

1 Static Electric Field

1.1 Electric Interaction

1.1.1 Basic electric experiments:

In ancient times Thales discovered that amber rubbed with fur or a piece of glass rubbed with a silk cloth would attract light objects such as small bits of paper. As a result of rubbing, materials may acquire a new property that is called electricity. These materials are electrified or charged with electricity.

Experiments show:

- two electrified piece of glass repel each other
- two electrified piece of amber repel each other
- an electrified piece of glass and an electrified piece of amber attract each other

Due to experiments, there are two kinds of electrified states:

- glasslike – called positive
- amberlike – called negative

Two bodies with the same kind of electrification (either positive or negative) repel each other, but if they have different kinds of electrification (one positive and the other negative), they attract each other.

In neutral state the material contains equal amounts of positive and negative electricity.

The experiments show that the electrified state can be transferred from one body to another, by taking them into contact. As there is electric interaction between the charged or electrified objects when they are not in contact, we can say that this interaction is carried out through an electric field.

To characterize the state of electrification of a body we define the electric charge (simply charge) represented by a symbol Q .

The electric charge characterizes the participation of the electrified objects in the electric interaction.

Place a unit charge particle into electric field and measure the force acting on it.

Place another charge into the same field and measure the force on it again. We say that its charge is k -times as the unit charge if it experiences a force that is k -times as the force acting on the unit charge.

Since there are two kinds of electrification, there are also two kinds of electric charge: positive and negative.

The fundamental building blocks of ordinary matter are the negatively charged electrons, the positively charged protons and the electrically neutral neutrons. The negative charge of the electron has the same magnitude as the positive charge of the proton. These three particles are arranged in the same general way in all atoms. The protons and neutrons form the nucleus which has a positive charge. Its measurement is about 10^{-14} m. Outside of the nucleus; at distances of the order of 10^{-10} m are the electrons. In a neutral atom the number of electrons equals the number of protons in the nucleus. The net electric charge of such an atom is zero. If

one or more electrons are removed, the remaining structure is called positive ion. A negative ion is an atom that has gained one or more electrons.

When a piece of bulk matter is electrically charged, this is almost always accomplished by adding or removing electrons. When we speak of the charge of a body, we always mean its net charge, which is always a very small fraction of the total positive or negative charge in the body.

The body is called neutral if the net charge is zero.

In some materials primarily in metals some of the atomic electrons are not bound to any particular atom but are free to move macroscopic distances within the material. These materials are called conductors.

Conductors permit the passage of charge through them. In glass, wood, paper, plastics... there are no free electrons, the electron can move only microscopic distances within a molecule. These materials are called insulators or dielectric.

There are some intermediate materials such as carbon, germanium, and silicon. They are called semiconductors.

The Earth is a good conductor and it acts as a giant reservoir to charge or discharge an object. If a body is connected to the ground with a metal (conductor) wire, it is called grounded.

Franklin was who discovered that in any process in which a positive charge is produced, an equal amount of negative charge must be produced at the same time.

For example: when plastic is rubbed with fur, opposite charges appear on the two materials. The electric charge is not created but is transferred from one body to another. The plastic acquires extra electrons, so it is negative, these electrons are taken from the fur, which this acquires a net positive charge.

The electric charge is conserved. The algebraic net charge of an isolated system of charges does not change. Charge can be transferred from one body to another but it cannot be created or destroyed.

1.1.2 The Electric Field

Many phenomena are most conveniently described in terms of field concept. Any region in space in which an electric charge experiences a force is called electric field. Consider a charged body at rest in our reference system. This charged body sets up an electric field in the space.



Put a point charge q into the point P and measure the force acting on it. Due to experiences the force acting on the point charge q is proportional to its charge. The ratio of the two physical quantity is independent of the point charge and depends only on the field:

$$\vec{E}(P) = \frac{\vec{F}(P)}{q}$$

By definition the intensity of the electric field numerically equals the force acting on a unit point charge at a given point of the field. The direction of the vector \vec{E} coincides with that of the force acting on a positive charge.

Unit of the electric field:

$$[E] = 1 \frac{N}{C}$$

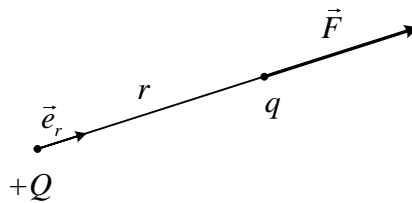
When several charges are present the resultant electric field at any point is the vector sum of the electric fields produced at the point by each charge. This statement expresses the principle of superposition or independent action.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_n = \sum_{i=1}^n \vec{E}_i$$

1.1.3 Coulomb's law, Force between Point Charges

In 1785 Coulomb studied the field due to a point charge. A point charge is defined as a charged body whose dimensions may be disregarded in comparison with the distances from this body to the other bodies carrying electric charge.

The Coulomb's Law states that a point charge Q at rest sets up a central and conservative electric field around itself. The centre of the field is the point charge. In this electric field the force acting on a point charge q is proportional to the magnitude of the two point charges and inversely proportional to the square of the distance between them. The direction of the force coincides with the straight line connecting the charges.



$$\vec{F} = k \frac{qQ}{r^2} \vec{e}_r$$

\vec{e}_r is the unit vector directed from Q to q , and $k > 0$ constant.

The value of the proportionality constant depends only on the choice of the charge unit. If the unit of charge is:

$$1 \text{ coulomb} = 1C$$

$$k = 9 \cdot 10^9 \frac{Nm^2}{C^2}$$

1 *coulomb* is the magnitude of the point charge if the force between two identical point charges equals $9 \cdot 10^9 N$ when they are 1 m apart.

The base unit of the electricity is the unit of the electric current. It means that the unit of charge is a derived unit. A *coulomb* is defined as the amount of charge that flows through any cross section of a wire in 1 *second* if there is a steady current of 1 *Ampere* in the wire.

$$1C = 1As$$

The proportionality factor:

$$k = \frac{1}{4\pi \epsilon_0},$$

ϵ_0 is called the permittivity constant of vacuum.

$$\epsilon_0 = 8,854 \cdot 10^{-12} \frac{C^2}{Nm^2}$$

1.1.4 Electrostatic field

The field due to a point charge at rest is conservative and the principle of superposition is valid for electric fields. That is the electric field due to any charge distribution, at rest in our inertial system, is conservative field. This field is called electrostatic field.

The work done by the field on a point charge as it moves from point 1 to point 2 is:

$$W_{1,2} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 q\vec{E} \cdot d\vec{r} = q \int_1^2 \vec{E} \cdot d\vec{r}$$

The voltage between two points:

$$U_{1,2} = \frac{W_{1,2}}{q} = \int_1^2 \vec{E} \cdot d\vec{r}$$

The voltage or potential difference depends only on the field and the two points but independent of the charge. The potential difference or voltage between two points in an electric field is the work done by the field on a unit positive charge as it moves between the points.

The unit of potential difference:

$$[U] = \frac{1J}{1C} = 1 \text{ volt} = 1V$$

As the electrostatic field is conservative the elementary work done by the field is the negative total differential of the potential energy:

$$\delta W = \vec{F} \cdot d\vec{r} = -dV$$

$$q\vec{E} \cdot d\vec{r} = -dV$$

$$\vec{E} \cdot d\vec{r} = -d\left(\frac{V}{q}\right)$$

$$\vec{E} \cdot d\vec{r} = -dU$$

$$U = \frac{V}{q}$$

This physical quantity U is called potential. The potential is equal numerically to the potential energy of a unit positive test charge. The potential is accurate to an arbitrary additive constant. According to the agreement the potential is chosen to be zero at the infinity.

$$U_{1,2} = \int_1^2 \vec{E} \cdot d\vec{r} = -\int_1^2 dU = U_1 - U_2$$

In electrostatic field the potential difference depends only on the initial and final points and equals to the negative difference of the potentials, (potential of initial point – potential of final point).

The potential of an arbitrary point P :

$$-dU = \vec{E} \cdot d\vec{r}$$

$$-\int_P^\infty dU = \int_P^\infty \vec{E} \cdot d\vec{r}$$

$$U(P) - U(\infty) = \int_P^{\infty} \vec{E} \cdot d\vec{r}$$

if $U(\infty) = 0$

$$U(P) = \int_P^{\infty} \vec{E} \cdot d\vec{r}$$

The potential numerically equals the work done by the field on a unit positive charge when it is moved from the point P to infinity.

In electrostatic field the work done on any closed path is zero that is the potential difference is also zero. This is the integrated form of the first law of electrostatics.

$$\oint_g \vec{E} \cdot d\vec{r} = 0$$

The circle on the integral sign indicates that integration is performed along a closed contour. Apply the Stokes's theorem. The line integral of a vector around an arbitrary contour g equals the flux of the *curl* of the vector through the arbitrary surface A surrounded by the given contour.

$$\oint_g \vec{E} \cdot d\vec{r} = \int_A \text{curl} \vec{E} \cdot d\vec{A} = \int_A (\nabla \times \vec{E}) \cdot d\vec{A} = 0$$

This condition must be observed for any surface resting on the contour g , this is possible only if:

$$\text{curl} \vec{E} = 0$$

or:

$$\nabla \times \vec{E} = 0$$

This is the differential form of the first law of electrostatics.

As the force related to the potential energy in conservative field:

$$\vec{F} = -\text{grad} V \quad /: q$$

$$\vec{E} = -\text{grad} U$$

The intensity of the electric field is the negative gradient of the potential.

The locus of points have the same potential is called equipotential surface.

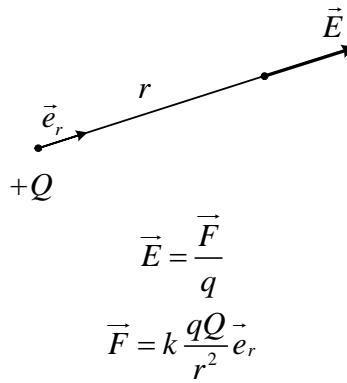
$$-dU = \vec{E} \cdot d\vec{r}$$

if $U = \text{constant} \quad dU = 0$

$$\vec{E} \cdot d\vec{r} = 0$$

This is the differential equation of the equipotential surface. The elementary arc vector $d\vec{r}$ is along the surface. As the dot product of two vectors is zero, the vectors are perpendicular. The electric field intensity vector is always perpendicular to the equipotential surface.

1.1.4 Electric field and potential due to a point charge



The field:

$$\vec{E} = k \frac{Q}{r^2} \vec{e}_r$$

$$-dU = \vec{E} \cdot d\vec{r} = k \frac{Q}{r^2} \vec{e}_r \cdot d\vec{r}$$

$$-\int_r^\infty dU = \int_r^\infty k \frac{Q}{r_*^2} dr_*$$

$$U(r) - U(\infty) = kQ \left[-\frac{1}{r_*} \right]_r^\infty$$

The potential:

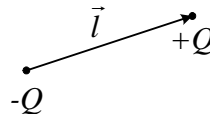
$$U(r) = k \frac{Q}{r}$$

1.1.5 Electric dipole

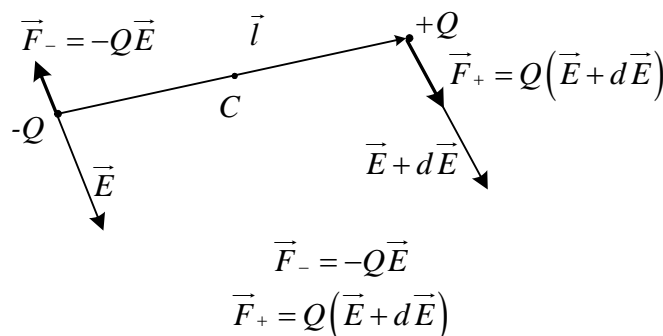
The electric dipole is positive and a negative charge of equal magnitude Q placed a distance l apart. The electric dipole moment vector is defined as:

$$\vec{p} = Q\vec{l}$$

The direction of \vec{l} is from the negative to the positive charge.



Consider now a very small dipole (point like dipole), suppose that $|\vec{l}| \approx 10^{-9} m$. This is the size of a molecule. Determine the torque acting on this dipole when it is placed in an external electric field.



The torque (the momentum of force) about point C

$$\vec{M}_c = -\frac{\vec{l}}{2} \times (-Q\vec{E}) + \frac{\vec{l}}{2} \times Q(\vec{E} + d\vec{E}) = Q\vec{l} \times \vec{E} + Q\frac{\vec{l}}{2} \times d\vec{E} = \vec{p} \times \vec{E}$$

due to the point like dipole $|\vec{l}|$ is very small and $|d\vec{E}|$ is also negligible.

$$\vec{M}_c = \vec{p} \times \vec{E}$$

The torque is zero if $\vec{p} \parallel \vec{E}$, that is if the dipole turned into the direction of the field. The torque is tending to align the electric dipole.

If $\vec{E} \uparrow \uparrow \vec{p}$ then $\vec{M}_c = 0$ the equilibrium is stable, if $\vec{E} \uparrow \downarrow \vec{p}$ then $\vec{M}_c = 0$, but the equilibrium is unstable.

1.2 Electric polarization, Electric field in dielectrics

Dielectrics or insulators are defined as substances not capable of conducting an electric current. Consider now a molecule of a dielectric; this is a system with a total charge of zero. Take the analogy of the centre of mass and introduce the centre of charge:

$$\vec{r}_c = \frac{\sum Q_i \vec{r}_i}{\sum Q_i}$$

In case of a molecule we can define the centre of charge for the positive charge and for the negative charge as well.

- In symmetrical molecule the centre of the positive and negative charges coincide in the absence of an external field. Such molecules have no intrinsic dipole moment and are called non-polar molecules: H₂, O₂, N₂
- In asymmetrical molecules the centre of positive and negative charges are displaced relative to each other. So these molecules have an intrinsic dipole moment, and are called polar molecules: CO, HN, HCl, H₂O ...

1.2.1 Induced polarization

Under the action of an external electric field the centre of charges in a non-polar molecule become displaced relative to one another, the positive ones in the direction of the field, the negative ones against the field. So the molecule acquires a dipole moment. In case of a polar molecule the dipole moment increases.

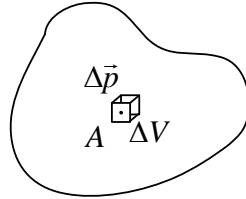
1.2.2 Polarization due to alignment

The action of an external field on a polar molecule consists mainly in tending to rotate the molecule so that its dipole moment is arranged in the direction of the field.

In the absence of an external electric field the dipole moments of the molecules of a dielectric either equal to zero (non-polar molecules) or are distributed in space by directions chaotically (polar molecules) so the total dipole moment of a dielectric equals zero.

A dielectric becomes polarized under the action of an external electric field. To characterize the polarization of a dielectric at a given point, introduce the polarization vector \vec{P} .

Suppose that $\Delta\vec{p}$ is the electric dipole momentum of the volume ΔV .



Then:

$$\vec{P} = \lim_{\Delta V \rightarrow 0} \frac{\Delta\vec{p}}{\Delta V}$$

The polarization of isotropic dielectrics is associated with the field strength at the same point by the next relation:

$$\vec{P} = \chi \varepsilon_0 \vec{E},$$

where χ is a quantity independent of \vec{E} and called electric susceptibility of a dielectric.

χ is a dimensionless quantity.

In case of vacuum and conductor:

$$\chi = 0$$

In dielectrics:

$$\chi > 0$$

Calculations of the electric fields is often simplified if we introduce an auxiliary quantity which is the linear combination of the electric field \vec{E} and the polarization vector \vec{P} . This linear combination is called electric displacement or electric induction vector.

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

If we use the first approaching, that \vec{E} is proportional to the polarization:

$$\vec{P} = \chi \varepsilon_0 \vec{E}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \chi \varepsilon_0 \vec{E}$$

$$\vec{D} = \varepsilon_0 (1 + \chi) \vec{E}$$

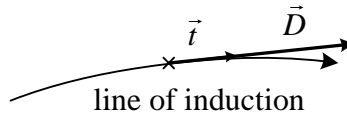
$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

$$\vec{D} = \varepsilon \vec{E}$$

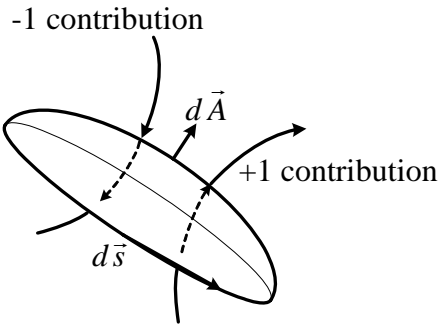
The dimensionless quantity $\varepsilon_r = 1 + \chi$ is called the relative permittivity of a medium, $\varepsilon = \varepsilon_0 \varepsilon_r$ is called absolute permittivity. According to this equation in homogeneous isotropic dielectric the vector \vec{D} is proportional to the vector \vec{E} .

1.2.3 Flux of the Electric Induction, Gauss's Law

To visualize the electric field patterns we use the lines of induction. The lines of induction are drawn so that a tangent to them at every point coincided with the direction of the vector \vec{D} . According to the agreement, the lines of induction are drawn so that the number of lines on unit perpendicular cross section area is equal to the magnitude of \vec{D} .

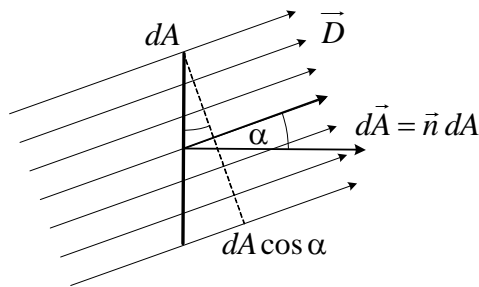


The electric flux refers always to a surface with given orientation, and equals to the total number of lines of induction passing through the surface.



The contribution is +1 if the direction of the surface vector $d\vec{A}$ is the same as the direction of the line of induction. The contribution is -1 in opposite case.

The elementary outward flux:



$$d\Psi = D dA \cos \alpha = \vec{D} \cdot d\vec{A}$$

$$(d\vec{A} = dA \vec{n})$$

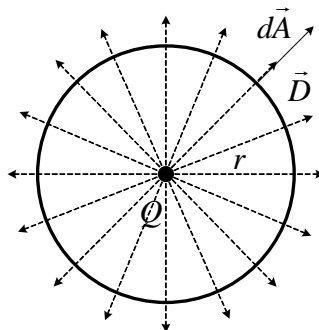
The total outward flux is the integral of \vec{D} over a surface:

$$\Psi = \int_A \vec{D} \cdot d\vec{A}$$

Units:

$$[D] = [P] = 1 \frac{C}{m^2} \text{ that is } [\Psi] = 1C$$

Consider a point charge in vacuum, and determine the flux through a sphere:



$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \vec{e}_r \\ \vec{D} &= \epsilon_0 \vec{E} \\ \vec{D} &= \frac{Q}{4\pi r^2} \vec{e}_r \\ \Psi^0 &= \oint_A \vec{D} \cdot d\vec{A} = \oint_A \frac{Q}{4\pi r^2} \vec{e}_r \cdot \vec{e}_r dA = \frac{Q}{4\pi r^2} 4\pi r^2 = Q \\ \oint_A \vec{D} \cdot d\vec{A} &= Q\end{aligned}$$

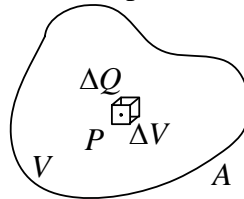
Easy to realize that this connection is valid for any closed surface if Q is enclosed. And it is valid at the presence of any substance. The above equation is called Gauss's Law or the Second Law of electrostatics.

The flux of the electric induction through a closed surface equals the algebraic sum of the charges enclosed by this surface. Denote by V the volume enclosed by the closed surface:

$$\oint_A \vec{D} \cdot d\vec{A} = Q_V$$

- positive charges are the sources of the field
- negative charges are the sinks of the field

Let's suppose that the charge is distributed in space continuously with a finite volume density.



The volume charge density ρ is defined as:

$$\rho(P) = \lim_{\Delta V \rightarrow 0} \frac{\Delta Q}{\Delta V}$$

The unit is:

$$[\rho] = 1 \frac{C}{m^3}$$

The enclosed charge:

$$\begin{aligned}Q_V &= \int_V \rho dV \\ \oint_A \vec{D} \cdot d\vec{A} &= \int_V \rho dV\end{aligned}$$

Replacing the closed surface integral with a volume one in accordance with the Gauss-Ostogradsky theorem:

$$\begin{aligned}\oint_A \vec{D} \cdot d\vec{A} &= \int_V \nabla \cdot \vec{D} dV \\ \int_V \nabla \cdot \vec{D} dV &= \int_V \rho dV \\ \int_V (\nabla \cdot \vec{D} - \rho) dV &= 0\end{aligned}$$

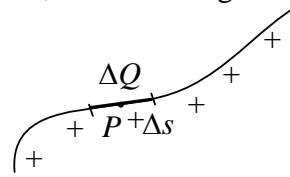
This relation must be true for any volume, so the integrand must be zero:

$$\nabla \cdot \vec{D} = \rho$$

This is the differential form of the Gauss's Law.

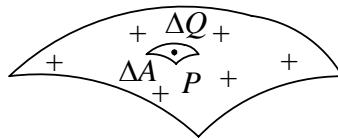
Let's introduce the line and the surface charge density.

If the charge is distributed along a line, the line charge density is defined as:



$$\lambda(P) = \lim_{\Delta s \rightarrow 0} \frac{\Delta Q}{\Delta s}$$

If the charge is distributed along a surface the surface charge density is defined as:



$$\sigma(P) = \lim_{\Delta A \rightarrow 0} \frac{\Delta Q}{\Delta A}$$

Units:

$$[\sigma] = 1 \frac{C}{m^2}$$

$$[\lambda] = 1 \frac{C}{m}$$

1.3 The two fundamental Laws of Electrostatics and the Poisson equation

Integral forms:

$$\oint_S \vec{E} \cdot d\vec{r} = 0$$

$$\oint_A \vec{D} \cdot d\vec{A} = Q_v$$

Differential or local forms:

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{D} = \rho$$

Consider now a homogeneous isotropic dielectric: $\vec{D} = \epsilon \vec{E}$ and ϵ is constant.

$$\nabla \times \vec{E} = 0 \rightarrow \vec{E} = -\nabla U$$

$$\nabla \cdot \vec{D} = \rho \rightarrow \nabla \cdot (\epsilon \vec{E}) = \rho$$

$$\epsilon \nabla \cdot \vec{E} = \rho$$

$$\epsilon \nabla \cdot (-\nabla U) = \rho$$

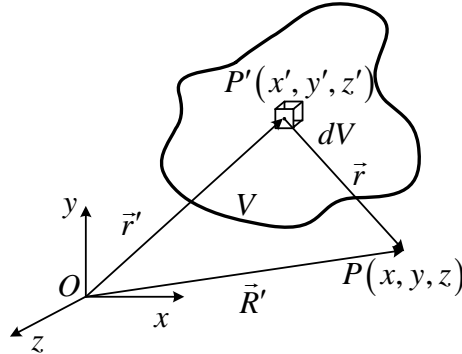
$$\nabla^2 U = -\frac{\rho}{\epsilon}$$

This equation is called Poisson equation.

Laplace operator:

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Poisson equation is a second order partial differential equation for the potential distribution $U(x, y, z)$, but we have to know the charge distribution in the space $\rho(x, y, z)$, the permittivity and the so called boundary conditions.



$$dQ = \rho dV$$

$$|\vec{r}| = |\vec{R} - \vec{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$dU = \frac{1}{4\pi\epsilon} \frac{dQ}{r} = \frac{1}{4\pi\epsilon} \frac{\rho}{r} dV$$

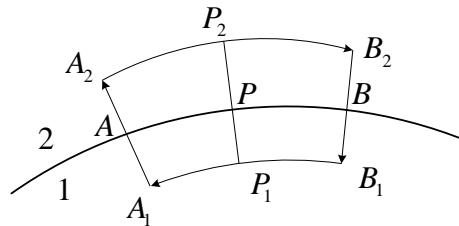
$$U(P) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho}{r} dV = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(x', y', z') dx' dy' dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

The integration must be taken for the volume of the charge distribution.

1.3.1 Boundary conditions at the interface between two different media

Let's consider the interface between two dielectrics with the permittivity ϵ_1 and ϵ_2 . Choose an arbitrary curve on the interface (\widehat{AB}) .

Set up a closed curve $A_1 A_2 B_1 B_2$ and apply the first law of electrostatics:



$$\oint_g \vec{E} \cdot d\vec{r} = 0$$

$$\int_{A_1 A_2} \vec{E} \cdot d\vec{r} + \int_{A_2 B_2} \vec{E} \cdot d\vec{r} + \int_{B_2 B_1} \vec{E} \cdot d\vec{r} + \int_{B_1 A_1} \vec{E} \cdot d\vec{r} = 0$$

Approach P_1 and P_2 points to point P and the $\widehat{A_1 B_1}$ and $\widehat{A_2 B_2}$ arcs to \widehat{AB} .

$$\int_{A_1 A_2} \vec{E} \cdot d\vec{r} \rightarrow 0, \text{ and } \int_{B_1 B_2} \vec{E} \cdot d\vec{r} \rightarrow 0$$

because the intervals of the integrations tend to zero.

$$\int_{\widehat{A_2 B_2}} \vec{E} \cdot d\vec{r} \rightarrow \int_{\widehat{AB}} \vec{E}_2 \cdot d\vec{r}$$

\vec{E}_2 is the electric field at the interface in the second medium.

$$\int_{\widehat{B_1 A_1}} \vec{E} \cdot d\vec{r} \rightarrow \int_{\widehat{BA}} \vec{E}_1 \cdot d\vec{r} = - \int_{\widehat{AB}} \vec{E}_1 \cdot d\vec{r}$$

\vec{E}_1 is the electric field at the interface but in the first medium.

We have:

$$\int_{\widehat{AB}} (\vec{E}_2 - \vec{E}_1) \cdot d\vec{r} = 0$$

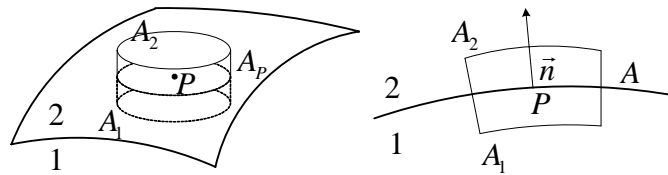
but $d\vec{r} = \vec{t} ds$ where \vec{t} a tangential unit vector

$$\int_{\widehat{AB}} (\vec{E}_2 \cdot \vec{t} - \vec{E}_1 \cdot \vec{t}) ds = 0$$

$$E_{t2} = E_{t1}$$

That is the tangential components of the electric field at the interface between two media are the same (continuous).

Let's consider now the interface between two different dielectrics. Set up a closed curve on this interface and draw the normal at all points of the curve. Finally make a closed surface. Denote the surface of the superificies by A_p and the top surface by A_2 the bottom one by A_1 .



For the closed surface apply the second law of electrostatics:

$$\oint_A \vec{D} \cdot d\vec{A} = Q_v$$

$$\int_{A_1} \vec{D} \cdot d\vec{A} + \int_{A_p} \vec{D} \cdot d\vec{A} + \int_{A_2} \vec{D} \cdot d\vec{A} = \int_V \rho dV + \int_A \sigma dA$$

Approach A_1 and A_2 surfaces to A . It means that $A_p \rightarrow 0$ and $V \rightarrow 0$

$$\int_{A_p} \vec{D} \cdot d\vec{A} \rightarrow 0 \text{ ,and } \int_V \rho dV \rightarrow 0$$

$$\int_{A_1} \vec{D} \cdot d\vec{A} \rightarrow \int_A \vec{D}_1 \cdot d\vec{A} \rightarrow - \int_A \vec{D}_1 \cdot \vec{n} dA$$

$$\int_{A_2} \vec{D} \cdot d\vec{A} \rightarrow \int_A \vec{D}_2 \cdot d\vec{A} \rightarrow \int_A \vec{D}_2 \cdot \vec{n} dA$$

\vec{D}_1 is the induction at the interface in the first medium, \vec{D}_2 is the induction at the interface in the second medium.

$$- \int_A \vec{D}_1 \cdot \vec{n} dA + \int_A \vec{D}_2 \cdot \vec{n} dA = \int_A \sigma dA$$

$$\int_A (D_{n2} - D_{n1} - \sigma) dA = 0$$

and $D_{n2} - D_{n1} = \sigma$

That is the normal component of the electric induction at the interface between two media is not continuous. Its jumping is just the surface charge density. If $\sigma = 0$ the normal component is continuous.

$$-\int_A \vec{D}_1 \cdot \vec{n} dA + \int_A \vec{D}_2 \cdot \vec{n} dA = \int_A \sigma dA$$

$$\int_A (D_{n2} - D_{n1} - \sigma) dA = 0$$

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That is the normal component of the electric induction at the interface between two media is not continuous. Its jumping is just the surface charge density. If $\sigma = 0$, the normal component is continuous.

1.2 Conductors in electrostatic field

An initially uncharged insulated conductor must have $\vec{E} = 0$ for all internal points. If this were not so, electrical forces would act on the free charge carriers producing motion of these charges. In electrostatics the charge carriers must be at rest so inside the conductor $\vec{E} = 0$. If the two points and the curve of integration is inside the conductor:

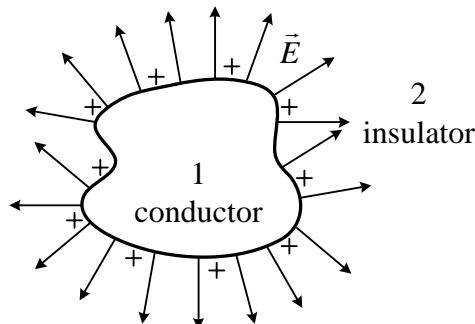
$$U_{12} = \int_1^2 \vec{E} \cdot d\vec{r} = 0, \text{ or}$$

$$U_{12} = 0$$

The potential difference between any two points is zero. The conductor is equipotential.

In the conductor $\vec{E} = 0 \rightarrow E_t = 0$. As the tangential component of the electric field intensity at the interface between two media is continuous, it means that in the surrounding insulator $E_t = 0$, that is the electric field is normal to the surface.

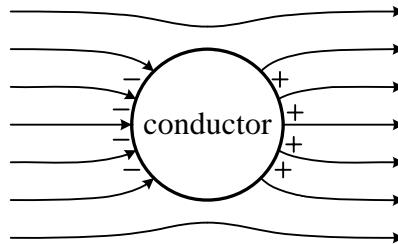
Inside the conductor $\vec{D} = 0$, and $\nabla \cdot \vec{D} = \rho$ so $\rho = 0$. In electrostatics inside the conductor there is no excess charge. If a charge Q is imparted to the conductor, the charge will be distributed on the outer surface of the conductor with a certain surface charge density σ .



Due to the boundary condition $D_{n2} - D_{n1} = \sigma$, but $D_{n1} = 0 \rightarrow D_{n2} = \sigma$. The normal component of the electric induction in the dielectric at the surface of the conductor is equal to the surface charge density.

1.2.1 Conductor in an External Electric Field

When an uncharged conductor is introduced into an electric field, the charge carriers come into motion: the positive ones in the direction of \vec{E} , the negative ones in the opposite direction. This process is called charging by induction and the excess charges are called induced charges. The charge carriers will be redistributed until the intensity of the electric field inside the conductor vanishes and the field lines outside the conductor are perpendicular to the surface. The induced charges distribute themselves over the outer surface of the conductor.



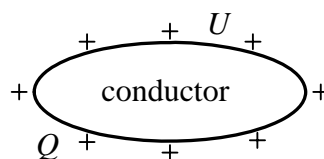
The electrostatic shielding is based on this phenomenon. Any instrument can be protected from the action of external fields, if it is surrounded by a conducting shield.

1.2.2 Capacitance

Let's consider an insulated conductor very far from the other conductors. A charge Q imparted to a conductor distributes itself over its outer surface so that the field intensity inside vanishes. If another charge Q of the same magnitude is imparted to the conductor this charge must distribute itself over the conductor in exactly the same way.

Considering the superposition: if the charge is doubled, the field set up is also doubled and the potential of the conductor is also doubled.

The zero potential is chosen in the infinity. The potential of an isolated conductor is proportional to the charge on it, and the proportionality factor is defined as the capacitance of the conductor.



$$U = \int_P^{\infty} \vec{E} \cdot d\vec{r}$$

$$Q = CU$$

The definition of capacitance of an insulated conductor:

$$C = \frac{Q}{U}$$

$$[C] = 1 \frac{C}{V} = 1 \text{ farad} = 1F$$

The submultiples are frequently used:

$$1mF = 10^{-3} F$$

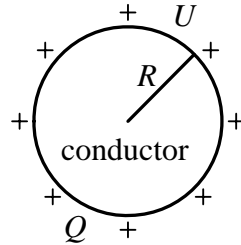
$$1\mu F = 10^{-6} F$$

$$1nF = 10^{-9} F$$

$$1\text{pF} = 10^{-12} \text{ F}$$

C depends on the shape of the conductor and depends on the medium in which it is immersed.

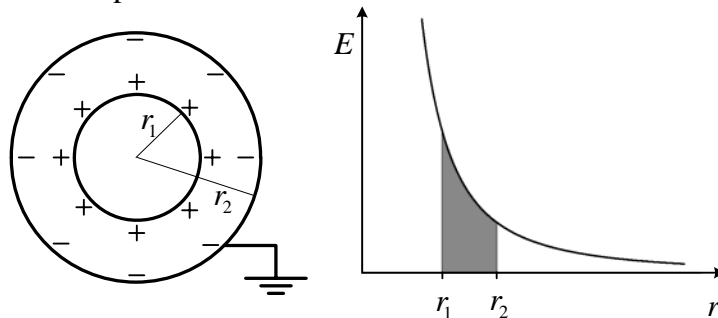
Determine the capacitance of an isolated conductor sphere:



$$C = \frac{Q}{U} = \frac{Q}{k \frac{Q}{R}} = \frac{R}{k} = 4\pi\epsilon_0 R$$

The capacitance is proportional to the radius of the sphere.

As the permittivity ϵ_0 is a very small number so the isolated conductors have a small capacitance. There are two ways to increase the capacitance. The first is to place a grounded conductor close to the charged conductor; the potential of the latter diminishes in absolute value. A capacitor consists of two isolated conductors carry equal and opposite charges. The conductors are called its plates. Depending on the shape, it should be parallel-plate -, cylindrical-, or spherical capacitors.



The capacitance of a capacitor is defined as the positive charge over the potential difference between the plates:

$$C = \frac{Q^+}{U_{12}}$$

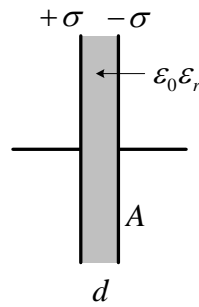
The second way to increase the capacitance, to apply a dielectric instead of vacuum between the plates. As we know in the insulator:

$$D_{n2} = \sigma \rightarrow E_{n2} = \frac{\sigma}{\epsilon_0 \epsilon_r}$$

where ϵ_r is the relative permittivity of the filling dielectric. As $\epsilon_r > 1$ the electric field, and the potential difference diminishes, so the capacitance increases.

1.2.3 Capacitance of a parallel-plate capacitor

Consider a parallel-plate capacitor, and suppose that $d \ll$ than the linear dimension of the plate:



$$D = \sigma \rightarrow E = \frac{\sigma}{\epsilon_0 \epsilon_r}$$

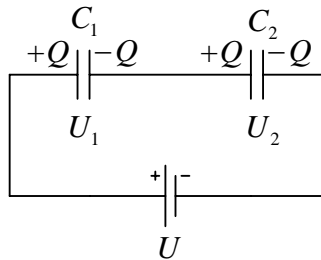
$$U_{12} = \int_1^2 \vec{E} \cdot d\vec{r} = \int_1^2 \frac{\sigma}{\epsilon_0 \epsilon_r} dr = \frac{\sigma}{\epsilon_0 \epsilon_r} d$$

The integral is evaluated for a straight line.

$$C = \frac{Q^+}{U_{12}} = \frac{\sigma A}{\frac{\sigma d}{\epsilon_0 \epsilon_r}} = \epsilon_0 \epsilon_r \frac{A}{d}$$

1.2.4 Capacitors in Series and Parallel

1. Series connection



If the capacitors are in series they always have the same charge.

$$C_1 = \frac{Q}{U_1} \quad U = U_1 + U_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

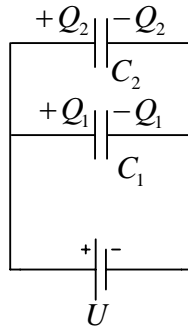
$$C_2 = \frac{Q}{U_2}$$

The equivalent capacitance C of a series combination is defined as the capacitance of a single capacitor whose charge is Q and its potential drop is U

$$C = \frac{Q}{U} \rightarrow \frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

2. Parallel Connection



If the capacitors are in parallel, the potential difference is the same for both capacitors and equal to the applied voltage.

$$\begin{aligned} Q_1 &= C_1 U \\ Q_2 &= C_2 U \end{aligned} \quad Q = Q_1 + Q_2 = C_1 U + C_2 U$$

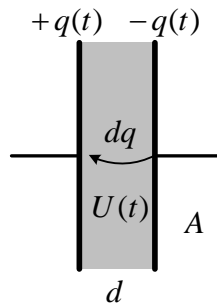
The equivalent capacitance C of the parallel combination is defined as a single capacitor whose charge is the total charge Q and its potential difference is U

$$\begin{aligned} C &= \frac{Q}{U}, \text{ so } Q = CU \\ CU &= (C_1 + C_2)U \\ C &= C_1 + C_2 \end{aligned}$$

1.2.5 Energy storage in electric field

It is known that all charge configurations have a certain electric potential energy which is equal to the work that must be done to assemble the system. So a charged capacitor has stored also an electric potential energy which is equal to the work required to charge it.

Consider now a capacitor and determine the work as it is charged.



At a stage of a charging process denote the potential difference by $U(t)$ and the charge by $q(t)$. The elementary work required to transfer the next dq charge is:

$$dW = U dq$$

If the capacitance is C then:

$$U = \frac{q}{C}$$

and the elementary work:

$$dW = \frac{q}{C} dq$$

The total work needed to increase the charge from zero to a final value Q is:

$$W = \int dW = \int_0^Q \frac{q}{C} dq = \frac{1}{C} \left[\frac{q^2}{2} \right]_0^Q = \frac{Q^2}{2C}$$

$$W = \frac{Q^2}{2C} = \frac{1}{2} QU = \frac{1}{2} CU^2$$

This is the work done to charge the capacitor. The potential energy of an uncharged capacitor is defined to be zero, then W is equal to the potential energy of the capacitor.

It is reasonable to suppose that this energy is stored in the electrostatic field. Consider now a parallel-plate capacitor, if we neglect fringing, the electric field intensity is the same at all points between the plates. This is a homogeneous field.

$$W = \frac{1}{2} QU = \frac{1}{2} \sigma AEd = \frac{1}{2} DEAd = \frac{1}{2} DEV$$

Because:

$$Q = \sigma A, \quad E = \frac{U}{d} \text{ and } V = Ad$$

The electric energy density is defined as:

$$W = \frac{W}{V} = \frac{1}{2} \vec{D} \cdot \vec{E}$$

$$[W_E] = 1 \frac{J}{m^3}$$

Although we have derived this energy density only for a specific situation, it turns out to be valid in general.

If an electric field \vec{E} exists at any point in space and the induction is \vec{D} then the energy stored in an elementary volume dV around the point in question is:

$$dW = \frac{1}{2} \vec{D} \cdot \vec{E} dV$$

In a finite volume the total electric energy is:

$$W = \int_V W dV = \int_V \frac{1}{2} \vec{D} \cdot \vec{E} dV$$