5 Maxwell equations, and Electromagnetic waves

5.1 The Ampère-Maxwell equation

For stationary field (i.e. not varying with time) Ampère's Law relates a steady current to the magnetic field, the current produces. Its differential form is:

$$\nabla \times \vec{H} = \vec{J}$$

The differential form of the continuity equation which expresses the charge conservation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

Take the divergence of the Ampère's Law:

$$\nabla \cdot \left(\nabla \times \vec{H} \right) = \nabla \cdot \vec{J}$$

The divergence of a curl must equal zero, it is an identity so:

$$\nabla \cdot \vec{J} = 0$$

At the same time the continuity equation is

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

There is a contradiction between the two equations. Indeed, in non-stationary processes, ρ may change with time (this is what happens with the charge density on the plates of a capacitor being charged). We suppose that the continuity equation is right, so Ampère's Law need a revision if it is to be applied to time-dependent fields. The modification of Ampère's Law was suggested by Maxwell and the modified equation is called Ampère-Maxwell Law. Maxwell introduced an additional term into the right hand side, and he called it displacement current density $\vec{J}_D$.

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_D,$$

$$\nabla \cdot \left(\nabla \times \vec{H} \right) = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}_D$$

The divergence of a curl must equal zero:

$$\nabla \cdot \left(\nabla \times \vec{H} \right) = 0$$

$$0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}_D$$

$$0 = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \vec{J}_D = \frac{\partial \rho}{\partial t}$$

Consider now the Gauss's Law for electric field, $\nabla \cdot \vec{D} = \rho$, and substitute it:

$$\nabla \cdot \vec{J}_D = -\frac{\partial}{\partial t}\left(\nabla \cdot \vec{D} \right)$$

Change the sequence of differentiation with respect to time and to coordinates:

$$\nabla \cdot \vec{J}_D = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

Hence we arrived that:

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t},$$

and so the differential form of Ampere-Maxwell equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \text{ where } \vec{J} = \rho \vec{v} + \vec{j}\,.$$

We must underline the fact that the displacement current is not a real current, it is a time-varying electric field. The only reason for calling it displacement current density is that the dimension of this quantity coincides with that of current density.

This phenomenon and equation is the symmetrical pair of the Faraday's Law of induction. We have already seen that a varying magnetic field sets up an electric field. It follows from the Ampère-Maxwell Law, that a varying electric field sets up a magnetic field. The Ampère's Law indicates that a steady current sets up magnetic field. The Ampère-Maxwell Law goes a step further and indicates that the time dependent electric field also contributes to the magnetic field. The displacement current sets up a magnetic field in exactly the same way as an ordinary conduction current. The fact that displacement current acts as a source of magnetic field plays an essential role in the understanding of electromagnetic waves. A changing electric field in a region of space induces a magnetic field in neighbouring regions even when no conduction current and no matter are present.

Take the integral of the differential form of Ampère-Maxwell Law for an open and fixed oriented surface:

$$\int_A \left(\nabla \times \vec{H}\right) \cdot d\vec{A} = \int_A \vec{J} \cdot d\vec{A} + \int_A \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A}$$

Applying the Stokes' theorem, the integrated form of Ampère-Maxwell Law:

$$\oint_g \vec{H} \cdot d\vec{s} = I_A + \frac{d}{dt} \int_A \vec{D} \cdot d\vec{A}$$

The line integral of the magnetic field strength for a fixed closed curve is equal to the algebraic sum of the currents plus the time rate of change of the electric flux passes through the surface boundered by the closed curve. There is a displacement current wherever there is a time-varying electric field. It also exists inside conductors, carrying an alternating electric current. Compare the conduction and diplacement current density in a good conductor:

$$\gamma \cong 10^7 \frac{1}{\Omega m}$$

Let it be:

$$\vec{E} = \vec{E}_0 \sin \omega t, \text{ and } \omega = 2\pi\nu$$

$$\vec{D} = \varepsilon_0 \vec{E}_0 \sin \omega t$$

The conduction current density:

$$\vec{j} = \gamma \vec{E} = \gamma \vec{E}_0 \sin \omega t$$

The displacement current density:

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \vec{E}_0 \omega \cos \omega t$$

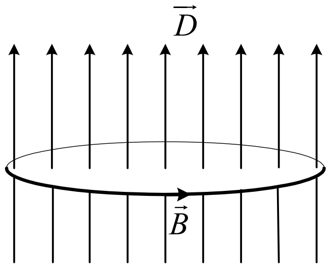
Take the ratio of the magnitudes of the maximum values:

$$\frac{j_0}{J_{D0}} = \frac{\gamma E_0}{\varepsilon_0 E_0 \omega} = \frac{\gamma}{\varepsilon_0 \omega} = \frac{10^7}{8.85 \cdot 10^{-12}\, 2\pi\nu} \approx \frac{2 \cdot 10^{17}}{\nu}$$

$$\varepsilon_0 = 8{,}85 \cdot 10^{-12} \frac{As}{Vm}$$

The displacement current inside a conductor in case of technical alternating currents is usually negligibly small in comparison with the conduction current. Not negligible at optical frequencies and when there are no conduction current.

The direction of the magnetic field produced by time dependent electric field:



If $\vec{D}$ increases $\frac{d}{dt}\int_A \vec{D} \cdot d\vec{A}$ positive, if $\vec{D}$ decreases $\vec{B}$ oppositely directed.

5.2 The Maxwell's Equations

At this point let us summarize our discussion of the electromagnetic field. The entire theory of the electromagnetic field is condensed into four basic laws. These are called Maxwell's equations:

1. Ampère-Maxwell Law:

$$\oint_g \vec{H} \cdot d\vec{s} = I_A + \frac{d}{dt}\int_A \vec{D} \cdot d\vec{A}, \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

2. Faraday's Law of induction:

$$\oint_g \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}, \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

3 Gauss's Law for electric field:

$$\oint_A \vec{D} \cdot d\vec{A} = Q, \qquad \nabla \cdot \vec{D} = \rho$$

4. Gauss's Law for magnetic field:

$$\oint_A \vec{B} \cdot d\vec{A} = 0 \qquad \nabla \cdot \vec{B} = 0.$$

Maxwell's equations form a consistent set of equations. These equations are used in integral or in differential form depending on the problem to be solved.

Base vectors: electric field: $\vec{E}$
magnetic induction: $\vec{B}$
Polarization vectors: electric polarization: $\vec{P}$
Magnetization: $\vec{M}$
Linear combination vector, electric induction: $\vec{D}$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Linear combination vector, magnetic field strength: $\vec{H}$

$$\overrightarrow{H} = \frac{\vec{B}}{\mu_0} - \overrightarrow{M}$$

$\vec{D}$ and $\vec{H}$ are introduced due to presence of chemical substance.
The first two Maxwell equations are vector-equations. Each vector equations are equivalent to three scalar equations. We get a total of 8 equations including 12 functions, three components each of the vectors $\vec{E}, \vec{D}, \vec{B}, \vec{H}$.
Since the number of equations is less then the number of unknown functions, to calculate the fields we must add equations relating $\vec{D}$ and $\vec{J}$ to $\vec{E}$ and $\vec{H}$ $\vec{B}$. These are the constitutive equations:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}, \quad \left(\vec{P} = \chi \varepsilon_0 \vec{E}\right)$$

$$\vec{B} = \mu_0 \mu_r \vec{H}, \quad \left(\vec{M} = \chi \vec{H}\right)$$

$$\vec{j} = \gamma\left(\vec{E} + \vec{v} \times \vec{B} + \vec{E}_i\right)$$

$$\vec{E}^* = \vec{v} \times \vec{B} + \vec{E}_i$$

These equations are also called material equation. The above equations are valid for isotropic media containing no ferromagnetics. These equations are not as general as Maxwell's equations.
The boundary conditions are:

$$E_{t1} = E_{t2}, \qquad D_{n2} - D_{n1} = \sigma$$

$$H_{t1} = H_{t2}, \qquad B_{n1} = B_{n2}$$

5.3 Electromagnetic Waves

When either an electric or magnetic field is changing with time a field of the other kind is induces in adjacent regions of space. We are led (as Maxwell was) to consider the possibility of an electromagnetic disturbance, consisting of time-varying electric and magnetic fields that can separate from their sources that is form charges and currents, and can propagate through space even when no matter is present in the region. So this continuous inter-conversation of the field preserves them and an electromagnetic perturbation propagates in space.

The existence of electromagnetic waves had been predicted by Maxwell as a result of a careful analysis of the basic equations of electromagnetic field. Maxwell proved in 1865 that the electromagnetic waves propagate in free space with the speed of light so the light waves are very likely to be electromagnetic nature.

In 1887 Hertz actually produced electromagnetic waves, and verified Maxwell's theory.
The development of our knowledge of electromagnetic waves is a beautiful example of the close relationship between theory and experiment in the evolution of physical ideas.

Consider now vacuum or neutral (not charged $\rho = 0$) insulator, that is $\vec{j} = 0$. Suppose that the medium is homogeneous and isotropic and not ferromagnetic. The permittivity ε and the permeability μ_0 is constant. The linear constitutive equations are:

$$\overrightarrow{D} = \varepsilon \overrightarrow{E} \text{ , and } \overrightarrow{B} = \mu_0 \overrightarrow{H} \text{ ,}$$

Write the differential form of the Maxwell's equations and apply the above conditions:

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t},\qquad \nabla\times\frac{\vec{B}}{\mu_0}=\frac{\partial\left(\varepsilon\vec{E}\right)}{\partial t},\qquad \nabla\times\vec{B}=\varepsilon\,\mu_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\qquad\qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{D}=\rho\qquad \nabla\cdot\left(\varepsilon\vec{E}\right)=0\qquad \nabla\cdot\vec{E}=0$$

$$\nabla\cdot\vec{B}=0\qquad\qquad \nabla\cdot\vec{B}=0$$

To eliminate the magnetic induction $\vec{B}$ from the system, let us take the curl of the second equation:

$$\nabla\times\left(\nabla\times\vec{E}\right)=-\nabla\times\frac{\partial\vec{B}}{\partial t}$$

Due to a mathematical identity:

$$\nabla\times\left(\nabla\times E\right)=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}$$

$$\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\nabla\times\vec{B}\right)$$

Using the first and third equation:

$$\nabla\times\vec{B}=\varepsilon\,\mu_0\frac{\partial\vec{E}}{\partial t},\ \nabla\cdot\vec{E}=0$$

$$-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\varepsilon\mu_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla^2\vec{E}=\varepsilon\mu_0\frac{\partial^2\vec{E}}{\partial t^2}.$$

This equation is called wave equation for $\vec{E}$. Similar equation can be obtained for $\vec{B}$, starting this procedure with the Ampère's law:

$$\nabla^2\vec{B}=\varepsilon\mu_0\frac{\partial^2\vec{B}}{\partial t^2}$$

The linear homogeneous second order partial differential equation for E_x is:

$$\frac{\partial^2E_x}{\partial x^2}+\frac{\partial^2E_x}{\partial y^2}+\frac{\partial^2E_x}{\partial z^2}=\varepsilon\mu_0\frac{\partial^2E_x}{\partial t^2}.$$

Similar equations can be written for E_y, E_z, B_x, B_y, B_z.
It is conceivable that $E_x=f\left(\varphi\right)$ is the solution of the differential equation, where *f* is any twicw differentiable function, and φ the argumentum is:

$$\varphi=\omega\left(t-\frac{\vec{N}\cdot\vec{r}}{u}\right)$$

ω is a positive constant called cyclic frequency, $\vec{N}$ is a unit vector $\vec{N}\cdot\vec{N}=1$, $\vec{r}$ is the position vector, and *u* is a suitable positive constant.

$$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{N}=N_x\vec{i}+N_y\vec{j}+N_z\vec{k}$$

$$\varphi=\omega\left(t-\frac{N_xx+N_yy+N_zz}{u}\right)$$

$$E_x = f\left(\omega\left(t - \frac{N_x x + N_y y + N_z z}{u}\right)\right)$$

Take the derivatives, and denote the derivative with respect to the argumentum by dash:

$$\frac{df}{d\varphi} = f'$$

The first derivative:

$$\frac{\partial E_x}{\partial x} = f'\left(-\omega\frac{N_x}{u}\right)$$

The second derivative:

$$\frac{\partial^2 E_x}{\partial x^2} = f''\omega^2\frac{N_x^2}{u^2}$$

$$\frac{\partial^2 E_x}{\partial y^2} = f''\omega^2\frac{N_y^2}{u^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} = f''\omega^2\frac{N_z^2}{u^2}$$

The first derivative with respect to time:

$$\frac{\partial E_x}{\partial t} = f'\omega$$

The second derivative with respect to time:

$$\frac{\partial^2 E_x}{\partial t^2} = f''\omega^2$$

Inserting into the differential equation:

$$f''\frac{\omega^2}{u^2}\left(N_x^2 + N_y^2 + N_z^2\right) = \varepsilon\,\mu_0 f''\omega^2$$

As $\vec{N}$ is a unit vector:

$$N_x^2 + N_y^2 + N_z^2 = 1$$

So the wave equation is satisfied with the function above if u is given by the next formula:

$$\frac{1}{u^2} = \varepsilon\mu_0\,,$$

$$u = \frac{1}{\sqrt{\varepsilon\,\mu_0}}$$

It is easy to see that the value of the phase at a given place and time is the same as Δt time later and $\Delta\vec{r} = u\,\vec{N}\Delta t$ distance away:

$$\varphi\left(t + \Delta t, \vec{r} + \Delta\vec{r}\right) = \omega\left(t + \Delta t - \frac{\vec{N}\cdot\left(\vec{r} + \Delta\vec{r}\right)}{u}\right) = \omega\left(t + \Delta t - \frac{\vec{N}\vec{r}}{u} - \frac{\vec{N}\,u\cdot\vec{N}}{u}\Delta t\right) =$$

$$= \omega\left(t - \frac{\vec{N}\cdot\vec{r}}{u}\right) = \varphi\left(t, \vec{r}\right)$$

We can conclude that the phase propagates into the $\vec{N}$ direction and its velocity is:

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{u\,\vec{N}\,\Delta t}{\Delta t} = u\,\vec{N} = \frac{1}{\sqrt{\varepsilon\,\mu_0}}\vec{N}$$

That is the quantity denoted by *u* is just the so called phase velocity:

$$u = \frac{1}{\sqrt{\varepsilon\,\mu_0}}$$

In case of vacuum $\varepsilon' = 1$, and $\varepsilon = \varepsilon'\varepsilon_0$, so:

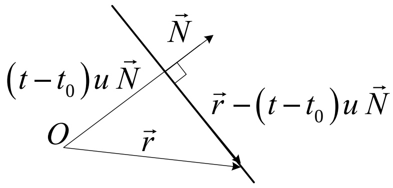
$$c = \frac{1}{\sqrt{\varepsilon_0\,\mu_0}} = 3\cdot 10^8\, m/s$$

The phase velocity in vacuum equals the velocity of light in vacuum.
In chemical substance:

$$u = \frac{1}{\sqrt{\varepsilon\,\mu_0}} = \frac{1}{\sqrt{\varepsilon'}\sqrt{\varepsilon_0\mu_0}} = \frac{c}{\sqrt{\varepsilon'}} = \frac{c}{n}$$

As $\varepsilon' > 0$, in chemical substance. $u < c$.
$n = \sqrt{\varepsilon'}$ is called absolute refractive index.

The locus of all adjacent points at which at a given instant the phase is constant is called wave front. It is generally a moving surface. In our case the wave front is a plane whose normal is $\vec{N}$ and the wave front travels in the direction of $\vec{N}$ parallel with itself with a speed of *u*. Let's suppose that the wave front passes the origin at the moment t_0 and travels in the $\vec{N}$ direction. At time *t* the wave front is $(t - t_0)u$ distance from the origin. $\vec{r}$ is an arbitrary vector of the plane.



Because the $\vec{r} - (t - t_0)u\,\vec{N}$ vector always lies at the plane, it means that the scalar product with $\vec{N}$ is zero:

$$\vec{N}\cdot\left(\vec{r} - (t - t_0)u\,\vec{N}\right) = 0$$

$$\vec{N}\cdot\vec{r} - (t - t_0)u = 0$$

$$\frac{\vec{N}\cdot\vec{r}}{u} = t - t_0 \to t_0 = t - \frac{\vec{N}\cdot\vec{r}}{u}$$

That is the phase along this plane is really the same as at t_0 in the origin.

$$E_x = f(\omega t_0) = f\left(\omega\left(t - \frac{\vec{N}\cdot\vec{r}}{u}\right)\right).$$

This formula is a travelling plane wave solution.

5.4 Monochromatic plane wave

Monochromatic means that is contains only one frequency. Suppose that *f* is a sinusoidal function, namely sine or cosine:

$$E_x = E_{x0}\cos\omega\left(t - \frac{\vec{N}\cdot\vec{r}}{u}\right)$$

ω is a positive constant called cyclic frequency E_{x0} is constant, and called amplitude of the x component of the electric field.

$$\varphi = \omega\left(t - \frac{N \cdot \vec{r}}{u}\right)$$

called phase, dimensionless quantity.
The monochromatic plane wave solution is periodical both in time and in space:
As the cosine function is periodical by 2π, at a given point:

$$\varphi\left(t+T,\vec{r}\right) = \varphi\left(t,\vec{r}\right) + 2\pi$$

$$\omega\left(t + T - \frac{\vec{N}\cdot\vec{r}}{u}\right) = \omega\left(t - \frac{\vec{N}\cdot\vec{r}}{u}\right) + 2\pi$$

$$\omega T = 2\pi$$

Therefore the period in time:

$$T = \frac{2\pi}{\omega}.$$

The period in space is called wave length and this is the distance of two wave fronts at a given moment whose phase difference is 2π.

$$\varphi\left(t,\vec{r}+\lambda\vec{N}\right) = \varphi\left(t,\vec{r}\right) - 2\pi$$

$$\omega\left(t - \frac{\vec{N}\cdot\left(\vec{r}+\lambda\vec{N}\right)}{u}\right) = \omega\left(t - \frac{\vec{N}\cdot\vec{r}}{u}\right) - 2\pi$$

$$\frac{\omega}{u}\lambda = 2\pi$$

The cyclic frequency is:

$$\omega = 2\pi\nu$$

$$\frac{2\pi\nu}{u}\lambda = 2\pi \rightarrow \lambda = \frac{u}{\nu}$$

The circular wave number k is defined as the number of wave lengths in a length 2π unit:

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{u}$$

$\vec{k} = k\,\vec{N}$ is the (circular) wave number vector so the phase:

$$\varphi = \omega\left(t - \frac{\vec{N}\cdot\vec{r}}{u}\right) = \omega t - \frac{\omega}{u}\vec{N}\cdot\vec{r} = \omega t - \vec{k}\cdot\vec{r}$$

It is conceivable that in the electromagnetic wave the electric and the magnetic part waves are in phase, so in general:

$$\vec{E} = \vec{E}_0\cos\left(\omega t - \vec{k}\cdot\vec{r} + \delta\right)$$

$$\vec{B} = \vec{B}_0\cos\left(\omega t - \vec{k}\cdot\vec{r} + \delta\right)$$

Using the electric and magnetic Gauss's Laws it is possible to prove, that both the electric field intensity and the magnetic induction is always perpendicular to the direction of propagation:

$$\nabla\cdot\vec{E} = 0 \Rightarrow \vec{E}_0 \perp \vec{k}\text{, or } \vec{E}_0 \perp \vec{N}$$

$$\nabla\cdot\vec{B} = 0 \Rightarrow \vec{B}_0 \perp \vec{k}\text{, or } \vec{B}_0 \perp \vec{N}$$

The electromagnetic wave is a transverse wave. The two part waves are called linearly (plane) polarized waves.
Remark: the natural light is not linear but circularly polarized wave but it is possible to polarize in plane.

From the first Maxwell's equation we can obtain the connection between the amplitude vectors:

$$\nabla \times \vec{B} = \varepsilon\,\mu_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{E}_0 = -u\,\vec{N} \times \vec{B}_0$$

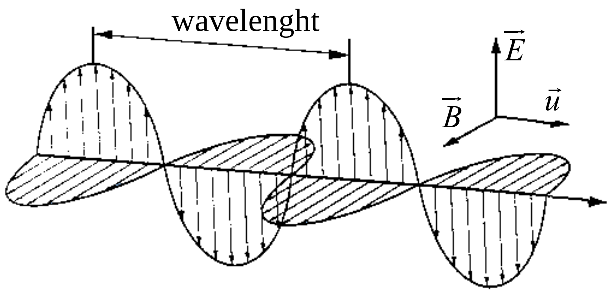
$$\vec{E}_0 = -\vec{u} \times \vec{B}_0$$

$\vec{u}$ is the velocity vector of propagation, therefore $\vec{E}_0$ and $\vec{B}_0$ are also perpendicular vectors so the next three vectors form a right hand system:

$$\left\{\vec{N}, \vec{E}_0, \vec{B}_0\right\}$$

The connection between the magnitudes:

$$E_0 = u\,B_0$$



This is a "snapshot" representation of the travelling electromagnetic wave showing the alternating electric and magnetic vectors along the propagation.

5.5 Propagation of the energy in electromagnetic waves

The vectors $\vec{E}$ and $\vec{B}$ at a given point of space vary in the same phase:

$$\vec{E} = \vec{E}_0 \cos\varphi$$

$$\vec{B} = \vec{B}_0 \cos\varphi$$

The phase:

$$\varphi = \omega t - \vec{k} \cdot \vec{r}\,.$$

The connection between the amplitudes, and the phase velocity:

$$E_0 = u\,B_0\,,\; u = \frac{1}{\sqrt{\varepsilon\,\mu_0}}$$

$$E_0 = \frac{1}{\sqrt{\varepsilon\,\mu_0}} B_0 \rightarrow E_0^2 \varepsilon = \frac{B_0^2}{\mu_0}$$

Consider the electromagnetic energy density:

$$w = w_e + w_m = \frac{1}{2}\vec{D}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{H} = \frac{1}{2}\varepsilon\,E^2 + \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\varepsilon\,E_0^2\cos^2\varphi + \frac{1}{2}\frac{B_0^2}{\mu_0}\cos^2\varphi$$

Due to the relation between the amplitudes it follows that the electric and magnetic energy densities of the wave are identical at each moment.

$$w = w_e + w_m = 2w_e = \varepsilon E_0^2 \cos^2 \varphi$$

Introduce the energy current density or energy flow rate vector called Poynting vector, by the next definition:

$$\vec{S} = \vec{E} \times \vec{H}$$

The magnitude of the Poynting vector gives the flow of energy through a cross section perpendicular to the propagation direction per unit area per unit time. The direction of the vector gives the direction of the electromagnetic energy propagation.
It's unit:

$$[S] = 1\frac{J}{m^2 s} = 1\frac{W}{m^2}$$

$$S = E\frac{B}{\mu_0} = E_0 \cos\varphi \frac{B_0 \cos\varphi}{\mu_0} = E_0 \cos\varphi \frac{E_0 \sqrt{\varepsilon \mu_0} \cos\varphi}{\mu_0} = \varepsilon E_0^2 \cos^2 \varphi \frac{\sqrt{\varepsilon \mu_0}}{\varepsilon \mu_0} =$$

$$= w \frac{1}{\sqrt{\varepsilon \mu_0}} = wu$$

$S = wu$, or in vector form $\vec{S} = w\vec{u}$

In an electromagnetic wave in homogeneous isotropic dielectric the electromagnetic energy propagates in the direction as the wave propagates, with the phase velocity.

As the frequency of the electromagnetic wave is very high, for example at visible light ~10^{14} *Hz*, only the time average of the energy flow density can be observed.
The time-averaged of the magnitude of the Poynting vector is called intensity:

$$I = \overline{S} = \overline{\varepsilon E_0^2 \cos^2 \varphi \frac{1}{\sqrt{\varepsilon \mu_0}}} = \sqrt{\frac{\varepsilon}{\mu_0}} E_0^2 \overline{\cos^2 \varphi} = \sqrt{\frac{\varepsilon}{\mu_0}} E_0^2 \overline{\frac{1 - \cos 2\varphi}{2}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_0}} E_0^2$$

$$I = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_0}} E_0^2$$

The intensity is proportional to the square of the amplitude of the field.

$$\overline{E^2} = \overline{E_0^2 \cos^2 \varphi} = E_0^2 \overline{\frac{1 - \cos 2\varphi}{2}} = E_0^2 \frac{1}{2}$$

The intensity:

$$I = \sqrt{\frac{\varepsilon}{\mu_0}} \frac{E_0^2}{2} = \sqrt{\frac{\varepsilon}{\mu_0}} \overline{E^2}$$

5.6 Interference

Interference occurs when two or more waves coincide in space and time and causing a geometrically regular interference pattern in space where maximum and minimum intensities can be observed, and the regular interference pattern is a standing picture so we can observe. The interference is one of the most important wave properties. By interference we mean the superposition of waves from a finite number of sources.

Consider now the superposition of two monochromatic plane waves having the same frequency.

$$\vec{E}_1 = \vec{E}_{10}\cos\varphi_1 = \vec{E}_{10}\cos\left(\omega t - \overrightarrow{k_1}\cdot\vec{r}\right)$$
$$\vec{E}_2 = \vec{E}_{20}\cos\varphi_2 = \vec{E}_{20}\cos\left(\omega t - \overrightarrow{k_2}\cdot\vec{r} + \delta\right)$$

where δ is the phase difference between the two waves.
The superposition:

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$
$$E^2 = E^2{}_1 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2 = E^2{}_1 + E_2^2 + 2\vec{E}_{10}\cdot\vec{E}_{20}\cos\varphi_1\cos\varphi_2$$

The time average:

$$\overline{E^2} = \overline{E^2{}_1} + \overline{E_2^2} + 2\vec{E}_{10}\cdot\vec{E}_{20}\overline{\cos\varphi_1\cos\varphi_2} \qquad /\cdot\sqrt{\frac{\varepsilon}{\mu_0}}$$
$$\sqrt{\frac{\varepsilon}{\mu_0}}\,\overline{E^2} = \sqrt{\frac{\varepsilon}{\mu_0}}\,\overline{E^2{}_1} + \sqrt{\frac{\varepsilon}{\mu_0}}\,\overline{E_2^2} + 2\vec{E}_{10}\cdot\vec{E}_{20}\sqrt{\frac{\varepsilon}{\mu_0}}\,\overline{\cos\varphi_1\cos\varphi_2}$$
$$I = I_1 + I_2 + I_{12}$$

If the term $I_{12} \neq 0$ is not zero, the resultant intensity is not equal to the sum of the individual intensities. In such a case we speak about interference, and I_{12} is called interference term.

$$I_{12} = 2\vec{E}_{10}\cdot\vec{E}_{20}\sqrt{\frac{\varepsilon}{\mu_0}}\,\overline{\cos\varphi_1\cos\varphi_2}$$

Transform the $\cos\varphi_1\cos\varphi_2$ product to a sum:

$$\cos\left(\varphi_1+\varphi_2\right) = \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2$$
$$\cos\left(\varphi_1-\varphi_2\right) = \cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2$$
$$\cos\left(\varphi_1+\varphi_2\right) + \cos\left(\varphi_1-\varphi_2\right) = 2\cos\varphi_1\cos\varphi_2$$
$$I_{12} = \sqrt{\frac{\varepsilon}{\mu_0}}\,\vec{E}_{10}\cdot\vec{E}_{20}\cdot\left(\overline{\cos\left(\varphi_1+\varphi_2\right)+\cos\left(\varphi_1-\varphi_2\right)}\right)$$

The sum of the two phases:

$$\varphi_1+\varphi_2 = 2\omega t - \overrightarrow{k_1}\cdot\vec{r} - \overrightarrow{k_2}\cdot\vec{r} + \delta$$

The difference:

$$\varphi_1-\varphi_2 = \overrightarrow{k_2}\cdot\vec{r} - \overrightarrow{k_1}\cdot\vec{r} - \delta$$

The first term is periodic with time so the time average is zero. The second term is constant so the time average is itself.

$$I_{12} = \sqrt{\frac{\varepsilon}{\mu_0}}\vec{E}_{10}\cdot\vec{E}_{20}\cos\left(\left(\overrightarrow{k_2}-\overrightarrow{k_1}\cdot\right)\vec{r}-\delta\right) = \sqrt{\frac{\varepsilon}{\mu_0}}\,\vec{E}_{10}\cdot\vec{E}_{20}\cos\left(\varphi_1+\varphi_2\right)$$

The intensity has a maximum if $\cos\left(\varphi_1+\varphi_2\right)=1$ that is the phase difference is $\varphi_1-\varphi_2 = 2m\pi$ where $m = 0,\pm1,\pm2,...$. In other words the phase difference is even-number multiple of π. It is called constructive interference or reinforcement.
The intensity has a minimum if $\cos\left(\varphi_1-\varphi_2\right)=-1$ that is the phase difference is $\varphi_1-\varphi_2 = \left(2m+1\right)\pi$, where $m = 0,\pm1,\pm2,...$.So the phase difference is add-number multiple of π. It is called destructive interference or cancellation, or maximum attenuation.

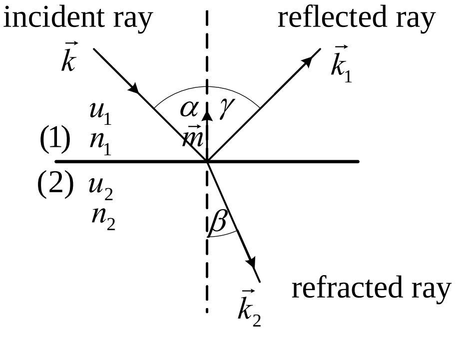
The conditions of the interference to get a long time wave pattern in the wave space:

1. The frequencies of the waves must be equal
$$\nu_1 = \nu_2 \text{, or } \omega_1 = \omega_2 .$$
2. The two amplitude vectors are not perpendicular to each other
$$\vec{E}_{10} \cdot \vec{E}_{20} \neq 0$$
3. The phase difference between the two waves must be constant in time. With common light sources it is impossible to reach interference pattern in space, with two independent light rays. If one light ray is divided into two parts and later they are combined then we can obtain interference.
4. The path difference between the two waves must not be too much because the wave trains have limited length and the first train must be the place when the second arrives.

The waves which fulfil the above conditions of interference are called coherent waves. In case of a time-dependent phase difference between the sources would result in a varying phase difference between the superposed wave trains. The interference pattern would be continuously moving, perhaps too rapidly for detection by an eye. No stationary interference pattern is observed in this case (incoherent waves).

5.7 Behaviour of waves at the interface between two media

In many familiar optical phenomena a wave strikes an interface between two optical materials such as air and glass or water and glass. When the interface is smooth the wave is in general partly reflected and partly transmitted into the second material.
The phase velocity in chemical substance:
$$u = \frac{c}{\sqrt{\varepsilon'}} = \frac{c}{n}$$
$n = \sqrt{\varepsilon'}$ is called absolute refractive index.
The ratio of the velocity of a light wave in vacuum to the phase velocity in a medium is known as the absolute refractive index.
$$n = \frac{c}{u}$$
The segments of plane waves can be represented bundles of rays forming beams of light, for simplicity we will consider only one ray characterized by its wave number vector $\vec{k}$ directed into the propagation.



α: is the angle of incidence, β: is the angle of refraction, γ: is the angle of reflection, u_1 and u_2 the phase velocities, and n_1 and n_2 the absolute refractive indexes.

Experimental studies lead to the following results:

The incident, reflected and refracted rays and the normal to the surface, all lie in the same plane.
The frequency of the incident ray is the same as the frequency of the reflected and refracted rays:

$$\nu = \nu_1 = \nu_2 \text{, or } \omega = \omega_1 = \omega_2 .$$

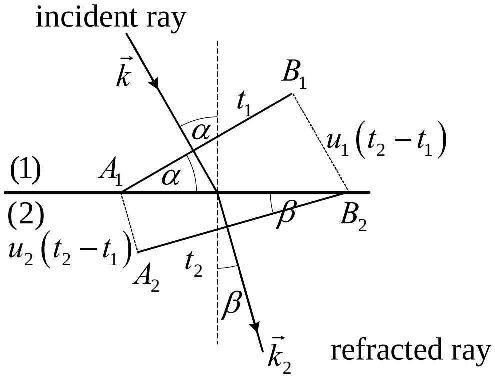
In case of reflection the wavelength of the reflected ray or wave does not change, and the angle of incidence is equal to the angle of reflection:

$$\lambda = \lambda_1 \text{, and } \alpha = \gamma$$

In case of refraction:
As light passes from one material to another its wavelength changes due to the change of the phase velocity

$$\left.\begin{aligned} u_1 &= \nu\,\lambda_1 \\ u_2 &= \nu\,\lambda_2 \end{aligned}\right\} \qquad \frac{\lambda_1}{\lambda_2} = \frac{u_1}{u_2} = \frac{\frac{c}{n_1}}{\frac{c}{n_2}} = \frac{n_2}{n_1}$$

Consider the next figure:



A_1B_1 is the part of the incident wave front at moment t_1 and A_2B_2 is the wave front at t_2.

$$\sin\alpha = \frac{u_1(t_2 - t_1)}{A_1B_2},\ \sin\beta = \frac{u_2(t_2 - t_1)}{A_1B_2}$$

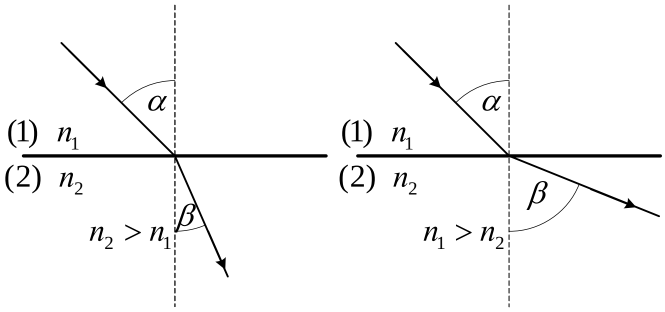
$$\frac{\sin\alpha}{\sin\beta} = \frac{u_1}{u_2} = \frac{n_2}{n_1} = n_{21}$$

This is the so called Snell's Law of refraction.
n_{21} is the relative index of refraction. It is the relative index of the second medium to the first. It is easy to prove that:

$$n_{21} = \frac{1}{n_{12}}$$

In speaking of optical materials, we often use the term of optical density. A material in which the speed of light is smaller then in the other is called optically denser. The substance is optically more dense if its absolute refractive index is grater then the other.

When light passes from an optically less dense transparent medium into an optically denser, the light is bent toward the normal. If light passes from an optically denser medium into an optically less dense medium the ray is bent away from the normal.



When light travels from a medium to an optically denser medium, $n_1 > n_2$, the maximum angle of refraction occurs when $\beta = 90^\circ$, then the angle of incidence is called critical angle of total internal reflection.

$$\frac{\sin\alpha_{cr}}{\sin 90^\circ} = \frac{n_2}{n_1} = n_{21}$$

If the angle of incidence increases and approaches to the critical value α_{cr}, the intensity of the refracted ray decreases and approaches to zero.

5.8 Dispersion

Light waves of different frequency (and therefore different wave length in a vacuum) are deviated by a prism through different angles and are said to be dispersed. $n = \sqrt{\varepsilon'}$, at high frequency ε' is not strictly constant but depends on the frequency of the light, that is the refractive index depends also:

$$n = \sqrt{\varepsilon'(\nu)}, \quad n = n(\nu)$$

If the incident beam is composed of several frequencies, each component will be refracted through a different angle, so due to dispersion the mixed light can be decomposed into monochromatic components.