coefficients for the *Q(t)* function.

4.2.1 Energy in an inductor, magnetic energy density

Consider a serial RL circuit shown on figure, and apply the generalized loop equation:



$$L\frac{dI}{dt} + RI = \mathcal{E}$$

Multiplying this equation by *I*, we have:

$$LI\frac{dI}{dt} + RI^2 = \mathcal{E}I$$

To maintain a current in a circuit, energy must be supplied. The energy required per unit time, in other words the power is $\mathcal{E}I$, the power of the seat of emf. The term RI^2 is the energy spent in moving the electrons through the crystal lattice of the conductor and is transferred to the ions that make up the lattice, so the power dissipated in the resistor. The last term $LI\frac{dI}{dt}$ is then interpreted as the energy required per unit time to build up the magnetic field in space. Therefore the rate of increase of the magnetic energy is the first term , and can be written as:

$$LI\frac{dI}{dt} = \frac{d}{dt}\left(\frac{1}{2}LI^2\right)$$

That is the magnetic energy:

$$W_m = \frac{1}{2}LI^2 + \text{constant}$$

Due to agreement, if $I = 0$ the energy of the magnetic field is zero:

$$W_m = 0 \quad \Rightarrow \text{constant=0}$$

The energy due to magnetic field:

$$W_m = \frac{1}{2}LI^2$$

Consider now a long solenoid:

$$\Phi = LI \text{, and } H = \frac{NI}{l} \Rightarrow I = \frac{Hl}{N} \text{ finally } \Phi = BAN$$

$$W_m = \frac{1}{2}LII = \frac{1}{2}\Phi I = \frac{1}{2}BAN\frac{Hl}{N} = \frac{1}{2}BH\,Al$$

The volume of the coil and the volume of the space of the magnetic field is: $Al = V$

$$W_m = \frac{1}{2}\vec{B}\cdot\vec{H}V$$

The magnetic energy density is defined as:

$$w_m = \frac{W_m}{V} = \frac{1}{2}\vec{B}\cdot\vec{H} :$$

Although this expression has been justified for the magnetic energy density in a very special case, a more detailed analysis would indicate that the result is completely general.

If $\vec{B}$ is the magnetic induction and $\vec{H}$ is the magnetic field strength at a point then the energy stored in an elementary volume dV is:

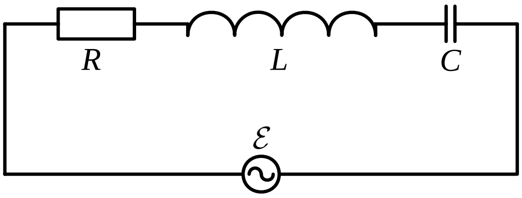
$$dW_m = \frac{1}{2}\vec{B}\cdot\vec{H}\,dV\,,$$

and in a finite volume:

$$W_m = \int_V w_m dV = \int_V \frac{1}{2}\vec{B}\cdot\vec{H}\,dV$$

4.3 Forced electrical oscillations in a serial RLC circuit

Consider the next electrical circuit:



The generalized loop equation is:

$$L\dot{I} + RI + \frac{Q}{C} = \mathcal{E}$$

The generalized loop equation for the charge:

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \mathcal{E}$$

Let's suppose that an alternating emf is applied as:

$$\mathcal{E} = \mathcal{E}_0 \cos\omega t$$

$\mathcal{E}_0$ is the maximum value of the applied emf, ω is the cyclic frequency.

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \mathcal{E}_0 \cos\omega t$$

The loop equation coincides with the differential equation of forced mechanical oscillations.

$$m\ddot{x} + \kappa\dot{x} + Dx = F_0 \cos\omega t$$

The correspondence between the forced oscillation and the electrical circuit is

$$x \to Q$$
$$m \to L$$
$$\kappa \to R$$
$$D \to \frac{1}{C}$$
$$\omega_0 = \sqrt{\frac{D}{m}} \to \omega_0 = \frac{1}{\sqrt{LC}}$$
$$\alpha = \frac{\kappa}{2m} \to \alpha = \frac{R}{2L}$$
$$F_0 \to \mathcal{E}_0$$

We know that the general solution of a non homogeneous equation equals the sum of the general solution of the corresponding homogeneous equation and a partial solution of the non homogeneous equation:

$$Q_{\text{inh.gen.}} = Q_{\text{hom.gen.}} + Q_{\text{inh.part.}}$$

As the general solution of the homogeneous equation contains an exponential decreasing term:

$$Q_{\text{hom.gen.}} \sim e^{-\frac{R}{2L}t},$$

after sufficient time elapses, becomes very small and it may be disregarded. This is called transient process. So the stationary solution of the non-homogeneous equation is a partial solution of this non homogeneous equation.

We look for this solution and denote it by *Q*. Set up a helping equation, multiply it by the complex unit *i* and add it to the original loop equation:

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \mathcal{E}_0 \cos\omega t$$

$$L\ddot{Q}' i + R\dot{Q}' i + \frac{Q'}{C} i = \mathcal{E}_0\, i \sin\omega t\,,$$

$$L\left(\ddot{Q} + i\ddot{Q}'\right) + R\left(\dot{Q} + i\dot{Q}'\right) + \frac{1}{C}\left(Q + iQ'\right) = \mathcal{E}_0\left(\cos\omega t + i\sin\omega t\right)$$

Use the Euler-relation:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Introduce the complex charge:

$$\overline{Q} = Q + iQ',$$

$$L\ddot{\overline{Q}} + R\dot{\overline{Q}} + \frac{1}{C}\overline{Q} = \mathcal{E}_0 e^{i\omega t}$$

Use the complex emf:

$$\overline{\mathcal{E}} = \mathcal{E}_0 e^{i\omega t}$$

$$L\ddot{\overline{Q}} + R\dot{\overline{Q}} + \frac{1}{C}\overline{Q} = \overline{\mathcal{E}}$$

We shall try to find the partial solution in the next form:

$$\overline{Q} = \overline{Q}_0 e^{i\omega t}$$

$\overline{Q}_0$ is the complex amplitude of the charge.
Take the first and second derivatives of the complex charge:

$$\dot{\overline{Q}} = i\omega\overline{Q}_0 e^{i\omega t} = i\omega\overline{Q} = \overline{I}\,,$$

The first derivative of the complex charge is the complex current $\overline{I}$.

$$\ddot{\overline{Q}} = \left(i\omega\right)^2 \overline{Q}_0 e^{i\omega t} = -\omega^2\overline{Q}$$

Inserting into the differential equation:

$$-L\omega^2\overline{Q} + i\omega R\overline{Q} + \frac{1}{C}\overline{Q} = \overline{\mathcal{E}}$$

Instead of the differential equation we have a simple complex algebraic equation for $\overline{Q}$.

$$i\omega\overline{Q}\left(R + i\omega L + \frac{1}{i\omega C}\right) = \overline{\mathcal{E}}$$

Introduce the complex impedance as:

$$\overline{Z} = R + i\omega L + \frac{1}{i\omega C}\text{, or } \overline{Z} = R + i\left(L\omega - \frac{1}{\omega C}\right)$$

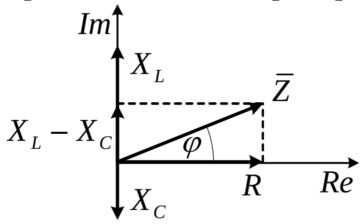
The equation we have obtained is called the complex Ohm's Law:

$$\overline{I}\,\overline{Z} = \overline{\mathcal{E}},$$

$$\overline{I} = \frac{\overline{\mathcal{E}}}{\overline{Z}}.$$

Introduce the inductive reactance: $X_L = L\omega$, and the capacitive reactance $X_C = \frac{1}{\omega C}$

The complex impedance may be represented on a complex plane:



$$\overline{Z} = R + i\left(L\omega - \frac{1}{\omega C}\right) = R + i\left(X_L - X_C\right)$$

The absolute value of the complex impedance is called the real impedance or simply impedance.

$$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2} = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

The so called trigonometric form of the complex impedance is:

$$\overline{Z} = Ze^{i\varphi}$$

φ is the argument:

$$tg\varphi = \frac{L\omega - \frac{1}{\omega C}}{R}\text{, or } \cos\varphi = \frac{R}{Z}$$

The complex current:

$$\overline{I} = \frac{\overline{\mathcal{E}}}{\overline{Z}} = \frac{\mathcal{E}_0 e^{i\omega t}}{Ze^{i\varphi}} = \frac{\mathcal{E}_0}{Z}e^{i(\omega t - \varphi)}$$

Initiate the maximum value of the real current:

$$I_0 = \frac{\mathcal{E}_0}{Z}$$

$$\overline{I} = I_0 e^{i(\omega t - \varphi)}$$

Apply the Euler-relation for the complex current:

$$\overline{I} = I_0 e^{i(\omega t - \varphi)} = I_0 \cos\left(\omega t - \varphi\right) + i\, I_0 \sin\left(\omega t - \varphi\right)$$

The real part of the current and so the solution is:

$$I\left(t\right) = I_0 \cos\left(\omega t - \varphi\right),$$

where

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}}$$

If $\varphi > 0$ then the current lags behind the applied emf with a phase lag of φ.
If $\varphi < 0$ then the current leads the emf with a phase of lead of φ

4.3.1 Instantaneous voltages across the different circuit element

1. Resistor:

$$\overline{U}_R = \overline{I}\,R = I_0 R\,\mathrm{e}^{i(\omega t-\varphi)} = U_{R0} e^{i(\omega t-\varphi)}$$

$$\overline{U}_R = U_{R0}\, e^{i(\omega t-\varphi)}, \text{ where } U_{R0} = I_0 R$$

The real part of the voltage across the resistor:

$$U_R(t) = U_{R0}\cos(\omega t - \varphi)$$

We have shown that the potential difference between the terminals of a resistor is in phase with the current.

2. Capacitor:

$$\overline{U}_C = \frac{\overline{Q}}{C} = \frac{\overline{I}}{i\omega C} = e^{-i\frac{\pi}{2}} \frac{1}{\omega C} I_0\, e^{i(\omega t-\varphi)} = X_C I_0 e^{i\left(\omega t-\varphi-\frac{\pi}{2}\right)} = U_{C0} e^{i\left(\omega t-\varphi-\frac{\pi}{2}\right)}$$

The real part of the voltage across the capacitor:

$$U_C(t) = U_{C0}\cos\left(\omega t - \varphi - \frac{\pi}{2}\right), \text{ where } X_C = \frac{1}{\omega C}, \text{ and } U_{C0} = I_0 X_C$$

We used the formula, that $\frac{1}{i} = e^{-i\frac{\pi}{2}}$.

The voltage across a capacitor lags the current by $\frac{\pi}{2}$.

3. Inductor

$$\overline{U}_L = L\dot{\overline{I}} = L I_0 e^{i(\omega t-\varphi)} i\omega = L\omega I_0 e^{i(\omega t-\varphi)} e^{i\frac{\pi}{2}} = X_L I_0 e^{i\left(\omega t-\varphi+\frac{\pi}{2}\right)} = U_{L0} e^{i\left(\omega t-\varphi+\frac{\pi}{2}\right)}$$

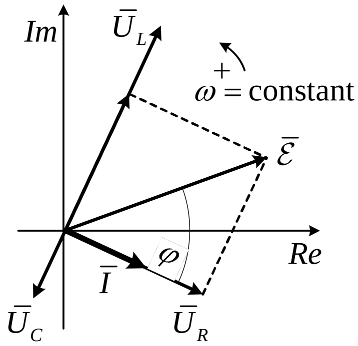
The real part of the voltage across the inductor:

$$U_L(t) = U_{L0}\cos\left(\omega t - \varphi + \frac{\pi}{2}\right), \text{ where } X_L = L\omega, \text{ and } U_{L0} = I_0 X_L$$

We used the formula, that $i = e^{i\frac{\pi}{2}}$.

The voltage across the inductor leads the current by $\frac{\pi}{2}$.

At the analysis of alternating-current circuits we often apply the rotating vector diagrams. In such diagrams the instantaneous value of a quantity that varies sinusoidally with time is represented by the projection onto the horizontal (real) axis. The length of the complex vector corresponds to the maximum value of the quantity and rotates counter-clockwise with constant angular speed ω. These rotating vectors are called phasors and the diagram containing them phasor diagrams. It is called complex phase diagram also.

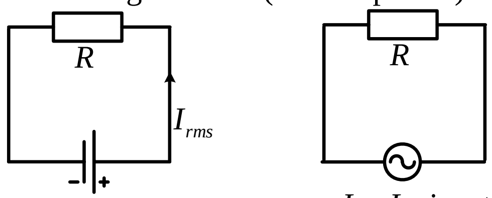


$$\overline{U}_L + \overline{U}_R + \overline{U}_C = \overline{\mathcal{E}}$$

The complex description cannot be used for the transient process and when the applied emf is not sinusoidal function.

4.3.2 The effective value of varying current

To characterize an alternating current we use the concepts of effective or root-mean-square value. The effective value of an alternating current is that steady current which would do the same work on the same resistor during a time *T* (time of period) as the alternating current.



$I = I_0 \sin \omega t$

Determine the effective value of sinusoidally varying current:

$$W = I_{rms}^2 RT\ , \quad W = \int_0^T I^2(t) R dt$$

The two works are equal to each other that is:

$$I_{rms}^2 RT = \int_0^T I^2(t) R\, dt$$

$$I_{rms}^2 = \frac{1}{T}\int_0^T I^2(t)\, dt$$

This is the root-mean-square value of an alternating current:

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I^2(t)\, dt}$$

This is the square root of the average value of the square of the current or voltage. If $I = I_0 \sin \omega t$, then:

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I_0^2 \sin^2 \omega t\, dt} = \sqrt{\frac{I_0^2}{2T}\int_0^T (1-\cos 2\omega t)\, dt} = \sqrt{\frac{I_0^2}{2T}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T} = \sqrt{\frac{I_0^2}{2}}$$

$$I_{rms} = \frac{I_0}{\sqrt{2}}\text{, similar way } \mathcal{E}_{rms} = \frac{\mathcal{E}_0}{\sqrt{2}}$$

4.3.3 Power in an RLC series circuit

When a source with an instantaneous emf $\mathcal{E}(t)$ supplies an instantaneous current $I(t)$, to a circuit, the instantaneous power it supplies is:

$$P(t)=\mathcal{E}(t)I(t)=\mathcal{E}_0\cos\omega t\ I_0\cos(\omega t-\varphi)=\frac{\mathcal{E}_0 I_0}{2}2\cos\omega t\cos(\omega t-\varphi)$$

Apply the next trigonometrical expressions, and add the equations:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$$

If $\alpha=\omega t$, and $\beta=\omega t-\varphi$, then $\alpha+\beta=2\omega t-\varphi$, and $\alpha-\beta=\varphi$.
The instantaneous power is:

$$P(t)=\frac{\mathcal{E}_0 I_0}{2}\left[\cos(2\omega t-\varphi)+\cos\varphi\right]$$

The time average of this instantaneous power is:

$$\overline{P}=\frac{\mathcal{E}_0 I_0}{2}\cos\varphi=\frac{\mathcal{E}_0}{\sqrt{2}}\frac{I_0}{\sqrt{2}}\cos\varphi=\mathcal{E}_{rms}I_{rms}\cos\varphi$$

The time average of the power is denoted by over bar, and $\cos\varphi$ is called power factor.