Lecture Note

Non-perturbative renormalization of sine-Gordon type models

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Abstract

The goal of this lecture note is to give a presentation of some key issues regarding the non-perturbative renormalization of periodic quantum field theories, i.e., sine-Gordon models in the framework of the functional renormalization group (FRG) method. Sine-Gordon type Lagrangians consist of scalar fields similarly to the simplest and well-studied scalar theory, the φ^{2n} model, however, they have an additional symmetry to the reflection one, i.e., the periodicity which makes their phase structure even more complex with a wider range of applicability. Thus, sine-Gordon models represent an excellent playground to discuss the subtleties of the FRG method which is one of the aims of this lecture note. The first part of the lecture note is devoted to applications of sine-Gordon models to Higgs, inflaton, branon and axion physics, topological phase transitions, superconductivity, superfluidity, bosonisation, spin-models, topological defects and conformal field theory. The second part stands for the introduction of the FRG method where the relation between the Wetterich, the Wegner-Houghton and the Polchinski equations is discussed and after that the FRG method is applied for sine-Gordon models.

Contents

1 Introduction, Motivation, Aims

1.1 Motivation

Quantum Field Theory (QFT) is a natural choice to describe the physics of elementary particles. In QFT the construction of models is based on symmetry considerations. However, the requirement of quantisation and relativistic description leads to energy (or length) scale-dependent parameters. In order to obtain the low-energy effective theory of a particular QFT one has to take into account that scale-dependence which requires renormalization. Therefore, at the heart of every quantum field theory, there is the need for renormalization.

In the framework of the well-known perturbative renormalization procedure, the interaction Lagrangians are decomposed in a Taylor series in the fields which generates the vertices of the theory. Keeping only a finite number of terms each interaction vertex can be treated independently. However, there are theories which cannot be considered in this traditional way since the symmetries of the Lagrangian impose the requirement of taking infinitely many interaction vertices into account and any truncation of these infinite series would lead to an unacceptable violation of essential symmetries of the model. Sine-Gordon type scalar quantum field theories contain a periodic self-interaction thus belong to these problematic models where the Lagrangian of the sine-Gordon model reads as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + u\cos(\beta\varphi) \tag{1}$$

where the Fourier amplitude u and the frequency β are the parameters of the model. Both the sine-Gordon theory and its generalizations have important physical realizations.

- For example, the *sine-Gordon* model has been used to describe the vortex dynamics and the topological phase transitions of superfluid films, it is equivalent to the Coulomb-gas in arbitrary dimensions which is a neutral gas of interacting point-like charges with Coulomb interaction, in $d = 2$ dimensions it is equivalent to the XY classical spin model and to the fermionic Thirring-model (via bosonisation). The sine-Gordon model has been received application in Higgs, inflaton, branon and axion physics and it is a paradigmatic example of Integrable and Conformal Field Theory thus its conformal properties such as the c-function have been studied in detail.

- One possible extension is the *massive sine-Gordon* model which contains an explicit mass term in addition to the periodic self-interaction and it has been used to describe the vortex dynamics of superconducting films and in Higgs, inflaton and branon physics. It is equivalent to the Yukawa-gas, to the XY spin model with external field and to the two-dimensional Quantum Electrodynamics (QED_2).

- Another example is the *layered sine-Gordon* model, i.e., system of coupled periodic scalar quantum field theories, which has been used to describe the vortex dynamics of magnetically coupled superconducting films, it is equivalent to the layered vortex-gas, to a layered XY spin model where the coupling between the layers mediated by the external field, finally it is the bosonic counterpart of the multi-flavor QED_2 and with a different coupling of the layers, to the multi-color two-dimensional Quantum Chromodynamics (QCD_2).

- The analytic continuation of the original periodic Lagrangian to imaginary frequencies $\beta \to i\beta$ which is called the *sinh-Gordon* theory is another paradigmatic model of Conformal Field Theory similarly to the interpolating models between the sine- and sinh-Gordon models, i.e., the so called *shine-Gordon* and *sn-Gordon* Lagrangians.

Therefore, sine-Gordon models are of relevance for both statistical physics, quantum field theory, conformal field theory, high energy physics and cosmology but they are perturbatively non-renormalizable if their Taylor series is truncated (however, I show that in two-dimensions all terms of the Taylor expansion can be summed up).

The general goal of this lecture note is to present the non-perturbative renormalization of sine-Gordon type theories in the framework of the Functional Renormalization Group (FRG). The history of FRG approach runs back over decades rooted from Kenneth G. Wilson and Leo P. Kadanoff. Starting from the Wegner-Houghton RG equation published in 1973, which is based on the Wilson-Kadanoff blocking, through the Polchinski RG equation formulated in 1984, one arrives at the modern form of FRG which is usually called the Wetterich equation,

$$k\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\frac{k\partial_k R_k}{\Gamma_k^{(2)}[\varphi] + R_k}\right], \tag{2}$$

where $\Gamma_k[\varphi] = \int \mathrm{d^d x}\, \mathcal{L}_k[\varphi]$ is the effective action depending on the running scale k, $\Gamma_k^{(2)}[\varphi]$ denotes its second functional derivative and the trace Tr stands for an integration over all the degrees of freedom of the field φ, while R_k is a regulator function, the choice of which is arbitrary within certain limits. When the running scale goes to zero $k \to 0$ the scale-dependent effective action $\Gamma_{k=0}[\varphi]$ is the exact effective action of the intended quantum field theory.

By inserting (1) into (2) one can perform the non-perturbative renormalization of the sine-Gordon scalar model which is the aim of this lecture note. The structure is chosen to be a lecture note for the FRG study of SG models which, I hope, helps the reader to understand the findings more easily.

1.2 Aims

Let me summarise the research goals, aims of this lecture note which are centered around the FRG method and its application to sine-Gordon (SG) type models.

1. **The application of the FRG method for sine-Gordon models**

   Symmetries and dimensionality can be used to determine the phase structure. SG type models are periodic, thus, one has to use a method which retains their essential symmetry, i.e. the periodicity. The FRG method is suitable to perform the renormalization non-perturbatively without violating the periodicity which motivates the use of the FRG method for SG type models.

   It was known that the SG model, which contains a periodic self-interaction, undergoes an infinite-order (topological) phase transition in $d = 2$ dimension if the frequency of the model is chosen as a critical value $\beta_c^2 = 8\pi$. However, a complete mapping of the phase structure, the examination of the model in higher dimensions, the renormalization and the phase structure of theories obtained by different generalizations of the SG model were not discussed in the literature. Such a generalization is, for example, the addition of a mass term to the SG theory or the coupling of SG models in various ways.

   One of my most important goals of this lecture note is to develop and apply a method suitable for the RG study of SG models in the framework of the FRG approach [1, 2, 3]. This new procedure allows to obtain the critical frequency $\beta_c^2 = 8\pi$ of the 2D-SG model [1] and to study whether it depends on the choice of the regulator function or the approximations used [4, 5]. It provides us the complete mapping of the phases of the massless, massive, layered (coupled) SG models [6, 7, 8] in different dimensions and the generalization for imaginary (sinh-Gordon) and complex (shine-Gordon) frequencies and the discussion on the elliptic deformation, i.e., the sn-Gordon model.

2. **Sine-Gordon models in low-dimensions**

   The use of bosonization transformations is an important tool for the study of fermion and gauge fields in low dimensions. These allow us to rewrite the action of a model containing fermion and gauge fields onto a scalar theory of bosons. Such low-dimensional models play an important role in Integrable and Conformal field theory. For example, the bosonic forms of the fermionic Thirring model, the single and multi-flavor QED_2 and the multiflavor QCD_2 are SG-like theories. Various results have been obtained for these models, including the mapping out their phase structure and the study of conformal properties of their SG counterparts but no systematic renormalization group study of the corresponding SG theories was performed in the literature.

In addition SG models receive application in the vortex dynamics of superconductors with high transition temperatures which are mostly layered, so the resulting superconducting condensate shows a strong anisotropy. Such a layered arrangement can be formed not only due to the crystal structure, but also from artificially layered superconducting films separated by insulating layers where the elementary excitations with the lowest energy are superconducting eddy currents, which have magnetic moment, i.e., vortices and antivortices. It was known that SG models, like the pure SG and the massive SG theories can be used to map out the vortex dynamics of superfluid and superconducting films. Their generating functional can be rewritten into the partition function of the two-dimensional Coulomb and Yukawa gases and can be mapped onto two-dimensional XY spin models, too. However, it was not known in the literature which SG model should be used to study the vortex dynamics of a system with superconducting layers. Will there be a difference if we assume a magnetic or Josephson coupling between the layers? How does the phase transition temperature depend on the number of layers?

Thus, one of the aims of this lecture note is to investigate the low-energy behavior of the two-dimensional SG models obtained by bosonization by using the FRG method and to study how their phase structure depends on the number of flavors and colors [9]. In addition, the FRG method provides us a framework to consider the conformal properties such as the c-function and central charges of these SG-type models [10, 11]. Another goal here is to examine the phase structure of layered superconducting systems in the framework of the FRG method [12, 13, 14] and to study the dependence of the critical temperature on the number of layers and to compare the coupled SG models which are the bosonised version of multiflavor QED_2 and multicolor QCD_2 to the layered SG models used for the vortex dynamics of high transition temperature superconductors and to investigate the phase structure of the corresponding XY spin models [15, 16] too.

3. **Sine-Gordon models in higher dimensions**

SG type scalar field theories are known to play an important role in low dimensions. One might expect no room for any physical application for SG models in $d > 2$ dimensions, however, scalar fields naturally appear in various cases which opens the door for possible applications of SG type models in $d = 4$ dimensions, too. The four most natural situations among these are the followings (i) the physics of large extra dimensions with an SG type Randall-Sundrum warp factor which results in an SG type effective Branon action, (ii) SG type inflationary potentials, (iii) mass generation by SG type Higgs potentials, (iv) the axion potential which naturally appears as a periodic function.

The main concepts of these cases are the followings. In inflationary cosmology the so called natural inflation i.e., a periodic potential has already been used as a competing inflationary model and one can generalise this idea by using the massive SG model which contains an explicit mass term in addition to the periodic one thus it can also be used as a possible extension of the standard model Higgs potential. In the framework of models of large extra dimensions, in particular in the so-called Brane World Scenario, the usual choice for the Randall-Sundrum warp factor are the absolute value and the quadratic functions which can be replaced by the SG or massive SG models which result in SG type effective Branon actions. Finally, one has to mention the periodic axion potential which was proposed to retain the CP conserving nature of QCD where gauge symmetry and renormalizability allow the inclusion of CP violating terms but experimental data do not favour such an extension.

The same SG type model can be used in the above examples and each of these can be associated to a particular energy (or length) scale such as (i) the transplanckian physics of the early Universe (ii) the cosmic inflation at the GUT scale, (iii) the Higgs physics at the Electroweak scale. Thus, it is a natural question whether the RG running [17, 18] can be used to connect these energy scales which is considered in this lecture note in the framework of the FRG method after discussing the applications of the massive SG model in Branon, Inflaton and Higgs physics [18, 19, 20, 21].

4. **Methodical issues of the FRG approach**

   The functional RG method is suitable to perform renormalization non-perturbatively, however, the use of approximations cannot be avoided which has important consequences. For example, physical quantities (such as critical exponents) obtained by approximated FRG equations could become regulator-dependent, i.e, they depend on the choice of the so-called regulator function. In addition, the usual (standard) perturbative RG equations are derived with a particular choice of the renormalization scheme, thus, β-functions (with higher loop coefficients) are scheme-dependent, so the scheme-dependence of the perturbative RG approach is expected to be connected to the regulator-dependence of the FRG method. Thus, it is unavoidable to study the regulator-dependence of the FRG method.

   Indeed in order to make the FRG method predictive, the optimization of the regulator-dependence is required. A rather general optimization procedure (the Litim–Pawlowski method) leads to the Litim regulator which gives the closest theoretical propositions to the experimental results, at least in the lowest order of the gradient expansion, however, it is not a smooth function, so it cannot be used at higher orders. Another commonly used optimization is the Principle of Minimum Sensitivity (PMS), where the optimal parameters of a given regulator are chosen such as to make physical quantities as insensitive as possible to any conceivable changes of the parameters entering the regulator. It can be applied at any order of the gradient expansion, however, one cannot compare different regulators in this way.

   The aims in this lecture note are to study the regulator dependence by using known results on critical value of the single flavour QED_2 [22] and to introduce the so called Compactly Supported Smooth (CSS) regulator [23] which solves the problem of differentiability (in case of the Litim–Pawlowski optimization) and the comparability (in case of the PMS method)[24]. In addition, to discuss new optimisation strategies based on SG type models [25] by using the appearance of Spontaneous Symmetry Breaking for cases where it is not allowed but the truncated FRG equation signals it [26]. Finally, to investigate the FRG treatment of the field independent term [27] which is used in the RG evolution of the cosmological constant.

2 The classical and the quantum sine-Gordon model

2.1 The classical sine-Gordon model

Classical field theory can be used to describe the mechanics of continuous systems. In this section, the classical sine-Gordon scalar field theory is introduced by taking the continuous limit of a discrete model, i.e., torsional harmonic oscillators that can oscillate with a rotational motion around the axis of the torsion spring, clockwise and counterclockwise. This system contains a chain of massless rigid rods (pendulums) spaced a distance a apart and connected by an infinitely long horizontal torsion spring wire. The displacement of each pendulum rod is perpendicular to the axis of the wire. At the end of each pendulum rod with a uniform length l one finds point like particles with equal mass m which feel gravitational force towards the Earth. The kinetic and potential energies of such classical and discrete system with coupled pendulum rods read as [194]

$$T = \sum_n \frac{1}{2}\Theta\,\dot{\varphi}_n^2, \qquad V = \sum_n \frac{1}{2}\kappa(\varphi_{n+1} - \varphi_n)^2 + \sum_n mgl(1 - \cos\varphi_n), \tag{3}$$

with $\Theta = ml^2$ where φ is the angle of the twist from its equilibrium position ($\dot{\varphi}$ is its time derivative) and κ denotes the torsion coefficient. Let me note, φ is not a compact variable, thus, its value is not restricted into the regime $[-\pi, \pi]$ but can be arbitrary. The Lagrangian of the system $L = T - V$ is the following, [194]

$$\begin{aligned} L &= \sum_n \left[\frac{1}{2}\Theta\dot{\varphi}_n^2 - \frac{1}{2}\kappa(\varphi_{n+1} - \varphi_n)^2 - mgl(1 - \cos\varphi_n)\right] \\ &= \sum_n a\left[\frac{1}{2}\frac{\Theta}{a}\dot{\varphi}_n^2 - \frac{1}{2}\kappa a\left(\frac{\varphi_{n+1} - \varphi_n}{a}\right)^2 - \frac{m}{a}gl(1 - \cos\varphi_n)\right]. \end{aligned} \tag{4}$$

By introducing the mass-density $\mu = m/a$ and $\kappa' = \kappa a$ (using the fact that κ depends inversely on the length a, so, κ' is independent of a), one can take the continuous limit, i.e., $a \to 0$ which results in the Lagrangian [194]

$$L = \int dx \left[\frac{1}{2}(\mu l^2)\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}\kappa'\left(\frac{\partial\varphi}{\partial x}\right)^2 - \mu gl(1 - \cos\varphi)\right], \tag{5}$$

from which one can obtain the equation of motion, i.e., the Euler-Lagrangian equation which is a wave equation with the propagation velocity $v = \sqrt{\kappa'/(\mu l^2)}$. This Lagrangian can be easily extended to higher (spacial) dimensional cases and by introducing the variables $x_0 = vt$, $x_1 = x$, $x_2 = y$ and $x_3 = z, ...$, and rescaling the Lagrangian density ($\mathcal{L} \to \mathcal{L}/\kappa'$) which can be written as

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_0}\right)^2 - \sum_{j=1}^{d-1}\frac{1}{2}\left(\frac{\partial\varphi}{\partial x_j}\right)^2 - u(1 - \cos\varphi), \quad S[\varphi] = \int d^dx\mathcal{L}, \quad d^dx = dx_0dx_1...dx_{d-1} \tag{6}$$

with the action $S[\varphi]$ where, for the sake of simplicity, u denotes the prefactor of the periodic term, i.e., the Fourier amplitude. The covariant form of the above Lagrangian density can be taken by assuming that the propagation velocity in the wave equation is the speed of light, i.e., $v = c$. Thus, the corresponding relativistic (but not quantised) field theory and its equation of motion in $d = 1 + 1$ dimensions reads

$$\mathcal{L}_{\mathrm{SG}} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - u\,(1 - \cos\varphi), \qquad \partial_{x_0}^2\varphi - \partial_{x_1}^2\varphi + u\sin(\varphi) = 0, \tag{7}$$

which is the celebrated sine-Gordon model [28] and sine-Gordon equation where the field can be redefined by a rescaling in order to introduce the frequency β,

$$\mathcal{L}_{\mathrm{SG}} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{u}{\beta^2}\,(1 - \cos(\beta\varphi)). \tag{8}$$

The "name" of the former is related to the well-known Klein-Gordon model (free massive field theory),

$$\mathcal{L}_{\rm KG} = \frac{1}{2}\,(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{M^2}{2}\varphi^2 \tag{9}$$

and it is a field theoretical integrable model. The sine-Gordon model was know since the beginning of the previous century and was investigated in great detail. It is a non-linear wave-equation with a feature of having soliton and multisoliton solutions. The general solution of the sine-Gordon equation (7) can be obtained through the solution of the corresponding equation for the static case, $\varphi \equiv \varphi(x_1)$ by applying a Lorentz boost, $x_1 \to (x_1 - vx_0/c)/\sqrt{1-v^2/c^2}$, [29]. The 1-soliton (static) solution of (7) reads as, see for example [30]

$$\varphi_{\rm soliton} = 4\arctan\left(e^{\pm\sqrt{u}(x_1-x_c)}\right), \tag{10}$$

which is considered as a particle (excited state) localised at the point x_c which represents a stable configuration with a well defined energy. Since solitons are the solutions of non-linear wave equations, the superposition principle does not hold for them. The 1-soliton solution with a positive (negative) sign in the exponent called a *kink* (*antikink*) with the asymptotic conditions $\partial_{x_0}\varphi(x_0,\pm\infty) = 0$ and $\varphi(x_0,\pm\infty) = 0(\mathrm{mod}\ 2\pi)$. Thus, the vacuum states, related to these asymptotic cases are constant solutions with zero energy. In the kink configuration the field approaches these different asymptotic values in opposite directions in space, i.e., the kinks (antikinks) represent twists in the field variable which take the system from one asymptotic case $\varphi = 0$ to another $\varphi = 2N\pi$ and the difference is $2N\pi$, characterized by the integer N, called the kink number. The stability of the kinks is the consequence of the topology. There is a conservation law which corresponds to the stability of the kinks and the corresponding topological current j^μ $(\mu = 0,1)$:

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi(x_0,x_1) \tag{11}$$

(where $\epsilon_{\mu\nu}$ is antisymmetric, with $\epsilon_{01} = -\epsilon_{10} = 1$ and $\epsilon_{00} = \epsilon_{11} = 0$) is conserved in the semi-classical expansion: $\partial_\mu j^\mu = 0$. The conserved charge Q is equal to the kink number N:

$$Q = \int_{-\infty}^{\infty} \mathrm{d}x_1\; j^0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x_1\;\frac{\partial\varphi}{\partial x_1} = \frac{1}{2\pi}\left[\varphi(x_0,\infty) - \varphi(x_0,-\infty)\right] = N. \tag{12}$$

This conservation law can be violated in quantum theory, and the topological current can become anomalous.

2.2 The quantum sine-Gordon model

The quantisation and the study of properties of the quantised sine-Gordon model are the main goals of the present lecture note, thus, in this section I give a brief introduction to QFT with a special attention paid on the sine-Gordon scalar field theory. The quantisation of a classical field theoric model, such as the previously introduced sine-Gordon or Klein-Gordon scalar theories can be done in the traditional way in two steps: (i) the field φ and the canonically conjugated momenta $\pi = \partial\mathcal{L}/\partial\dot\varphi$ are considered operators over the Fock space, (ii) between them the following commutation relations are assumed,

$$[\hat\varphi(x_0,x),\hat\pi(x_0,x')] = i\hbar\delta(x-x'), \qquad [\hat\varphi(x_0,x),\hat\varphi(x_0,x')] = [\hat\pi(x_0,x),\hat\pi(x_0,x')] = 0, \tag{13}$$

where x denotes the set of all spacial coordinates and $\hbar = h/2\pi$ is the reduced Planck-constant. This is, however, not the most convenient way of obtaining the quantised field theory, at least, it is not suitable for calculating a cross section which is one of the key issue in high energy physics. Instead, I use the more reliable path-integral (Feynman or functional integral) formalism which is the standard choice in particle physics. It requires the introduction of the Euclidean spacetime metric,

$$x_0 = i(x_E)_0, \quad d^dx = i\,d^dx_E, \quad x_\mu x^\mu = -(x_E)_0^2 - \sum_{j=1}^{d-1}(x_E)_j^2, \quad \partial_\mu\partial^\mu = -\partial^2_{(x_E)_0} - \sum_{j=1}^{d-1}\partial^2_{(x_E)_j} \tag{14}$$

which assures the convergence of the path-integral used in the definition of the (Euclidean) partition function

$$Z_E[J] = \mathcal{N} \int \mathcal{D}\varphi \exp\left[-\frac{1}{\hbar}\int d^d x_E \left(\mathcal{L}_E + \hbar J\varphi\right)\right] \tag{15}$$

where J is the source term, $\mathcal{N}$ is an appropriately chosen normalization ($Z_E[J=0]=1$). The letter E denotes quantities which are calculated in the Euclidean metric, however, in the following I do not indicate it explicitly and this convention has been used along the lecture note. From Eq. (15) one can obtain observable quantities, but the presentation of the details of the path-integral method is not the aim of this lecture note, so, I just briefly summarise its main properties. For example, the path-integral in (15) can be evaluated if the Lagrangian density contains only quadratic terms of the field and its derivatives which is the case for the Klein-Gordon model (9) where the result reads as

$$Z_{\mathrm{KG}}[J] = \exp\left[\frac{\hbar}{2}\int d^d x\, d^d y\, J(x)\Delta(x-y)J(y)\right] \tag{16}$$

where $\Delta(x-y)$ is the solution of the differential equation $(-\partial_\mu\partial^\mu + M^2)\Delta(x) = \delta^4(x)$ and called as the scalar propagator (Green's function) which has a simple form in momentum space, $\Delta(k) = 1/(k^2+M^2)$. Let me now calculate the partition function of the φ^4 scalar theory (continuous Ising model) which contains a quartic self-interaction term in addition to the Klein-Gordon Lagrangian

$$\mathcal{L}_{\varphi^4} = \mathcal{L}_{\mathrm{KG}} + \mathcal{L}_{\mathrm{int}}, \qquad \mathcal{L}_{\mathrm{int}} = -\frac{g_4}{4!}\varphi^4. \tag{17}$$

Using the following expression

$$-\frac{\delta Z_{\mathrm{KG}}[J]}{\delta J(x)} = \mathcal{N}\int \mathcal{D}\varphi\ \varphi(x)\ \exp\left[-\frac{1}{\hbar}\int d^d x\left(\mathcal{L}_{\mathrm{KG}} + \hbar J\varphi\right)\right]. \tag{18}$$

the partition function of the φ^4 theory is written as

$$\begin{aligned} Z_{\varphi^4}[J] &= \mathcal{N}\int \mathcal{D}\varphi \exp\left[-\frac{1}{\hbar}\int d^d x \mathcal{L}_{\mathrm{int}}(\varphi)\right]\exp\left[-\frac{1}{\hbar}\int d^d x\left(\mathcal{L}_{\mathrm{KG}} + \hbar J\varphi\right)\right] \\ &= \mathcal{N}\exp\left[-\frac{1}{\hbar}\int d^d z \mathcal{L}_{\mathrm{int}}\left(\frac{\delta}{\delta J(z)}\right)\right] Z_{\mathrm{KG}}[J] \end{aligned} \tag{19}$$

which can be evaluated in the framework of perturbation theory using the Taylor expansion of the interaction exponential generating a series in terms of the coupling g_4. Let me note, the calculation can be done in a more convenient way by using Feynman rules, i.e., graphical representations. For example, if the scalar propagator $\Delta(x-y)$ is represented by a line, $\Delta(0) = \Delta(x-y)|_{x=y}$ is equivalent to a closed loop. This perturbative treatment of scattering amplitudes (or S-matrices which are operators mapping the incoming free particles to the free outgoing ones) leads to Feynman diagrams. Let me apply the above scenario for the sine-Gordon model (7) which requires the Taylor expansion of the interaction Lagrangian,

$$\mathcal{L}_{\mathrm{SG}} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - u(1-\cos\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + u\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\varphi^{2n} \tag{20}$$

however, in order to preserve the essential symmetry of the model one has to keep infinitely many interaction terms. Thus, if for any reason one has to truncate the summation, the perturbative treatment cannot be applied for the sine-Gordon theory.

Let me give an example for the case when every term of the Taylor expansion can be summed up. In particular the ϕ^4 theory is considered in $d = 1+1$, i.e., in a single spatial dimension [31]. As a first step, calculate the scattering amplitude of a $2 \to 4$ process (when we have 2 incoming and 4 outgoing particles). At the so-called tree-level, i.e., when no loops are included in the Feynman diagrams this is found to be a constant and proportional to $-g_4^2/M^2$. Thus, by adding $-\frac{1}{6!}\frac{g_4^2}{M^2}\varphi^6$

term to the original Lagrangian (17) it provides us a theory where the $2 \to 4$ process vanishes. Introducing $\beta^2 = g_4/M^2$ this new model reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{M^2}{\beta^2}\left[\frac{1}{2}\beta^2\varphi^2 + \frac{1}{4!}\beta^4\varphi^4 + \frac{1}{6!}\beta^6\varphi^6\right]. \tag{21}$$

Repeating this procedure and summing up all the monomials of the field, one finds that the scattering amplitudes of all $2 \to 2n$ processes vanish, i.e., there is no particle production at tree level in the following Lagrangian,

$$\mathcal{L}_{\mathrm{ShG}} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{M^2}{\beta^2}\left[\sum_{n=1}^{\infty}\frac{1}{(2n)!}\beta^{2n}\varphi^{2n}\right] = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{M^2}{\beta^2}\left[\cosh(\beta\varphi) - 1\right], \tag{22}$$

which is the so-called sinh-Gordon scalar theory and by the replacement $\beta \to i\beta$ it recovers the sine-Gordon model (in case of a periodic interaction the sign of the Fourier amplitude is irrelevant). The details of the calculation are given in [31] and it was also shown that no particle production is found in the sinh-Gordon theory model even beyond tree-level, i.e., at one-loop. Thus, it represents an example for the case where infinitely many terms are summed up and as a result, the sine-Gordon model is obtained.

2.3 Renormalization, Renormalization Group

Here, some properties of the renormalization and renormalization group are presented with a special attention on the sine-Gordon model but similarly to the previous subsection, leaving the details for the rest of the lecture note. Before I discuss the renormalization let us introduce the so called effective action $\Gamma[\varphi]$ which is related to the classical action

$$S[\varphi] = \int d^dx \mathcal{L}[\varphi]. \tag{23}$$

In order to find that relation, one has to define $\Gamma[\varphi]$ by the following Legendre transformation

$$\Gamma[\varphi] + W[J] - \int d^dx J\varphi = 0, \qquad \varphi(x) = \frac{\delta W}{\delta J(x)}, \qquad J(x) = \frac{\delta\Gamma}{\delta\varphi(x)}, \tag{24}$$

where $W[J] = \ln Z[J]$ is the generating functional of the so called connected Green's functions. In the previous section the partition function (15) of a scalar field theoric model was introduced which is considered as a generating functional, i.e., generator of the n-point Green's functions $G^{(n)}(x_1, ..., x_n)$,

$$G^{(n)}(x_1, ..., x_n) \equiv \frac{\int \mathcal{D}\varphi\ \varphi(x_1)\cdots\varphi(x_n)\exp\left[-\frac{1}{\hbar}S[\varphi]\right]}{\int \mathcal{D}\varphi\ \exp\left[-\frac{1}{\hbar}S[\varphi]\right]}, \quad G^{(n)}(x_1, ..., x_n) = \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)} \tag{25}$$

where $x_1, ..., x_n$ represent d-dimensional spacetime variables. Green's functions have great importance in QFT since scattering amplitudes of physical processes can be given by them. Moreover, Feynman diagrams can also be related to Green's functions. The connected Green's functions generated by $W[J]$ are related to connected components of Feynman diagrams. Let me come back to the effective action which generates one particle irreducible (1PI) correlation functions. Every measurable physical quantity (observables) can be derived from $\Gamma[\varphi]$ or can be expressed by 1PI correlation functions and quantum effects are incorporated in the effective action. Indeed, the relation between the classical and the (full quantum) effective action is given by the perturbation series of $\Gamma[\varphi]$ in terms of $\hbar$,

$$\Gamma[\varphi] = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \mathcal{O}(\hbar^3), \qquad \Gamma_0 = S[\varphi], \qquad \Gamma_1 = \frac{1}{2}\mathrm{Tr}\ln\left[\frac{\delta^2 S[\varphi]}{\delta\varphi^2}\right] \tag{26}$$

and the trace in Γ_1 is understood as integration over the momentum space. The angular part in the d-dimensional momentum integral can be evaluated and the remaining integral runs from zero to infinity which could be divergent with respect to its upper (UV – ultraviolet divergent) and lower (IR – infrared divergent) bound. In order to obtain reliable results for measurable quantities these divergencies/infinities should be handled somehow and the renormalization is the procedure which takes care of it. Similar divergencies appear in Feynman, and phase-space integrals. Due to the Kinoshita-Lee-Nauenberg theorem the Standard Model is IR safe (IR divergences from phase space and Feynman loop integrals cancel each other), but still in practice, an appropriately chosen subtraction method is required for numerical purposes. Thus, perturbation series taken either in the coupling constant of the model (e.g. g_4 in case of the φ^4 model) or in $\hbar$ require the renormalization of the model. The expansion in $\hbar$ up to the n-th order corresponds to Feynman diagrams with up to n loops. Series expansions in the coupling and in $\hbar$ are related to each other: higher order terms in the coupling requires higher loops in the corresponding diagrams.

The first step in renormalization is to introduce a regulator in order to observe the UV and IR divergences of momentum integrals. Various types of regulators are known: (i) momentum cutoff Λ (finite upper bound for the integral), (ii) lattice regularization with a lattice space a (discrete spacetime), (iii) Pauli-Villars regularization by introducing an auxiliary field with a mass (which suppresses the path-integral), (iv) dimensional regularization by the analytic continuation of the integrals to $d = 4 - \epsilon$ dimensions (where no divergences observed) etc. Of course, regulators are unphysical parameters, thus, they should be removed from the theory but before doing that one has to re-define (renormalize) the original parameters of the Lagrangian which are called "bare" couplings. The bare couplings such as u and β in the sine-Gordon model, or M^2 and g in the φ^4 polynomial model have no direct physical meanings, so, they can be redefined and they could be even infinite. For example, the physical mass (in the quantised theory) is not the one found in the Lagrangian but defined by the pole of the propagator which is the 2-point 1PI function. Therefore, in the second step of renormalization the bare couplings are changed by adding appropriately chosen counter terms to the action (same type of monomials of the field found in the original model) which keep the physical quantities finite even when the regularization parameters are removed from the theory, e.g. $\Lambda \to \infty$ or $\epsilon \to 0$. The new couplings are the so called renormalised ones.

Physical observables can be expressed as a function of bare couplings and the regularization parameter, i.e., $P(\Lambda, M_\mathrm{B}, g_{4,\mathrm{B}})$ which have finite values in the limit $\Lambda \to \infty$ and can also be expressed by renormalized parameters M_R and $g_{4,\mathrm{R}}$. However, as a consequence of the procedure a dependence on a new, artificial energy (momentum) scale k (usually denoted by μ) appears in the renormalized physical quantities, i.e., $P_\mathrm{R}(k, M_\mathrm{R}, g_{4,\mathrm{R}})$. Since the original bare quantities do not depend on k, the renormalized physical quantities are invariant against the change of the k-scale,

$$0 = k\frac{\partial}{\partial k}P(\Lambda, M_\mathrm{B}, g_{4,\mathrm{B}}) = k\frac{\partial}{\partial k}P_\mathrm{R}(k, M_\mathrm{R}, g_{4,\mathrm{R}}) \tag{27}$$

which results in differential equations (with the so called β-functions) for the couplings

$$k\frac{\partial g_{4,\mathrm{R}}}{\partial k} = \beta_{g_4}(g_4, M), \qquad k\frac{\partial M_\mathrm{R}^2}{\partial k} = \beta_M(g_4, M), \tag{28}$$

called renormalization group (RG) equations. It is important to note that the above scenario of renormalization has been done in the framework of perturbation theory, so, perturbative RG equations are obtained up to a given order of the coupling. As an example, one can mention the perturbative renormalization around the Gaussian fixed point ($M^2 = g_4 = 0$) of the 4-dimensional φ^4 model where the RG flow equation for the quartic coupling in the so-called minimal subtraction (MS) scheme, at 1-loop order reads as (see for example Eq. (10.52) of [34] or Eq. (45) of [33]),

$$1-\mathrm{loop}, \mathrm{MS}-\mathrm{scheme}: \qquad k\partial_k g_{4,k} = \frac{3}{16\pi^2}g_{4,k}^2 + \mathcal{O}(g_{4,k}^3), \quad \to \quad g_{4,k} = \frac{g_{4,\Lambda}}{1 - \frac{3g_{4,\Lambda}}{(4\pi)^2}\log\frac{k}{\Lambda}} \tag{29}$$

where its solution signals the appearance of the Landau pole at very high energies while in the low energy ($k \to 0$) limit the quartic coupling tends to zero. The dependence of the couplings on the energy scale predicted by RG equations was confirmed by experiments. For example, it was known from RG arguments that the value of the fine structure constant of QED scales logarithmically as the energy scale increased which was observed at the LEP accelerator: it was measured to be about 1/127 at 200 GeV, as opposed to its known value close to 1/137 at zero energy.

It was shown that RG equations emerge from the renormalization of the quantum field theories where one has to handle the problems of divergences, however, it is important to note that RG exists independently of the infinities. The physics behind RG equations is related to the requirement of quantisation and relativistic description at the same time which results in the so called vacuum polarisation effect which turns the parameters of the theory to be scale-dependent. A quantum field theoric model at zero-temperature is usually equivalent to a classical (not quantised) model of statistical physics at non-zero temperature. Quantum fluctuations in the former play the same role as thermal fluctuations in the latter case. Thus, the argument on the scale-dependence of the parameters is valid also in classical statistical physics. Indeed, the canonical partition function of a statistical model (canonical ensemble)

$$Z = \mathcal{N}\ \mathrm{Tr} \exp\left(-\frac{1}{k_{\mathrm{B}}T} H\right) \tag{30}$$

with a Hamiltonian H, can be related to the generating functional for the Green's function of a QFT. The Helmholtz free-energy $F[T] = -k_{\mathrm{B}} T \ln[Z]$ corresponds to $W[J]$ and by introducing the Gibbs free-energy (i.e. the internal or total energy) $U[S]$ as a function of the entropy S one finds the Legendre transformation

$$U[S] - F[T] = T\,S, \qquad T = \frac{dU[S]}{dS}, \qquad S = -\frac{dF[T]}{dT} \tag{31}$$

similar to that of written for the effective action in QFT. The RG equations found in QFT have the same importance in statistical physics because the thermodynamical limit cannot be performed in a reliable manner without taking into account the scale-dependence of the parameters of the microscopic theory.

2.4 Perturbative renormalization for the sine-Gordon model

Finally, let me note that a perturbative RG treatment is possible for the sine-Gordon model but as it was argued, not in the case when the potential is expanded in Taylor series and only finite terms are kept. In $d = 2$ dimensions, either one follows the scenario of [28, 32, 34, 33] when all terms of the Taylor expansion summed up and the perturbative renormalization can be performed, or by using the idea of an auxiliary mass term [35, 36].

In the first case the action for the SG model in $d = 2$ dimensions should be written as

$$\begin{aligned} S &= \int d^2x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{M^4}{\lambda}\left(1 - \cos\left(\frac{\sqrt{\lambda}}{M}\varphi\right)\right)\right] \\ &= \int d^2x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + M^2\frac{1}{4!}\frac{\lambda}{M^2}\varphi^4 - M^2\frac{1}{6!}\frac{\lambda^2}{M^4}\varphi^6 + \ldots\right]. \end{aligned} \tag{32}$$

and using the rescaling of the field $(\sqrt{\lambda}/M)\,\varphi \to \varphi$ and the spacetime $M\,x^\mu \to x^\mu$, it reduces to the usual definition of the sine-Gordon model. Considering the expanded form, one can treat it as a polynomial scalar field theory where the (UV) divergent graphs contains only a single vertex, i.e. tadpole and its trivial generalisations, see Fig. 1. This is because the scalar field is dimensionless in $d = 2$, so, canonical dimension of any vertex V is simply $\delta[V] = -2$, consequently the canonical dimension of a graph $\delta[\gamma] = 2 + \sum v_i \delta[V_i]$ (where v_i is the number of vertices) cannot be positive and can be zero only if it contains a single vertex. Using a normal ordered form with respect to the mass, no UV divergencies appear. This means that the potential term has to be modified in a following way

$$V = (M^2 + \delta M^2)\frac{M^2}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{M}\varphi\right)\right] = \left(1 + \frac{\delta M^2}{M^2}\right)\frac{M^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{M}\varphi\right)\right] \tag{33}$$

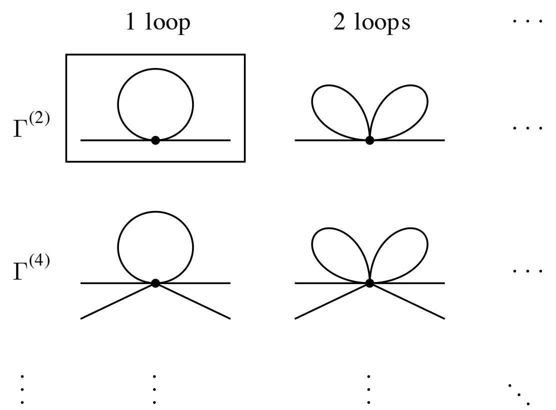

Figure 1: The only UV divergent graph for scalar fields in $d = 2$ dimensions is the one with a single vertex (see the boxed graph) and its trivial generalization (see the other graphs). Both λ and M^2 pick up divergent corrections but these corrections cancel in λ/M^2 thus only the mass term M^2 requires the renormalization as an overall prefactor.

where the counter term δM^2 is calculated at 1-loop order by the boxed graph of Fig. 1,

$$\begin{aligned}
\frac{\lambda}{2}\int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2+M^2} =& \frac{\lambda}{8\pi}\int dp^2 \frac{1}{p^2+M^2} = \frac{\lambda}{8\pi}\int_0^{\Lambda^2/k^2} d\left(\frac{p^2}{k^2}\right)\frac{1}{\frac{p^2}{k^2}+\frac{M^2}{k^2}} \\
=& \frac{\lambda}{8\pi}\left[-\log\left(\frac{M^2}{k^2}\right)+\log\left(\frac{\Lambda^2+M^2}{k^2}\right)\right] \\
\Longrightarrow 1-\text{loop}: \qquad & \delta M^2 = -\frac{\lambda}{8\pi}\log\left(\frac{\Lambda^2}{k^2}\right)
\end{aligned}$$

where k is an arbitrary momentum scale and Λ is the momentum cutoff. (Either $\Lambda \to \infty$ at fixed k or $k \to 0$ at fixed Λ). Substituting it into (33) one finds

$$\begin{aligned}
V^{1-\text{loop}} =& \left[1-\frac{\lambda}{M^2 8\pi}\log\left(\frac{\Lambda^2}{k^2}\right)\right]\frac{M^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{M}\varphi\right)\right] \\
=& \left[1+\log\left[\left(\frac{k}{\Lambda}\right)^{\frac{\lambda}{M^2 4\pi}}\right]\right]\frac{M^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{M}\varphi\right)\right]
\end{aligned} \tag{34}$$

thus, at 1-loop one finds $1+\log(k/\Lambda)^{\lambda/(M^2 4\pi)} \approx \exp[\log(k/\Lambda)^{\lambda/(M^2 4\pi)}]$ which becomes exact summing up all loop corrections, so, the RG running, $\beta^2 \equiv \frac{\lambda}{M^2}$, $u_k \equiv \frac{M^4}{\lambda}\left(\frac{k}{\Lambda}\right)^{\frac{\lambda}{M^2 4\pi}}$, gives the critical frequency $\beta_c^2 = 8\pi$. If β^2 larger or smaller then this critical value, the dimensionless Fourier amplitude $\tilde{u}_k = u_k k^{-2} \sim k^{\frac{\beta^2}{4\pi}-2}$ is either decreasing or increasing in the limit $k \to 0$, i.e., the model has two phases which is separated by $\beta_c^2 = 8\pi$. Thus the perturbative treatment produces the correct phase structure for the sine-Gordon model in $d = 2$ dimensions. Indeed, in Ref. [33] one finds a perturbative RG approach using the following parametrisation of the sine-Gordon theory

$$S[\phi] = \int d^2x\left[\frac{1}{2t}(\partial_\mu\phi)^2 - \frac{\alpha}{t}\cos(\phi)\right] \tag{35}$$

with Eqs. (159) and (160) of [33] where $\tilde{\alpha}_k = \tilde{u}_k\beta_k^2$ and $t_k = \beta_k^2$,

$$k\frac{d\tilde{\alpha}_k}{dk} = \tilde{\alpha}_k\left(\frac{t}{4\pi}-2\right), \qquad k\frac{dt_k}{dk} = \tilde{\alpha}_k^2\frac{t}{32\pi}, \tag{36}$$

$$\to \qquad k\frac{d\tilde{u}_k}{dk} = \tilde{u}_k\left(\frac{\beta_k^2}{4\pi}-2\right) - \tilde{u}_k^3\frac{\beta_k^4}{32\pi}, \qquad k\frac{d\beta_k^2}{dk} = \tilde{u}_k^2\frac{\beta_k^6}{32\pi}. \tag{37}$$

In this lecture note I perform the non-perturbative renormalization of the sine-Gordon model (and its generalizations) in the framework of the Functional Renormalization Group method.

3 Sine-Gordon type models, bosonization and conformal properties

Sine-Gordon (SG) type models considered in this lecture note are reviewed in this section putting emphasis on their symmetries and their phase diagrams. In addition, I show that some of the SG models are the co-called bosonised versions of two-dimensional fermionic and gauge theories which demonstrates their importance. Finally, I argue that SG models have received interest due to their conformal properties which are also summarised in this section.

3.1 Sine-Gordon type models and their symmetries

Symmetry considerations are important since together with the dimensionality, they can be used to determine the phase structure. Before going into the discussion of sine-Gordon models let me first analyse the the ϕ^{2n} polynomial scalar field theory defined by the Euclidean action

$$S_{\varphi^4}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{M^2}{2}\varphi^2 + \frac{g_4}{4!}\varphi^4\right] \tag{38}$$

which has the reflection (or Z_2) discrete symmetry. Depending on the dimensionality (above a lower critical dimension d_c) it undergoes a phase transition. At tree-level (when no quantum corrections are taken into account) the so called broken phase corresponds to $M^2 < 0$ where the potential has a double-well structure and the ground state of the system stays at one of these two minima (not at $\varphi = 0$), thus, the reflection symmetry has been broken spontaneously by the ground state of the system. It is known that the critical dimension of the Ising model is smaller than $d = 2$. However, if one extends the model replacing the single component field variable by an N-component vector, the discrete symmetry of the original Ising model is changed to a continuous $O(N)$ one which modifies the lower critical dimension. Indeed, in agreement to the Mermin-Wagner-Coleman theorem [37, 38, 39], the continuous symmetry of the extended model cannot be broken spontaneously in $d = 2$ dimensions, thus, the two-dimensional $O(N)$ theory has a single phase for $N > 2$ (the special case $N = 2$ is discussed later).

For dimensions $d > d_c$ (and below the upper critical dimension) the Ising model has two phases and known to undergo a so called second order phase transition which means that the second derivative of the thermodynamic potential becomes discontinuous near the transition point. In general, phase transitions can be classified according to the behaviour of the derivatives of the thermodynamic potential of the model, so one finds, for example, first, second and as special case (for the $O(N = 2)$ model), infinity order phase transitions.

3.1.1 Sine-Gordon model

The Euclidean action of the sine-Gordon (SG) model [28] contains a periodic self-interaction and reads as

$$S_{\rm SG}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + u\cos(\beta\varphi)\right]. \tag{39}$$

where u is the Fourier amplitude and β is the frequency. In addition to the reflection (Z_2) symmetry, the action of the SG model (39) under the transformation

$$\varphi(x) \to \varphi(x) + \frac{2\pi}{\beta} \tag{40}$$

remains unchanged, thus, it has another discrete symmetry: it is periodic in the field variable. Due to this additional symmetry one expects changes in the phase structure compared to the Ising model (38). Indeed, the SG model has two phases in $d = 2$ dimensions and it is known to undergo an infinity order (topological), i.e., Kosterlitz-Thouless-Berezinski (KTB) [40, 41] type phase transition which is controlled by the frequency, i.e., its critical value $\beta^2 = 8\pi$ separates the two phases. Let me note, both the periodicity and the reflection symmetry have been broken spontaneously in one of the phases of the model but this is in agreement to the Mermin-Wagner-Coleman theorem since they are discrete symmetries. Up till now the properties of the SG model has been discussed in $d = 2$ dimensions, however, it is interesting to study the dependence of its phase structure on the dimensionality which is also the goal of the present lecture note.

3.1.2 Massive Sine-Gordon model

The massive sine-Gordon (MSG) model [42] contains an explicit mass term in addition to the periodic self-interaction and its Euclidean action reads as

$$S_{\rm MSG}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}M^2\varphi^2 + u\cos(\beta\varphi)\right] \tag{41}$$

where the explicit mass term breaks the periodicity of the model. It has, however, a Z_2 symmetry, so one expects an Ising-type phase structure. This question is suitable to study in the framework of the FRG method which is presented here.

3.1.3 Layered Sine-Gordon model

Consider the extension of the original one-component SG model to a multi-component theory [42, 43] where the action is periodic in every field variable. In this case, an appropriately chosen mass term (mass matrix) can be used to study the consequences of a partial break down of the periodicity on the phase structure. I show that the number of the non-zero mass eigenvalues of the mass matrix determine whether the model undergoes a topological phase transition or not. Thus, the Euclidean action of the so called multi-layer or layered sine-Gordon (LSG) model in $d = 2$ dimensions reads as [9]

$$S_{\rm LSG}[\varphi] = \int \mathrm{d}^2 x \left[\frac{1}{2}(\partial_\mu \underline{\varphi})(\partial_\mu \underline{\varphi})^{\rm T} + \frac{1}{2}\underline{\varphi}\,\underline{\underline{M}}^2\,\underline{\varphi}^{\rm T} + \sum_{n=1}^{N} u_n \cos(\beta\varphi_n)\right] \tag{42}$$

with the $O(N)$ multiplet $\underline{\varphi} = (\varphi_1, \ldots, \varphi_N)$. The mass matrix couples the components of the field, i.e., layers and can be chosen arbitrarily. If all the mass-eigenvalues are non-zero than the periodicity of the LSG model (in every layers) has been broken by the mass term and one expects no infinite order phase transition. However, if one consider a mass matrix (let me call it magnetic-type interlayer interaction) [9, 14]

$$\frac{1}{2}\underline{\varphi}\;\underline{\underline{M}}^2_{\rm M-LSG}\;\underline{\varphi}^{\rm T} = \frac{1}{2}G\left(\sum_{n=1}^{N} a_n\varphi_n\right)^2,$$

$$\underline{\underline{M}}^2_{N=2} = \begin{pmatrix} G & -G \\ -G & G \end{pmatrix}, \quad \underline{\underline{M}}^2_{N=3} = \begin{pmatrix} G & -G & G \\ -G & G & -G \\ G & -G & G \end{pmatrix} \tag{43}$$

where the coupling strength between the layers denoted by G and $a_n = \pm 1$ are free parameters of the model, one finds a single non-vanishing mass-eigenvalue $M_N^2 = NG$ and an infinite order phase transition could be expected. (Based on symmetry considerations any choice with $a_n^2 = 1$ should reproduce exactly the same phase structure, as a consequence, the Fourier amplitudes (i.e. fugacities) $u_n \equiv u$ for $n = 1, 2, \ldots, N$.) The LSG model with magnetic type mass term interpolates between the SG and MSG models, since for $N = 1$ it reduces to the MSG model (41) and for $N \to \infty$ it is expected to recover the original SG theory (39) since the mass matrix has infinitely many zero and only single non-zero eigenvalue.

Another definition for the mass term of Eq.(42) (let me call it Josephson-type interlayer interaction) [9, 13]

$$
\frac{1}{2}\underline{\varphi}\ \underline{\underline{M}}^2_{\mathrm{J-LSG}}\ \underline{\varphi}^{\mathrm{T}} = \frac{1}{2}\sum_{n=1}^{N-1} J(\varphi_{n+1}-\varphi_n)^2,
$$

$$
\underline{\underline{M}}^2_{N=2} = \begin{pmatrix} J & -J \\ -J & J \end{pmatrix}, \quad \underline{\underline{M}}^2_{N=3} = \begin{pmatrix} J & -J & 0 \\ -J & 2J & -J \\ 0 & -J & J \end{pmatrix} \tag{44}
$$

has a single zero and $N-1$ non-zero mass-eigenvalues. Therefore, the Josephson coupled LSG model is invariant under the particular exchange of the layers $\varphi_n \leftrightarrow \varphi_{N-n+1}$, hence, $u_n \equiv u_{N-n+1}$.

In the latter case one finds a natural way to derive the mass matrix based on the discretised version of the 3D SG model which has the following action [8]

$$
S = \int d^3r \left[\frac{1}{2}(\partial_\mu\varphi_{3D})^2 + u_{3D}\cos(\beta_{3D}\varphi_{3D})\right], \tag{45}
$$

where $\varphi_{3D} \equiv \varphi_{3D}(x,y,z)$ is a one-component scalar field and β_{3D}, u_{3D} are the dimensionful parameters of the theory. The model is constructed in $d=3$ spatial dimensions with an Euclidean metric. The anisotropic 3D-SG model reads as

$$
S = \int d^3r \left[\frac{1}{2\beta_\parallel^2}[(\partial_x\theta)^2 + (\partial_y\theta)^2] + \frac{1}{2\beta_\perp^2}(\partial_z\theta)^2 + u_{3D}\cos(\theta)\right], \tag{46}
$$

where $\theta = \varphi_{3D}\beta_{3D}$ is introduced. In the isotropic limit $\beta_\parallel = \beta_\perp \equiv \beta_{3D}$ is assumed. Rescaling the field $\Phi = \theta/\beta_\parallel$, the action (46) becomes

$$
S = \int d^3r \left[\frac{1}{2}[(\partial_x\Phi)^2 + (\partial_y\Phi)^2] + \frac{\beta_\parallel^2}{2\beta_\perp^2}(\partial_z\Phi)^2 + u_{3D}\cos(\beta_\parallel\Phi)\right]. \tag{47}
$$

In case of very strong anisotropy, the continuous derivation and the integration in the z-direction is replaced by finite difference and summation, respectively,

$$
\partial_z\Phi(x,y,z) \to \frac{\Phi(x,y,z+s)-\Phi(x,y,z)}{s}, \qquad \int dz \to \sum_{z=1}^{N} s, \tag{48}
$$

where s is the interlayer distance. Using this discretisation, one arrives at the LSG model with N layers

$$
S = \int d^2r \left[\frac{1}{2}\sum_{i=1}^{N}(\partial\varphi_i)^2 + \frac{1}{2}J\sum_{i=1}^{N-1}(\varphi_{i+1}-\varphi_i)^2 + \sum_{i=1}^{N} u_i\cos(\beta\varphi_i)\right], \tag{49}
$$

where $\varphi_i(x,y) \equiv \sqrt{s}\Phi(x,y,z=is)$, $J \equiv \beta_\parallel^2/(\beta_\perp^2 s^2)$, $\beta \equiv \beta_\parallel/\sqrt{s}$ and $u \equiv su_{3D}$ are introduced. Therefore, in the continuum limit $N \to \infty$ the LSG model can be considered as the discretized version of the 3D-SG model and for $N=1$ the LSG model reduces to the 2D-SG model.

Finally, let me perform an $O(N)$ rotation of the layered models which diagonalises the mass matrix, consequently, the rotated models do not have interlayer interactions. Later I show that it enables us to perform an FRG analysis of the rotated models in the framework of the linearized FRG equations. The action of the rotated layered sine–Gordon model reads as [9]

$$
S_{\mathrm{rot}}[\underline{\alpha}] = \int \mathrm{d}^2r \left[\frac{1}{2}(\partial_\mu\underline{\alpha})(\partial_\mu\underline{\alpha})^{\mathrm{T}} + \frac{1}{2}\underline{\alpha}\,\underline{\underline{M}}^2_{\mathrm{rot}}\underline{\alpha}^{\mathrm{T}} + U_{\mathrm{rot}}(\alpha_1,...,\alpha_N)\right] \tag{50}
$$

with the rotated $O(N)$ multiplet $\underline{\alpha}^T = \underline{\underline{O}}^T\,\underline{\varphi}^T$ where $\underline{\underline{O}}$ represents the rotation which has the following form for the $N=3$-layers Josephson and magnetic LSG model

$$
\underline{\underline{O}}^T_{\mathrm{J-rot}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad \underline{\underline{O}}^T_{\mathrm{M-rot}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \tag{51}
$$

The general form of the rotated mass matrix for the Josephson and for the magnetically coupled LSG model are

$$\underline{\underline{M}}^2_{\mathrm{J-rot}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & M_2^2 & 0 & \cdots & 0 \\ 0 & 0 & M_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_N^2 \end{pmatrix} \qquad \underline{\underline{M}}^2_{\mathrm{M-rot}} = \begin{pmatrix} NG & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{52}$$

with the mass-eigenvalues M_i^2 for the Josephson coupled and $M^2 = NG$ for the magnetically coupled LSG model. In the rotated models the periodic part is no longer diagonal. For $N = 2$ layers this reads

$$U^{\mathrm{rot}}_{\mathrm{N=2}} = 2u \cos\left(\frac{\beta}{\sqrt{2}}\alpha_1\right) \cos\left(\frac{\beta}{\sqrt{2}}\alpha_2\right), \tag{53}$$

and for $N = 3$ the rotated periodic parts of the Josephson and of the magnetic LSG model are different,

$$\begin{aligned} U^{\mathrm{J-rot}}_{\mathrm{N=3}} =& u_2 \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{2\beta}{\sqrt{6}}\alpha_3\right) + 2u_1 \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \cos\left(\frac{\beta}{\sqrt{6}}\alpha_3\right) \\ &+ 2u_1 \sin\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \sin\left(\frac{\beta}{\sqrt{6}}\alpha_3\right) \\ &+ 2u_2 \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \sin\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \sin\left(\frac{\beta}{\sqrt{6}}\alpha_3\right), \end{aligned} \tag{54}$$

and

$$\begin{aligned} U^{\mathrm{M-rot}}_{\mathrm{N=3}} =& 2u \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \cos\left(\frac{\beta}{\sqrt{6}}\alpha_3\right) + u \cos\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{2\beta}{\sqrt{6}}\alpha_3\right) \\ &+ 2u \sin\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \cos\left(\frac{\beta}{\sqrt{2}}\alpha_2\right) \sin\left(\frac{\beta}{\sqrt{6}}\alpha_3\right) - u \sin\left(\frac{\beta}{\sqrt{3}}\alpha_1\right) \sin\left(\frac{2\beta}{\sqrt{6}}\alpha_3\right). \end{aligned} \tag{55}$$

3.1.4 Sinh-Gordon model

By using the analytic continuation of the frequency parameter of the original SG model to an imaginary one $\beta \to i\beta$, one finds the sinh-Gordon (ShG) model

$$S_{\mathrm{ShG}}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + u \cos(i\beta\varphi)\right] = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + u \cosh(\beta\varphi)\right]. \tag{56}$$

in which periodicity is lost but it has a Z_2 symmetry. Therefore, on the one hand one expects an Ising-type phase structure. However, even if the self-interaction term is Taylor expanded (which generates an Ising-type model) the expansion terms cannot be chosen arbitrarily, otherwise the $\cosh(\beta\varphi)$ is not recovered and this certainly modifies the phase diagram which has also been clarified in the present lecture note.

3.1.5 Shine- and Sn-Gordon models

Similarly to the LSG model where the mass matrix can be chosen to interpolate between the SG and MSG models, one can define an extension of the original SG theory where the resulting model interpolates between the previously introduced ShG and the original SG model. In the present lecture note I introduce and study two different classes of interpolating models.

The first class is built by considering the coupling constant β as a complex quantity which is the shine-Gordon (Shine) theory and defined by the action,

$$\begin{aligned} S_{\mathrm{Shine}}[\varphi] &= \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + u\, \mathrm{Re} \cos[(\beta_1 + i\beta_2)\varphi]\right] \\ &= \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + u \cos(\beta_1\varphi)\cosh(\beta_2\varphi)\right] \end{aligned} \tag{57}$$

where β_1 and β_2 are real value frequencies. Let me note, that the resulting theory can be treated for each non-zero β_2 as a scalar polynomial field theory.

In the second class, in order to construct an interpolating model first I introduce a periodic function written in terms of the Jacobi function [11],

$$S_{\mathrm{Sn}}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu\varphi)^2 + u \,\mathrm{sn}(\beta\varphi, m)\right] \tag{58}$$

which is termed as the *sn*-Gordon model which has the following limiting cases: for $m = 0$, it reduces to $u \sin(\beta\varphi)$ and for $m = 1$, it becomes $u \tanh(\beta\varphi)$. Using this idea, the interpolation between the sine-Gordon and sinh-Gordon models can be performed with potentials written in terms of Jacobi functions [11],

$$S_{\mathrm{SnG}}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu\varphi)^2 + u \,\mathrm{cd}(\beta\varphi, m)\,\mathrm{nd}(\beta\varphi, m)\right] \tag{59}$$

and term as the SnG model where the limiting cases are the required ones, i.e. *for m=0*: $u\cos(\beta\varphi)$ and *for m=1*: $u\cosh(\beta\varphi)$, thus m varies between zero and one. Let me note, that in this case the interpolating function is periodic (except for $m = 1$). Using the properties of the Jacobi functions $\mathrm{cd}(\varphi, m) = \mathrm{cn}(\varphi, m)/\mathrm{dn}(\varphi, m)$ and $\mathrm{nd}(\varphi, m) = 1/\mathrm{dn}(\varphi, m)$ it can also be written as [11],

$$S_{\mathrm{SnG}}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu\varphi)^2 + u \;\mathrm{cn}(\beta\varphi, m)\,[\mathrm{nd}(\beta\varphi, m)]^2\right]. \tag{60}$$

Eqs. (59) and (60) have periodic potential terms, so, they can be expanded in Fourier (Lambert) series

$$\begin{aligned}
\mathrm{cn}(\varphi, m) &= \frac{2\pi}{K\sqrt{m}} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos\left[(2n+1)\frac{\pi\varphi}{2K}\right], \\
\mathrm{nd}(\varphi, m) &= \frac{\pi}{2K\sqrt{1-m}} + \frac{2\pi}{K\sqrt{1-m}} \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^{2n}} \cos\left[2n\frac{\pi\varphi}{2K}\right],
\end{aligned}$$

where $q = \exp[-\pi K(1-m)/K(m)]$ and $K(m)$ is the quarter period (complete elliptic integral of the first kind) which can be expressed by the hypergeometric function

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\,\sin^2(\theta)}} = \frac{\pi}{2}\;{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, m\right)$$

which results in [11],

$$S_{\mathrm{SnG}}[\varphi] = \int d^d x \left[\frac{1}{2}(\partial_\mu\varphi)^2 + \sum_{n=1}^{\infty} u_n \cos(n\, b\, \varphi)\right], \qquad b = \frac{\beta}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, m\right)} \tag{61}$$

thus, the second class of interpolation models are expected to possess properties and phase structure similar to the SG model.

3.2 Bosonisation in $d = 2$ dimensions

It is an interesting property of sine-Gordon type models that some of them are the co-called bosonised versions of two-dimensional fermionic and gauge theories. The mapping of quantum field theories of interacting fermions onto an equivalent theory of interacting bosons called bosonization is well-established in the context of $d = (1+1)$ dimensional theories [47, 45, 42, 43, 44, 46, 48, 49]. The cornerstone of bosonization is the existence of a non-local transformation from local fermi fields to local Bose fields. The Lagrangian of a free massless Dirac fermion

$$\mathcal{L}_{\mathrm{fermi}} = \bar{\psi} i\gamma^\mu \partial_\mu \psi \tag{62}$$

is equivalent to the theory of a free massless scalar field

$$\mathcal{L}_{\rm Bose} = \frac{1}{2}\frac{1}{4\pi}(\partial_\mu \phi)^2, \tag{63}$$

and the fermion bilinears such as $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\psi$ can be expressed by bosonic degrees of freedom. The bosonization identities, which relate the fermionic current with the topological current of a bosonic theory is the consequence of a non-trivial current algebra.

3.2.1 Massive Thirring model

The massive Thirring (mT) model [50] is a theory of a single Dirac field defined in $d = (1+1)$ dimensions with dynamics determined by the Lagrangian density

$$\mathcal{L}_{\rm mT} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{2} g j^\mu j_\mu \tag{64}$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$ and g is a free parameter. Let us use the following identifications and conventions

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}, \qquad -\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi = j^\mu, \qquad u\cos(\phi) = -m\bar{\psi}\psi,$$
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{65}$$

and $\gamma^5 = \gamma^0\gamma^1$. Then, Eq. (64) can be mapped on the scalar theory

$$\mathcal{L} = \frac{1}{2}\frac{1}{4\pi}\left(1 + \frac{g}{\pi}\right)(\partial_\mu\phi)^2 + u\cos(\phi) \tag{66}$$

which is identical to the SG model (39) by performing the redefinition of the field variable $\phi = \beta\varphi$.

3.2.2 Massive Schwinger–Thirring model

The Lagrangian of QED_2 with a massive Dirac fermion is called the massive Schwinger (mS) model [51]. Its Lagrangian reads as

$$\mathcal{L}_{\rm mS} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{67}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using bosonization technique the fermionic theory (67) can be mapped onto an equivalent Bose form

$$\mathcal{L} = \mathcal{N}_M\left[\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{M^2}{2}\varphi^2 - cmM\cos\left(\sqrt{4\pi}\,\varphi - \theta\right)\right] \tag{68}$$

with $M^2 = e^2/\pi$, $c = \exp(\gamma)/(2\pi)$ where $\gamma = 0.5774$ is the Euler's constant, θ is the vacuum angle parameter, $\mathcal{N}_M$ denotes normal-ordering with respect to the boson mass M and φ is a one-component scalar field. The bosonized massive Schwinger model (68) can be considered as the specific form of the massive sine–Gordon (MSG) model whose Lagrangian density is written as (for $\beta^2 = 4\pi$) [9]

$$\mathcal{L}_{\rm MSG} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}M^2\varphi^2 + u\cos(\sqrt{4\pi}\varphi) \tag{69}$$

where the Fourier amplitude $u = e\,m\,\exp(\gamma)/(2\pi^{(3/2)})$ and the vacuum angle parameter has to be chosen as $\theta = \pm\pi$ for $u > 0$ and $\theta = 0$ for $u < 0$.

The Lagrangian of QED_2 with quartic self-interaction among massive Dirac fermions is called the massive Schwinger-Thirring (mST) model [52]. Its Lagrangian reads as

$$\mathcal{L}_{\rm mST} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g j^\mu j_\mu \tag{70}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $j^\mu = \bar{\psi}\gamma^\mu\psi$. Using bosonization technique the fermionic theory (70) can be mapped onto an equivalent Bose form which is an MSG model (41) where, $\frac{4\pi}{\beta^2} = 1 + \frac{g}{2\pi}$ and the Fourier amplitude is related to the fermion mass ($u \sim m$) and $M^2 = \frac{e^2}{\pi + g/2}$ [22].

3.2.3 Multi-flavor Massive Thirring model

The Lagrangian of massive Dirac fermions with SU(N_f) symmetry is called the N_f-flavor [or abelian SU(N)] massive Thirring (N-mT) model. Its Lagrangian reads as

$$\mathcal{L}_{\mathrm{N-mT}} = \sum_{n=1}^{N_f} \left[\bar{\psi}_n \left(i\gamma^\mu \partial_\mu - m\right) \psi_n\right] - \frac{1}{2} g^2 J^\mu J_\mu \tag{71}$$

where $J^\mu = \sum_{n=1}^{N_f} \bar{\psi}_n \gamma^\mu \psi_n$. The bosonized version of (71) for N_f=2 has the following form

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi_+\right)^2 + \frac{1}{2}\left(\partial_\mu \varphi_-\right)^2 + 2u\cos(\beta_+ \varphi_+)\cos(\sqrt{2\pi}\,\varphi_-) \tag{72}$$

where $\varphi_\pm = \frac{(\varphi_1 \pm \varphi_2)}{\sqrt{2}}$ and $\frac{2\pi}{\beta_+^2} = 1 + \frac{2g}{\pi}$. Let me notice that an O(2) rotation has been applied during the construction of (72). For $\beta_+^2 = 2\pi$ the unrotated form of Eq.(72) reads

$$\mathcal{L}_{2-\mathrm{SG}} = \frac{1}{2}\left(\partial_\mu \varphi_1\right)^2 + \frac{1}{2}\left(\partial_\mu \varphi_2\right)^2 + u[\cos(\sqrt{4\pi}\,\varphi_1) + \cos(\sqrt{4\pi}\,\varphi_2)] \tag{73}$$

with the unrotated frequency $\beta^2 = 4\pi$. (In this case the corresponding fermionic theory is non-interacting, i.e., $g = 0$.) This can be generalized for arbitrary N_f components

$$\mathcal{L}_{\mathrm{N-SG}} = \sum_{n=1}^{N_f} \frac{1}{2}\left(\partial_\mu \varphi_n\right)^2 + u \sum_{n=1}^{N_f} \cos(\sqrt{4\pi}\,\varphi_n). \tag{74}$$

3.2.4 Multi-flavor QED$_2$

The Lagrangian of QED$_2$ with massive N_f-flavor fermions is called the N_f-flavor (or multi-flavor) massive Schwinger (N_f-mS) model [44, 45, 42, 43, 46] and its Lagrangian reads as

$$\mathcal{L}_{\mathrm{N_f-mS}} = \sum_{n=1}^{N_f} \bar{\psi}_n \left(i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu\right) \psi_n - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{75}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using bosonization technique the fermionic theory (75) can be mapped onto an equivalent Bose form [44, 45, 42, 46, 43, 57, 58, 59, 60, 49, 53, 54, 55, 56, 61]

$$\mathcal{L} = \mathcal{N}_m \left[\sum_{n=1}^{N_f} \frac{1}{2}(\partial_\mu \varphi_n)^2 + \frac{M^2}{2}\left(\sum_{n=1}^{N_f} \varphi_n\right)^2 - cm^2 \sum_{n=1}^{N_f} \cos\left(\sqrt{4\pi}\,\varphi_n - \frac{\theta}{N_f}\right)\right] \tag{76}$$

with $M^2 = e^2/\pi$, $c = e^\gamma/(2\pi)$ where $\gamma = 0.5774$ is the Euler's constant, θ is the vacuum angle parameter, $\mathcal{N}_m$ denotes normal-ordering with respect to m and φ_n $n = 1, ..., N_f$ are one-component scalar fields. The bosonized N_f-flavor Schwinger model (76) can be considered as the specific form of the N-layer sine–Gordon model (42) with magnetic-type interlayer interaction (3.1.3) whose Lagrangian density is written as (for $\beta^2 = 4\pi$)

$$\mathcal{L}_{\mathrm{N-LSG}} = \sum_{n=1}^{N} \frac{1}{2}(\partial_\mu \varphi_n)^2 + \frac{1}{2} M^2 \left(\sum_{n=1}^{N} \varphi_n\right)^2 + u \sum_{n=1}^{N} \cos(\sqrt{4\pi}\varphi_n) \tag{77}$$

where the number of layers is identical to the number of flavors ($N = N_f$) and the Fourier amplitude related to the fermion mass ($u \sim m$) where the exact relation can be determined by using normal-ordering with respect to the boson mass. The vacuum angle parameter has to be chosen as $\theta = \pm N\pi$ for $u > 0$ and $\theta = 0$ for $u < 0$. One can generalise the multi-flavor QED$_2$ to the so-called multi-flavor massive Schwinger-Thirring model by adding a Thirring-type term to (75). The corresponding bosonic theory is a layered sine-Gordon model (77) but with an arbitrary frequency parameter β of the periodic self-interaction.

3.2.5 Multi-color QCD_2

It is tempting to find a bosonised form of QCD_2 in the same way I did for QED_2. It is indeed possible to arrive at a layered sine-Gordon model if one considers multi-color but single flavor QCD_2 [62, 63, 64]. The Hamiltonian of the QCD_2 with a single flavor $N_f = 1$ is

$$\mathcal{H} = g^2 \sum_{a,b=1}^{N_c} (E_a^b)^2 + \sum_{a,b=1}^{N_c} \bar{\psi}^a \gamma_1 (i\delta_a^b \partial_1 - A_a^b)\psi_b + m_g \sum_{a=1}^{N_c} \bar{\psi}^a \psi_a \tag{78}$$

in the gauge

$$A_0 = 0, \qquad A_b^a = 0 \text{ for } a = b, \qquad E_b^a = 0 \text{ for } a \neq b \tag{79}$$

Using the Gauss law the bosonized Lagrangian with one flavor becomes

$$\begin{aligned}\mathcal{L} = &\sum_a \left[\frac{1}{2}(\partial_\mu \phi_a)^2 - \frac{c m_g \mu}{\pi} \mathcal{N}_\mu \cos(2\sqrt{\pi}\phi_a)\right] + \frac{g^2}{8\pi N_c} \sum_{a,b} (\phi_a - \phi_b)^2 \\ &+ \frac{2c^2\mu^2}{\pi^{3/2}} \sum_{a,b} \frac{\sin(2\sqrt{\pi}(\phi_a - \phi_b))}{\phi_a - \phi_b},\end{aligned} \tag{80}$$

where the scale μ should satisfy $\mu = c'g$, with c' a constant, in order to take the interaction energy proportional to g^2 and $\mathcal{N}_\mu$ denotes the normal-ordering. The last term in (80) is non-periodic, hence, it can be expanded in Taylor series and keeping only the leading order terms one finds

$$\mathcal{L} = \sum_{a,b} \left[\frac{1}{2}(\partial_\mu \phi_a)^2 - \frac{cc' m_g g}{\pi} \mathcal{N}_\mu \cos(2\sqrt{\pi}\phi_a) + g^2 c_g (\phi_a - \phi_b)^2\right], \tag{81}$$

with $c_g = 1/8\pi N_c - 8c^2c'^2/3$. Let us note, that similarly to the bosonized QED_2, the scalar mass term in (81) can be written by a mass matrix $\mathcal{M}_{\mathrm{QCD}}$

$$\frac{1}{2}\Phi \mathcal{M}_{\mathrm{QCD}}^2 \Phi = g^2 c_g \sum_{a,b} (\phi_a - \phi_b)^2. \tag{82}$$

$\Phi = (\phi_1, \phi_2, ..., \phi_{N_c})$ and the summation runs from $a, b = 1$ to N_c. However, based on symmetry considerations the number of zero and non-zero eigenvalues of the mass matrix remains unchanged if $\mathcal{M}_{\mathrm{QCD}}$ is rewritten as

$$\frac{1}{2}\Phi \mathcal{M}_{\mathrm{QCD}}^2 \Phi = g^2 c_g \sum_{n=1}^{N_c - 1} (\phi_{n+1} - \phi_n)^2, \tag{83}$$

which reproduces exactly the same phase structure. Thus, one arrives at the layered sine-Gordon model (42) with Josephson-type interlayer interaction (3.1.3) where $N = N_c$.

3.3 Conformal field theory and sine-Gordon models

Statistical field theory has undergone an incredible improvement in the last four decades, most of this development has been due to two theoretical techniques: renormalization group (RG) and conformal field theory (CFT) which permitted a full understanding of the phase transition mechanism.

The RG approach has been originally introduced in hydrodynamics and later used to describe the critical phenomena of statistical systems near their phase transition point. The quantum field as well as the critical statistical system has infinitely many relevant (important) degrees of freedom; the quantum fluctuations and the thermal fluctuations, respectively. The $D + 1$ dimensional (D space-like and 1 time-like dimensional) QFT is (usually) equivalent to a statistical system taken in $D + 1$ space-like dimensions. Therefore, the understanding of the critical behavior of statistical systems close to their phase transitions provides us with a useful method to consider the phase structure of the equivalent QFT models. The cornerstone of RG is scale invariance: systems near their phase transition points are invariant under the global dilation of the observational scale (e.g. lattice space),

$$a \to b\ a. \tag{84}$$

More precisely, the fixed points of RG transformations correspond to phase transition points where the system is scale invariant. On the other hand, in $d = 1 + 1$ dimensions it is straightforward to extend the global dilatation symmetry to a local one,

$$a \to b(x)\ a \tag{85}$$

which changes the length of vectors but leave invariant their relative angles. These are called, the conformal transformations and they are part of a symmetry group, called conformal group. The conformal invariance has been demonstrated for two-dimensional field theories which allowed to obtain many, previously unknown, exact results and to complete the understanding of phase transition in two dimensions. The conformal group in d dimensions (for $d \neq 2$) has a number of independent generators equal to $\frac{1}{2}(d+1)(d+2)$, while for $d = 2$ the conformal group is infinite dimensional where the corresponding generators $L_n,\ n = 0, \pm 1, \pm 2, ...$ form a Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{86}$$

where the parameter c is the so-called central charge and can be used to characterise the corresponding CFT.

A bridge between CFT techniques and the RG description of field theories is provided in two dimensions by Zamolodchikov's c-theorem [65]. In particular the theorem states that it is always possible to construct a function of the couplings $c(g)$, the so-called c-function, which monotonically decreases when evaluated along the trajectory of the RG flow. Furthermore, at the fixed points this function assumes the same value as the central charge of the corresponding CFT,

$$c(g^\star) = c. \tag{87}$$

Based on the previous paragraph one can formulate the so-called sinh-Gordon puzzle which is the following. As a first step, let me discuss the conformal properties of the Ising (38), SG (39) and ShG (56) models. It is known that in $d = 2$ dimensions systems at criticality where they are scale invariant, give rise to invariance under the group of conformal transformations. As a consequence of conformal invariance the central charge c is well defined at any fixed points in the phase structure of the model and its difference Δc between the values at the trivial Gaussian and non-trivial fixed points characterises the theory. For example, it is known that $\Delta c = 1/2, 1, 1$ for the Ising, SG and ShG models respectively. The question to be addressed is: why the ShG model has $\Delta c = 1$ which is identical to the Δc of the SG model but differs from that of the Ising? This is unexpected since the sinh-Gordon model is not periodic, so, its self-interaction potential can be expanded in Taylor series which generates (ϕ^{2N}) terms and the model can be considered as an Ising type theory.

One of the goals of the present lecture note is to show how the functional RG method can be used to resolve the above "contradiction". Furthermore, if the functional RG treatment can clarify the issue above, there is a natural question to ask what happens if one consider models interpolating between the SG and ShG theories, i.e. shine- and sn-Gordon models.

4 Sine-Gordon models in condensed matter physics

Besides bosonisation which transforms two-dimensional fermionic and gauge models onto periodic scalar field theories various types of mappings of sine-Gordon type models can be discussed. For example, SG type generating functionals can be mapped onto partition functions of corresponding gas of topological defects (excitations) [66, 67]. Another example is the mappings between sine-Gordon theories and classical XY type spin models [68] which has a connection to the Ginzburg-Landau theory of superconductivity [69]. These equivalent models are known to undergo a topological (infinite order) i.e, Kosterlitz-Thouless-Berezinskii (KTB) type phase transition [40, 41]. In this section a brief overview of these mappings and the KTB phase transition of various models are given which sometimes depend on the dimensionality.

4.1 Ginzburg-Landau theory of superconductivity

In order to describe the phenomena of superconductivity one can use three different strategies. The most fundamental one is the microscopic description; in case of conventional superconductors this is the celebrated BCS theory. The second opportunity is the so-called Ginzburg–Landau (GL) model which can be derived from the microscopic theory. The third scenario is the electrodynamical description which is nothing but the equations of motion derived from the corresponding GL model. These three stages work for high transition temperature superconductors as well (e.g. the corresponding GL theory is the so-called Lawrence-Doniach model [70]), however, no well-accepted (single) microscopic model is available in the literature for high-T_c materials.

Here, I focus on the GL theory [69] which has been developed by applying a variational method to an assumed expansion of the free energy in powers of $|\psi|^2$ and $|\partial_\mu \psi|^2$ where ψ is a complex order parameter

$$\psi(r) = |\psi(r)| e^{i\theta(r)} \equiv \psi_0(r)\, e^{i\theta(r)} \tag{88}$$

(the inhomogeneous condensate of the superconducting electron pairs) and $|\psi|^2$ represents the local density of superconducting electron pairs (charged superfluid density). The total free energy has the form of a field theory

$$F = \int \mathrm{d}^3 r \left(\alpha |\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{2m_\star}\left| \left(\partial_\mu - \frac{i\, e_\star}{\hbar\, c} A\right)\psi\right|^2 + \frac{|B|^2}{8\pi}\right) \tag{89}$$

where α, β and $e_\star, m_\star$ are parameters, A is the electromagnetic vector potential and the last term stands for the magnetic field energy which does not depend on the material ($B = \nabla \times A$). In the absence of electromagnetic fields ($A \equiv 0$), the total free energy becomes

$$F = \int \mathrm{d}^3 r \left(\alpha \psi_0^2 + \frac{\beta}{2}\psi_0^4 + \frac{\hbar^2}{2m_\star}\left[(\partial_\mu \psi_0)^2 + \psi_0^2 (\partial_\mu \theta)^2\right]\right) \tag{90}$$

where the functional form of the order parameter has been substituted.

4.1.1 Layered GL theory – Josephson coupling

Let me first discuss the free energy functional in the absence of electromagnetic fields (uncharged superfluid). Since a strong spatial anisotropy is a typical property of high T_c materials the free energy functional should be discretised in one of the spatial direction which results in a layered structure [71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84]). Another important assumption is the so-called London-limit which requires that the superconducting state is homogenous in every layer, i.e., $\psi_0(r) \equiv \psi_0$ is constant (does not depend on the coordinate). Let me first use the London limit, then Eq. (90) reads as (in natural units: $\hbar = c = \epsilon_0 = 1$)

$$F = \frac{\psi_0^2}{2m_\star} \int \mathrm{d}^3 r (\partial_\mu \theta)^2 \tag{91}$$

and then apply the discertization of the z-coordinate which results in [81]

$$F = s\psi_0^2 \int \mathrm{d}^2 r \left(\sum_{n=1}^{N} \frac{1}{2m_{ab}} (\partial_\mu \theta_n)^2 + \sum_{n=1}^{N-1} \frac{1}{2m_c} \frac{(\theta_{n+1} - \theta_n)^2}{s^2} \right). \tag{92}$$

Here, m_{ab} and m_c represent the intralayer and interlayer effective masses, s is the interlayer distance, and N stands for the total number of layers. The gradient operator ∂_μ is two-dimensional where μ covers the spatial coordinates $\mu = x, y$.

The layered structure has important consequences on the phase structure. The elementary excitations are conducting electrons in the 3d bulk model but vortex-antivortex pairs (supercurrent rings with normal core) in the layered system, see Fig. 2. Furthermore, the phase structure

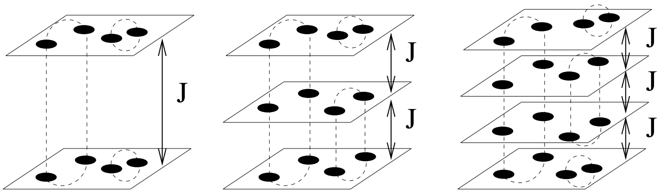

Figure 2: Schematic representation of the Lawrence-Doniach model with $N = 2, 3, 4$ layers which can describe the vortex properties of layered superconductors. The planes are coupled by the Josephson coupling $J \sim 1/m_c$. The solid discs represent the topological excitations of the model, the vortex-antivortex pairs. Two such pairs belonging to neighbouring layers can form vortex loops and rings due to weak Josephson coupling. The critical behaviour of the vortices is found to depend on the number of layers and is again different in the limit of an infinite number of layers.

and the vortex dynamics depend on the number of layers. For a single layer $N = 1$, (92) reduces to

$$F = \frac{\psi_0^2}{2m_\star} \int \mathrm{d}^2 r (\partial_\mu \theta)^2 \tag{93}$$

which is known to undergo a topological or also known as KTB type phase transition (strictly speaking if no spin-wave fluctuations are taken into account). In the molecular phase the vortices-antivortices form closely bound pairs, in the ionised phase they dissociate into a neutral plasma. Let me note, that this situation is related to an uncharged superfluid. A realistic description of vortex dynamics in a single superconducting layer requires the incorporation of electromagnetic field (charged superfluid) which will be discussed later. For a finite number of layers $1 < N < \infty$ the Josephson coupling modifies the phase structure and for infinite number of layers $N = \infty$ one expects that Eq. (92) recovers the phase structure of the bulk model.

It is constructive to show that Eq. (92) can also be obtained by the discretised version of (89) (in the absence of external fields) which is the Lawrence–Doniach model,

$$F = s \int \mathrm{d}^2 r \left(\sum_{n=1}^{N} \left(\alpha |\psi_n|^2 + \frac{\beta}{2} |\psi_n|^4 + \frac{|\partial_\mu \psi_n|^2}{2\, m_{ab}} \right) + \sum_{n=1}^{N-1} \frac{|\psi_{n+1} - \psi_n|^2}{2\, m_c\, s^2} \right), \tag{94}$$

and then taking it in the London-limit by introducing a complex layer-dependent order parameter as $\psi_n(r) = \psi_{0,n}(r) \exp[\mathrm{i}\theta_n(r)]$ with real $\psi_{0,n}(r)$, where the $\theta_n \in [0, 2\pi)$ are compact variables,

$$\begin{aligned} F = s \int \mathrm{d}^2 r \Bigg(& \sum_{n=1}^{N} \alpha \psi_{0,n}^2 + \frac{\beta}{2} \psi_{0,n}^4 + \frac{1}{2m_{a,b}} \left[(\partial_\mu \psi_{0,n})^2 + \psi_{0,n}^2 (\partial_\mu \theta_n)^2 \right] \\ & + \sum_{n=1}^{N-1} \frac{1}{s^2\, 2m_c} (\psi_{0,n+1}^2 + \psi_{0,n}^2) - \frac{1}{s^2\, m_c} \psi_{0,n+1} \psi_{0,n} \cos(\theta_{n+1} - \theta_n) \Bigg) \end{aligned} \tag{95}$$

and in the London approximation the moduli $\psi_{0,n}$ are assumed to be constant and identical in every layer (i.e. $\psi_{0,n}(r) = \psi_0$) which results in

$$F = s\psi_0^2 \int \mathrm{d}^2 r \left(\sum_{n=1}^{N} \frac{1}{2m_{ab}} (\partial_\mu \theta_n)^2 + \sum_{n=1}^{N-1} \frac{1}{s^2\, m_c} \left[1 - \cos(\theta_{n+1} - \theta_n)\right] \right) \tag{96}$$

and recovers Eq. (92) after expanding the cosine in Taylor series and keeping the quadratic terms only. This is the London-type form of the Lawrence-Doniach model [70] where the interaction between the compact fields θ_n of various layers is represented by the so-called Josephson coupling.

4.1.2 Layered GL theory – Magnetic coupling

Let me now turn to the analysis of the GL free energy in the presence of electromagnetic fields [80, 82, 84]. The London-type approximation of (90) in case of a non-vanishing vector potential A reads as,

$$F = \frac{\psi_0^2}{2m_\star} \int \mathrm{d}^3 r (\partial_\mu \theta - e_\star A)^2 \tag{97}$$

with a compact field θ. Strong anisotropy can be taken into account by discretising (97) in one spatial dimension,

$$\frac{1}{m_c} \partial_z \theta(x,y,z) \to \frac{1}{m_c} \frac{\theta(x,y,z+s) - \theta(x,y,z)}{s}, \qquad \int dz \to \sum_{z=1}^{N} s, \tag{98}$$

where $1/m_c$ is the Josephson coupling between the layers (m_c is the effective mass) which vanishes in the limit of infinite anisotropy ($m_c = \infty$) and leads to

$$F = s\psi_0^2 \int \mathrm{d}^2 r \left(\sum_{n=1}^{N} \frac{1}{2m_{ab}} (\partial_\mu \theta_n - e_\star A)^2 \right) \tag{99}$$

where the coupling between the layers is mediated by the vector potential A which represents a magnetic-type coupling between the vortices (antivortices) of each layers, see Fig. 3. The vortex

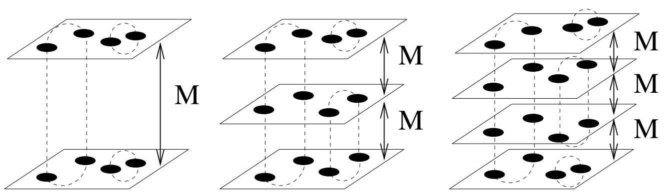

Figure 3: Schematic representation of the vortex properties of layered superconductors where the Josephson coupling vanishes $J \sim 1/m_c = 0$ and the vortices (antivortices) of each layers [77, 82] are coupled by magnetic-type coupling (99).

dynamics of the magnetically coupled model (99) depends on the number of layers (similarly to the Josephson coupled case). For $N = 1$ Eq. (99) reads as

$$F = \frac{\psi_0^2}{2m_\star} \int \mathrm{d}^2 r (\partial_\mu \theta - e_\star A)^2 \tag{100}$$

which is known to describe a real two-dimensional superconductor, i.e., charged superfluid. Due to the presence of the electromagnetic field, no KTB type phase transition is observed. For finite number of layers $1 < N < \infty$, the screening effect of A is partial and a model undergoes a KTB phase transition where the transition temperature depends on the number of layers. For $N = \infty$ the effect of the electromagnetic field can be neglected, thus (99) recovers (93).

In order to show how the layered GL theory with Josephson and magnetic couplings is related to various sine-Gordon type scalar field theories one has to determine the so-called gases of topological excitations which is my goal in the upcoming subsections.

4.2 Models in $d = 2$ dimensions

4.2.1 Uncharged superfluid

My starting point is the partition function of the 2d–SG model which reads as ($\hbar = 1$)

$$Z_{\rm 2d-SG} = \mathcal{N} \int \mathcal{D}[\varphi] \exp\left[- \int {\rm d}^2 r \, \left(\frac{1}{2} \left(\partial_\mu \varphi \right)^2 + u \cos(\beta\,\varphi) \right) \right] \tag{101}$$

where $\varphi \in [-\infty, \infty]$ is a one-component scalar field, u is a fundamental Fourier amplitude, and β is a dimensionless frequency. The partition function (101) can be identically rewritten as the partition function of the equivalent gas of topological excitations using the following steps [66]. One expands the periodic piece of the partition function (101) in a Taylor series and use the identity, $\cos(\beta\varphi) = [\exp{(i\beta\varphi)} + \exp{(-i\beta\varphi)}]/2$,

$$\exp\left[\int {\rm d}^2 r \; u \cos(\beta\varphi) \right] = \sum_{\nu_+, \nu_- = 0}^{\infty} \frac{(u/2)^{\nu_+} (u/2)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_+ + \nu_-} \int {\rm d}^2 r_j \right) \exp\left[i \rho_j \beta \varphi(r_j) \right], \tag{102}$$

where we introduced the integer valued topological charges $\rho_j = \pm 1$ which fulfil the neutrality condition $\sum_{j=1}^{\nu_+ + \nu_-} \rho_j = 0$. This leads to the intermediate result,

$$Z = \mathcal{N} \sum_{\nu_+, \nu_- = 0}^{\infty} \frac{(u/2)^{\nu_+} (u/2)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_+ + \nu_-} \int {\rm d}^2 r_j \right) \int \mathcal{D}[\varphi] \exp\left[- \int {\rm d}^2 r \left(\frac{1}{2} \varphi \left(-\partial^2 \right) \varphi + i \, \beta \, \rho \, \varphi \right) \right], \tag{103}$$

where $\partial^2 \equiv \partial_\mu \partial_\mu$ and $\rho(r) = -\rho_j \delta(r - r_j)$. I have thus placed the $(\nu_+ + \nu_-)$ vortices onto a two-dimensional plane (single layer). The Gaussian integration in Eq. (103) can now be performed easily, and the inversion of $-\partial^2$ can be accomplished by going to momentum space. Via a subsequent back-transformation to coordinate space and using $\sum_{i,j} = 2 \sum_{i<j}$ I finally arrive at the result

$$Z_{\rm 2d-SG} = \sum_{\nu_+, \nu_- = 0}^{\infty} \frac{(u/2)^{\nu_+} (u/2)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_+ + \nu_-} \int {\rm d}^2 r_j \right) \exp\left[\frac{\beta^2}{(2\pi)} \sum_{j<k=1}^{\nu_+ + \nu_-} \rho_j \rho_k \ln\left(\frac{r_{jk}}{a} \right) \right],$$

which is equivalent to the two-dimensional Coulomb gas (2d-CG)

$$Z_{\rm 2d-CG} = \sum_{\nu_+, \nu_- = 0}^{\infty} \frac{(z_+)^{\nu_+} (z_-)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_+ + \nu_-} \int {\rm d}^2 r_j \right) \exp\left[\frac{1}{k_{\rm B} T} \sum_{j<k=1}^{\nu_+ + \nu_-} \rho_j \rho_k \ln\left(\frac{r_{jk}}{a} \right) \right], \tag{104}$$

where $\rho_\alpha = \pm 1$ is the charge of the jth particle, and require the neutrality $z_+ = z_- = z$ is the dimensionful fugacity, $k_{\rm B}$ is the Boltzmann constant, T is the temperature and a stands for the lattice spacing which serves as a short-distance cutoff. The interaction potential between two charges depends on their relative distance ($r_{ij} = |\vec{r}_i - \vec{r}_j|$). The frequency parameter β^2 of the 2d–SG model can be identified as the inverse of the temperature of the equivalent Coulomb gas, $\beta^2 \equiv 2\pi/(k_{\rm B}T)$ and the Fourier amplitude plays the role of the fugacity, $u/2 \sim z$. The above procedure could be repeated in any dimension, therefore the equivalency between the SG and CG models holds for arbitrary dimensions as well.

It is generally assumed that the 2d–SG model belongs to the universality class of the two-dimensional classical XY spin-model [85] which is given by the partition function

$$Z_{\rm 2d-XY} = \int \mathcal{D}[\mathbf{S}] \, \delta(\mathbf{S}^{\,2} - 1) \, \exp\left[-\frac{1}{k_{\rm B} T} \sum_{<x,y>} (-J) \, \mathbf{S}_x \cdot \mathbf{S}_y \right], \tag{105}$$

where the classical spin $\mathbf{S}$ is a unit-vector in the two-dimensional internal space; $\sum_{<x,y>}$ stands for the sum over pairs of nearest neighbour lattice sites. Representing the classical unit spin vector by an angle $\mathbf{S}_x \equiv (\cos(\theta_x), \sin(\theta_x))$ the partition function of the 2D–XY model can be written as

$$Z_{2\mathrm{d-XY}} = \int \mathcal{D}[\theta] \exp\left[\frac{J}{k_\mathrm{B}T} \sum_{<x,y>} \cos(\theta_x - \theta_y)\right] \approx \mathcal{N} \int \mathcal{D}[\theta] \exp\left[-\int \mathrm{d}^2 r \left(\frac{J}{2k_\mathrm{B}T} (\partial_\mu \theta)^2\right)\right], \tag{106}$$

where the cosine is Taylor-expanded and the quadratic term generates $(\partial_\mu\theta)^2$ in the continuum limit and the higher order derivatives are neglected. This approximation works for the symmetry broken phase. The field-independent term has been built in the normalization constant $\mathcal{N}$. Thus, Eq.(106) is found to be the corresponding field theory of the XY spin model. Let me note, however, that the Hubbard-Strotonovich transformation is used to map the XY spin model onto an $O(2)$ symmetric φ^4 field theory exactly [86, 15], i.e, it is equivalent with a complex scalar theory where density fluctuations are taken into account which is not discussed here.

The structure of Eq.(106) is similar to Eq.(101) but there is an important difference, namely $\theta \in [0, 2\pi]$ is a compact variable. In case of the 2d–XY model, the compact nature of the field generates the topological excitations of the theory, which are the point-like vortices. In case of the 2d–SG model, the periodic self-interaction is responsible for the existence of the topological defects which are solitons.

Let me note, the partition function (106) is equivalent to the GL model (93) which describes a 2d superconducting film in the absence of electromagnetic fields taken in the London-type approximation (with $J \equiv \psi_0^2/m_\mathrm{ab}$). It is used to describe the vortex dynamics of an uncharged superfluid. Moreover, the 2d–XY model can also be mapped onto the 2d–CG. To show this the field variable is rewritten

$$\theta(r) = \theta_v(r) + \theta_\mathrm{sw}(r) \tag{107}$$

in terms of vortex (θ_v) and spin-wave (θ_sw) terms with the following properties

$$\int_V \partial_\mu \partial^\mu \theta_v = \int_{\partial V} \partial_\mu \theta_v = 2\pi \sum_i q_i, \qquad \int_V \partial_\mu \partial^\mu \theta_\mathrm{sw} = \int_{\partial V} \partial_\mu \theta_\mathrm{sw} = 0, \tag{108}$$

where the integrals are independent of the two-dimensional volume (V) and its surface (∂V) which is a closed contour in $d = 2$. The integer valued variable q_i is the so-called vortex charge (vorticity or winding number). Based on the analogy to electrostatic (where θ_v plays the role of the scalar electric potential in $d = 2$) it is easy to show that

$$\partial_\mu \partial^\mu \theta_v(r) = 2\pi\rho, \qquad \rho = \sum_i q_i \delta(r - r_i) \quad \to \quad \theta_v(r) = \sum_i q_i \ln\left(\frac{r - r_i}{a}\right). \tag{109}$$

The action for the continuous XY model can be rewritten in three terms

$$S = -\frac{J}{2k_\mathrm{B}T}\left[\int \mathrm{d}^2 r (\partial_\mu \theta_v)^2 + \int \mathrm{d}^2 r (\partial_\mu \theta_\mathrm{sw})^2 + 2\int \mathrm{d}^2 r (\partial_\mu \theta_v)(\partial_\mu \theta_\mathrm{sw})\right] \tag{110}$$

in which the last term is zero. If one neglect the second term (i.e. the spin-wave fluctuation) than the partition function of the model form a Coulomb gas.

Therefore, the continuous version of the 2d–XY (without spin-wave fluctuation) and the 2d–SG model are dual to each other. Indeed, the two models can be mapped onto each other by a suitable duality relation based on the Gaussian integration which inverts the coupling of the derivative term ($\beta^2 \sim J/T$).

4.2.2 Charged superfluid

Let me repeat the previous calculation for the two-dimensional massive sine-Gordon (2d–MSG) model

$$Z_{2\mathrm{d-MSG}} = \mathcal{N} \int \mathcal{D}[\varphi] \exp\left[-\int \mathrm{d}^2 r \left(\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} M^2 \varphi^2 + u \cos(\beta\,\varphi)\right)\right]. \tag{111}$$

The partition function (111) can be identically rewritten as the partition function of the equivalent gas of topological excitations

$$Z_{\rm 2d-MSG} = \sum_{\nu_+,\nu_-=0}^{\infty} \frac{(u/2)^{\nu_+}(u/2)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_++\nu_-} \int {\rm d}^2 r_j\right) \exp\left[-\beta^2 \sum_{j<k=1}^{\nu_++\nu_-} \rho_j \rho_k A_{jk}\right], \tag{112}$$

where the interaction potential between the vortices reads as [14],

$$A_{\alpha\,\gamma} = \frac{1}{2\pi}\left[K_0\left(\frac{r_{\alpha\gamma}}{\lambda_{\rm eff}}\right) - K_0\left(\frac{a}{\lambda_{\rm eff}}\right)\right] = \begin{cases} -\frac{1}{2\pi}\ln\left(\frac{r_{\alpha\gamma}}{a}\right) & (r_{\alpha\gamma} \ll \lambda_{\rm eff}) \\ \frac{1}{2\pi}\ln\left(\frac{\lambda_{\rm eff}}{a}\right) & (r_{\alpha\gamma} \gg \lambda_{\rm eff}) \end{cases} \tag{113}$$

with $r_{\alpha\gamma} = |\vec{r}_\alpha - \vec{r}_\gamma|$ and $K_0(r)$ stands for the modified Bessel function of the second kind, a is the lattice spacing which serves as an UV cutoff and an effective screening length $\lambda_{\rm eff}$ is introduced [87, 81, 88] which is related inversely to the mass $\lambda_{\rm eff}^{-1} = M$. The relation $K_0(r) = -\ln(r) + \ln 2 - \gamma_{\rm E} + \mathcal{O}(r)$ has been used in the derivation of the asymptotic short- and long-range forms and only the leading logarithmic terms are indicated ($\gamma_{\rm E} = 0.577216\ldots$ is Euler's constant). Thus, the 2d-MSG model is equivalent to the 2d Yukawa gas [87] and to the so-called 2d frustrated XY model which is identical to Eq. (100) and responsible for the description of the vortex dynamics for a real superconducting film (charged superfluid) [71, 72].

4.3 Models in $d = 3$ dimensions

The partition function of the three-dimensional sine–Gordon (3d–SG) model is

$$Z_{\rm 3d-SG} = \mathcal{N} \int \mathcal{D}[\varphi] \exp\left[-\int {\rm d}^3 r\, \left(\frac{1}{2}\,(\partial_\mu\varphi)^2 + u\cos(\beta\,\varphi)\right)\right] \tag{114}$$

where $\varphi \in [-\infty, \infty]$ is a one-component scalar field, $(\partial_\mu\varphi)^2 \equiv \sum_{\mu=1}^{3}(\partial_\mu\varphi)^2$. The partition function of the equivalent gas of topological excitations reads as

$$Z_{\rm 3d-SG} = \sum_{\nu_+,\nu_-=0}^{\infty} \frac{(u/2)^{\nu_+}(u/2)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_++\nu_-} \int {\rm d}^3 r_j\right) \exp\left[\frac{\beta^2}{2(\Omega_3)} \sum_{j<k=1}^{\nu_++\nu_-} \sigma(r_j)\,\frac{1}{r_{jk}}\,\sigma(r_k)\right],$$

where Ω_3 is the three-dimensional solid angle. Since the equivalence of the sine–Gordon field theory and the Coulomb gas holds in arbitrary dimensions, the partition function (115) is equivalent to the partition function of the 3d-CG

$$Z_{\rm 3d-CG} = \sum_{\nu_+,\nu_-=0}^{\infty} \frac{(z_+)^{\nu_+}(z_-)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_++\nu_-} \int {\rm d}^3 r_j\right) \exp\left[\frac{1}{2k_{\rm B}T} \sum_{j<k=1}^{\nu_++\nu_-} \sigma(r_j)\,\frac{1}{r_{jk}}\,\sigma(r_k)\right] \tag{115}$$

where $\sigma_\alpha = \pm 1$, z is the dimensionful fugacity, T is the dimensionful temperature and the interaction potential between two point-like charges depends on their relative distance ($r_{\alpha\beta} = |\vec{r}_\alpha - \vec{r}_\gamma|$). The dimensionful frequency parameter β^2 of the 3d–SG model can be identified as the inverse of the dimensionful temperature of the equivalent 3d–CG, $\beta^2 \equiv \Omega_3/(k_{\rm B}T)$ and again the Fourier amplitude plays the role of the fugacity, $u/2 \sim z$.

The 2d–SG and the 2d–XY model belong to the same universality class, however, this is not necessary true for the 3d counterparts since the topological defects of the XY model are point-like objects in $d = 2$ but $d-1$ surfaces in higher dimensions, e.g. vortex lines (or loops) in three dimensions. The partition function of the 3d–XY model taken in the continuum limit reads as

$$\begin{aligned} Z_{\rm 3d-XY} &= \int \mathcal{D}[\theta] \exp\left[\frac{J}{k_{\rm B}T} \sum_{<x,y>} \cos\left(\theta_x - \theta_y\right)\right] \\ &\approx \mathcal{N} \int \mathcal{D}[\theta] \exp\left[-\int {\rm d}^3 r\, \left(\frac{J}{2k_{\rm B}T}\,(\partial_\mu\theta)^2\right)\right], \end{aligned} \tag{116}$$

where $\theta \in [0, 2\pi]$. Let us note, Eq.(116) is equivalent to the 3d GL theory of superconductivity [69, 81, 89, 90] taken in the London-type approximation in the absence of electromagnetic fields (91). The partition function of the corresponding gas of topological excitations (vortex-loop gas) is [85, 91, 92]

$$
\begin{aligned}
Z_{\mathrm{3d-VLG}} = & \sum_{\nu_+,\nu_-=0}^{\infty} \frac{1}{(\nu_+)!(\nu_-)!} \left(\prod_{L=1}^{\nu_+ + \nu_-} \int d^3 r_L \; z^{(L)} \right) \\
& \exp\left[\frac{J\pi}{2k_{\mathrm{B}}T} \sum_{L,L'} \sum_{\alpha,\gamma} j_\mu^{(L)}(r_\alpha) \; U(r_{\alpha\gamma}) \; j_\mu^{(L')}(r_\gamma) \right]
\end{aligned}
\tag{117}
$$

where the interaction potential is $U(r_{\alpha\gamma}) \approx 1/r_{\alpha\gamma}$ asymptotically. Here the topological excitations are vortex lines (currents) and Eq.(117) can be considered as a Biot-Savart law for these “topological currents” $j_\mu(r)$. Although the asymptotic form of the interaction potentials are the same for the 3d–CG and for the 3d–VLG models, the topological defects are different, consequently, the 3d–SG and 3d–XY models belong to different universality classes.

It has been argued that the field theory equivalent to the 3d–XY model is a QED-type Abelian model which has the following partition function (see Eq.(3.11) of [85])

$$
Z = \mathcal{N} \sum_{j_\mu} \int \mathcal{D}[A_{\mu,l}] \exp\left[\sum_{\mu,\nu,l} -\frac{1}{2} \frac{k_{\mathrm{B}}T}{J} F_{\mu\nu,l} \, F_{\mu\nu,l} + i2\pi j_{\mu,l} A_{\mu,l} \right]. \tag{118}
$$

with integer-valued currents j_μ and $F_{\mu\nu,l} = \partial_\mu A_{\nu,l} - \partial_\nu A_{\mu,l}$. The above QED-type quantum field theory is not a sine–Gordon type scalar model. Therefore, one should conclude that the 3d–SG and the 3d–XY are not the dual theory of each other.

4.4 Layered models

Let me discuss the layered sine–Gordon (LSG) model (42) with two different couplings between the layers (i.e. different mass matrices couple the field components). The LSG model with a Josephson-type mass matrix (3.1.3) has been constructed from the 3d–SG model by a suitable discretization of the derivative term in one direction. The LSG model with a magnetic-type mass matrix (3.1.3) interpolates between the massive and massless SG models (for $N = 1$ layer the LSG model is equivalent to the massive SG and for $N \to \infty$ it becomes the massless SG). Let me note that LSG models with Josephson and magnetic couplings have the same form for $N = 2$ which is my starting point. The partition function of the 2-layer LSG model reads as [13],

$$
Z_{2-\mathrm{LSG}} = \mathcal{N} \int \mathcal{D}[\varphi] \exp\left[\int \mathrm{d}^2 r \left(\frac{1}{2} \sum_{n=1}^{2} (\partial_\mu \varphi_n)^2 + \frac{1}{2} J(\varphi_2 - \varphi_1)^2 + \sum_{n=1}^{2} u \cos(\beta\varphi_n) \right) \right] \tag{119}
$$

where $\varphi \in [-\infty, \infty]$. The LSG model with Josephson type coupling has been proposed as a candidate model for the description of the vortex properties of layered superconductors [93, 94] (in the presence of Josephson coupling). In order to be able to decide whether this statement is correct or not let me study he partition function of the equivalent gas of topological excitations of the 2-LSG model which reads as [13]

$$
\begin{aligned}
Z_{2-\mathrm{LSG}} \;=\; & \mathcal{N} \sum_{\nu_+,\nu_-=0}^{\infty} \frac{(u/2)^{\nu_+} (u/2)^{\nu_-}}{(\nu_+)!(\nu_-)!} \left(\prod_{j=1}^{\nu_+ + \nu_-} \int \mathrm{d}^2 r_j \right) \\
& \exp\left[-\frac{\beta^2}{2} \sum_{j<k=1}^{\nu_+ + \nu_-} \sigma_j \, \sigma_k \left\{ \delta_{n_j n_k} A(r_{jk}) + (1 - \delta_{n_j n_k}) \, B(r_{jk}) \right\} \right]
\end{aligned}
\tag{120}
$$

where the asymptotic forms of the in-plane (A) and inter-plane (B) interaction potentials are [13]

$$
\begin{aligned}
A(r_{\alpha\gamma} \ll \lambda_{\mathrm{eff}}) &\sim -\frac{1}{2\pi} \ln\left(\frac{r_{\alpha\gamma}}{a}\right), && \text{(121a)}\\
A(r_{\alpha\gamma} \gg \lambda_{\mathrm{eff}}) &\sim -\frac{1}{2\pi} \left(\frac{1}{2} \ln\left(\frac{r_{\alpha\gamma}}{\lambda_{\mathrm{eff}}}\right) + \ln\left(\frac{\lambda_{\mathrm{eff}}}{a}\right) + \frac{1}{2} \ln(2) - \frac{1}{2}\gamma_{\mathrm{E}}\right), && \text{(121b)}\\
B(r_{\alpha\gamma} \ll \lambda_{\mathrm{eff}}) &\sim 0, && \text{(121c)}\\
B(r_{\alpha\gamma} \gg \lambda_{\mathrm{eff}}) &\sim -\frac{1}{2\pi} \left(\frac{1}{2} \ln\left(\frac{r_{\alpha\gamma}}{\lambda_{\mathrm{eff}}}\right) + \frac{1}{2} \ln(2) - \frac{1}{2}\gamma_{\mathrm{E}}\right), && \text{(121d)}
\end{aligned}
$$

where $\lambda_{\mathrm{eff}} = 1/\sqrt{2J}$ and a is the short-distance UV cutoff. If one extends the partition function (120) for $N > 2$ then the magnetic and the Josephson type cases become different. For example, the LSG model (42) with magnetic type mass matrix (3.1.3) (for arbitrary number of layers) is equivalent to the layered GL theory with magnetic coupling (99), the layered Yukawa gas (extension of (120) for arbitrary numbers of layers with magnetic interaction), and the so-called layered frustrated XY model where the frequency β^2 is related inversely to the temperature. On the contrary, the LSG model (42) with Josephson type mass matrix (3.1.3) is equivalent to the layered version of 3d-CG.

Let me turn to the layered version of the 3d–XY model. The partition function of the layered XY (LXY) model in continuum limit reads as

$$
Z_{\mathrm{LXY}} = \mathcal{N} \int \mathcal{D}[\theta] \exp\left[-\int \mathrm{d}^2 r \left(\sum_{n=1}^{2} \frac{1}{2} J_{\parallel} \left[(\partial_x\theta_n)^2 + (\partial_y\theta_n)^2\right] + J_{\perp}[1 - \cos(\theta_2 - \theta_1)]\right)\right], \quad (122)
$$

where $\theta_n \in [0, 2\pi]$ is considered as a continuous variable in the xy-plane but discrete in the perpendicular direction. The partition function (122) is equivalent to the layered GL model in the absence of electromagnetic fields and in the London limit (96). After Taylor-expanding the cosine in (122) and keeping only the quadratic term, the partition function of the LXY model reduces to

$$
Z_{\mathrm{LXY}} \approx \mathcal{N} \int \mathcal{D}[\theta] \exp\left[-\int \mathrm{d}^2 r \left(\sum_{n=1}^{2} \frac{1}{2} J_{\parallel} \left[(\partial_x\theta_n)^2 + (\partial_y\theta_n)^2\right] + \frac{1}{2} J_{\perp}(\theta_2 - \theta_1)^2\right)\right]. \quad (123)
$$

The decisive question is whether the LXY and the LSG model (with Josephson type interlayer coupling) belong to the same universality class. On the one hand, one can argue that they should do so since the layered models consist of 2d–XY and 2d–SG structures which coupled by a similar quadratic term and the 2d models are dual to each other. On the other hand, one can argue that this conjecture is wrong because in the continuum limit ($N \to \infty$), the layered models become the discretised version of the corresponding 3d models which belong to different universality classes. In order to answer this question one can compare the equivalent gases of topological excitations. The partition function of the gas of topological excitation corresponds to the LXY model [92] has been constructed and reads as (see also Eq.(1) of [95])

$$
Z_{\mathrm{LXY}} = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(\nu!)^2} \sum_{n_1=1}^{2} \int \mathrm{d}^2 r_1 \ldots \sum_{n_{2\nu}=1}^{2} \int \mathrm{d}^2 r_{2\nu} \sum_{\sigma_1,\ldots,\sigma_\nu} \exp\left(-\frac{1}{2k_{\mathrm{B}}T} \sum_{\alpha\neq\beta} \sigma_\alpha \, \sigma_\beta \, V(r_{\alpha\beta}, \, n_{\alpha\beta})\right), \quad (124)
$$

where $\sigma_\alpha = \pm 1$ is the charge of the αth vortex, a stands for the lattice spacing and the interaction potential V between two vortices depends on their relative distance $r_{\alpha\beta}$ within the two-dimensional planes ($r_{\alpha\beta} = |\vec{r}_\alpha - \vec{r}_\beta|$) and on the distance $n_{\alpha\beta}$ across the planes ($n_{\alpha\beta} = |n_\alpha - n_\beta|$), where n_α is the layer in which the αth vortex is located. If one neglects interactions between vortices separated by more than one layer this results in intra- and interlayer interaction potentials which have commonly accepted short- and long-range asymptotic forms given by (see, Eqs. (2) and (3) of [95])

$$
\begin{aligned}
V(r_{\alpha\beta}, \, n_{\alpha\beta} = 0) &= -\ln\left(\frac{r_{\alpha\beta}}{a}\right) - \sqrt{\lambda}\, \frac{r_{\alpha\beta} - a}{a}, && \text{(125a)}\\
V(r_{\alpha\beta}, \, n_{\alpha\beta} = 1) &= b\sqrt{\lambda}\, \frac{r_{\alpha\beta}}{a}, && \text{(125b)}
\end{aligned}
$$

with a coupling $\lambda \sim a^2 J_\perp / J_\|$ where b is a constant of order unity. The intralayer interaction between the vortices is logarithmic for short distances, as in the case of the usual 2d–CG, but linear for large distances. The interlayer interaction is always linear and similar to the long-range intralayer interaction but with an opposite sign. Within a layer, vortices of opposite charge attract, whereas the positive prefactor of the linear term in the interlayer interaction implies the formation of vortex stacks of same charges.

Therefore, one should conclude that the following models, the LSG with Josephson interlayer coupling (3.1.3) and the LXY (123) model (which is equivalent to the layered GL with Josephson coupling (92)) belong to different universality classes, since the asymptotic behavior of the interaction potentials of Eq.(120) and of Eq.(124) are different, logarithmic for the LSG and linear for the LXY model. Thus, the assumption that the LSG model (42) with Josephson type mass matrix (3.1.3) is suitable for the description of the vortex dynamics of layered superconductors with Josephson interlayer coupling is rather questionable. A brief summary of equivalent models for the Josephson coupled case is shown in Table 1.

$N=\infty \quad \Rightarrow$	3d-CG (115)	$=$	3d-SG (114)	$\neq$	3d-GL (91)	$\approx$	3d-XY (116)
$1<N<\infty \Rightarrow$	layered CG	$=$	Josephson-LSG (3.1.3)	? $\neq$	Josephson layered GL (92)	$\approx$	layered XY (123)
$N=1 \quad \Rightarrow$	2d-CG (104)	$=$	2d-SG (101)	$=$	2d uncharged GL (93)	$\approx$	2d-XY (106)

Table 1: Equivalent models for the Josephson type interlayer coupling.

Let me turn the attention to magnetically coupled layered modes. The interaction potentials (121) have the same asymptotic behavior as the vortices of magnetically coupled superconducting layers [81, 75, 79, 77] [for the intralayer and interlayer interactions see Eqs. (86) and (89) of Ref. [77], under the substitution $\Lambda_D = \Lambda_s/N$]. This observation shows that the LSG model (42) with magnetic type mass matrix (3.1.3) is suitable to describe the vortex dynamics in magnetically coupled layered systems where no Josephson but magnetic interlayer interaction is assumed between vortices of neighbouring layers. A few remarks are now in order. (i) In the general N-layer case, the prefactor $(N-1)/N$ appearing in the intralayer interaction (for $N=2$ it is $\frac{1}{2}$ in (121b)) indicates the existence of vortices with fractional flux. (ii) For the case $N=1$, there exists no interlayer interaction, and the intralayer potential is logarithmic for small distances and vanishes for large distances. Consequently, there are always free, non-interacting vortices in the model which push the KTB transition temperature to zero. The LSG model (3.1.3) for a single layer reduces to the 2d-MSG model where the periodicity in the internal space is broken and the KTB transition is absent. (ii) In the bulk limit $N \to \infty$, the effective screening length and the interlayer interaction disappear ($\lambda_{\rm eff} \to 0$, $B_{\alpha\gamma} \to 0$), and the intralayer potential has a logarithmic behaviour with full flux, thus the LSG model (3.1.3) predicts the same behaviour as that of the pure 2d-SG model with $T^{(\infty)}_{\rm KTB} = T^\star_{\rm KTB}$. A brief summary of equivalent models for the magnetically coupled case is shown in Table 2. One of the goals of this lecture note is to show how one can determine the phase structure of these 2d, 3d, and layered models in the framework of FRG. A typical example for such FRG study is presented in Ref. [16] where the effect of a linear tunneling coupling between two-dimensional systems, each separately exhibiting the topological (KTB) transition is studied. It was found that in the uncoupled limit, there are two phases: one where the one-body correlation functions are algebraically decaying and the other with exponential decay. When the linear coupling is turned on, a third KTB-paired phase emerges, in which one-body correlations are exponentially decaying, while two-body correlation functions exhibit power-law decay. In Ref. [16] numerical simulations in the paradigmatic case of two coupled XY models at finite temperature was performed, finding evidences that for any finite value of the interlayer coupling, the KTB-paired phase is present. The complete picture of the phase diagram using the FRG approach was presented in Ref. [16].

$N = \infty \quad \Rightarrow$	2d-CG (104)	=	2d-SG (101)	=	2d-GL (93)	$\approx$	2d-XY (106)
$1 < N < \infty \Rightarrow$	layered YG	=	magnetic-LSG (3.1.3)	=	magnetic layered GL (99)	$\approx$	layered frustrated XY
$N = 1 \quad \Rightarrow$	2d-YG (112)	=	2d-MSG (111)	=	2d charged GL (100)	$\approx$	2d frustrated XY

Table 2: Equivalent models for the magnetic type interlayer coupling.

The vortex dominated properties of high transition temperature superconductors can be verified by several experimental techniques as an example one can mention the so called electrical transport measurement method which can be used to investigate the length-scale dependence and critical behaviour of vortices in layered systems.

4.5 The $O(2)$ model

In this short subsection I show the connection between the $O(2)$ scalar theory and the previously discussed two-dimensional models [15]. The Lagrangian of the $O(2)$ model reads as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \underline{\varphi})^2 + \frac{1}{2} g_2 \underline{\varphi}^2 + \frac{1}{4!} g_4 \underline{\varphi}^4 + \dots, \qquad \underline{\varphi} = (\varphi_1, \varphi_2) \tag{126}$$

where the two real scalar fields (φ_1, φ_2) can be rewritten as a single complex one

$$\phi = \varphi_1 + i\varphi_2, \;\; \phi^\star = \varphi_1 - i\varphi_2 \quad \to \quad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi^\star) + \frac{1}{2} g_2 (\phi\phi^\star) + \frac{1}{4!} g_4 (\phi\phi^\star)^2 + \dots \tag{127}$$

and the complex field can be parametrized by an amplitude (ρ) and a phase (θ)

$$\phi = \sqrt{\rho} e^{i\theta}, \;\; \phi^\star = \sqrt{\rho} e^{-i\theta} \quad \to \quad \mathcal{L} = \frac{1}{8\rho}(\partial_\mu \rho)^2 + \frac{1}{2}\rho(\partial_\mu \theta)^2 + \frac{1}{2} g_2 \rho + \frac{1}{4!} g_4 \rho^2 + \dots \tag{128}$$

which is identical to (90) using the notation $\psi_0 \equiv \sqrt{\rho}$. Important to note that the angle θ is a compact field, so, the integration measure should be chosen properly in the path integral of the model. The Lagrangian (128) can also be derived by using the identification $\varphi_1 = \sqrt{\rho}\cos(\theta)$, $\varphi_2 = \sqrt{\rho}\sin(\theta)$ in (126). In summary, neglecting the amplitude fluctuations, the original definition of the $O(2)$ model reduces to

$$\partial_\mu \rho = 0 \quad \to \quad \mathcal{L} = \frac{1}{2}\rho(\partial_\mu \theta)^2. \tag{129}$$

Thus, in the absence of amplitude fluctuations, the two-dimensional $O(2)$ theory is equivalent to the uncharged 2d-GL theory taken in the London limit, i.e., Eq. (93) which is assumed to be in the universality class of the sine-Gordon, XY and Coulomb-gas models in $d = 2$ dimensions. The effect of amplitude fluctuations on the phase structure could depend on the dimensionality.

5 Sine-Gordon models in high energy physics

So far, the application of SG type models is considered in low dimensions (mostly in $d = 2$). In this section I mention possible situations where periodic scalar fields can play a role in $d = 4$ dimensions. These cases can be associated to a particular energy (or length) scale such as (i) the transplanckian physics of the early Universe above the Planck scale, (ii) physics of large extra dimensions, (iii) axion physics, (iv) the cosmic inflation at the GUT scale, (v) the Higgs physics at the Electroweak scale. In these cases, scalar fields naturally appear which opens the door for possible applications of sine-Gordon type models.

5.1 Asymptotically Safe Quantum Gravity

The quantum field theory of gravity, i.e., Quantum Einstein Gravity (QEG) is perturbatively non-renormalizable. It requires infinitely many unknown parameters to be set by experiment. A possible solution to perform renormalization is the use of a nonperturbative treatment. Indeed, nonperturbative renormalizability, which is also referred to as Asymptotic Safety (AS), provides us with a nontrivial high energy, i.e., ultraviolet (UV) fixed point of the RG flow. The RG flow leads to a finite number of UV-attractive couplings; so, it is sufficient to perform only a finite number of measurements. In other words, it controls the UV behavior of the dimensionless couplings. They do not need to be small or tend to zero in the UV limit but tend to finite values at the nontrivial UV fixed point. It was shown in [96] that for the simplest truncation of QEG which is the Einstein–Hilbert action, such a nontrivial fixed point is indeed present. Up to the present, many different lecture notes have already confirmed that the AS scenario is possible, for a recent review, I refer to [97]. For applications to cosmology, one consults Ref. [98, 99], and for applications to black-hole physics and instructive explicit functional RG computations, I refer to [100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111].

Although, scalar fields play no direct role in AS gravity but the field independent term does and this term has great importance in cosmic inflation, too, where I discuss possible application of periodic scalar models. Therefore, it is useful to summarise the cornerstones of AS gravity which can be done in the framework of the Einstein-Hilbert truncation of the effective average action

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^4x \sqrt{-g}(R - 2\Lambda_k) \tag{130}$$

where g is the determinant of the metric tensor, R is the Ricci scalar and the scale-dependent parameters are the cosmological constant Λ_k and the Newton coupling G_k. The scale-dependence is analyzed in terms of dimensionless couplings, $\lambda_k \equiv \Lambda_k k^{-2}$, $g_k \equiv G_k k^2$ with the help of the β-functions, see for example [98]

$$k\partial_k g_k = \beta_g, \qquad k\partial_k \lambda_k = \beta_\lambda \tag{131}$$

which are calculated by the so called Litim regulator (specified later in the following sections)

$$\beta_g = (2 + \eta_N) g_k, \qquad \beta_\lambda = (\eta_N - 2)\lambda_k + \frac{g_k}{12\pi}\left[\frac{30}{1 - 2\lambda_k} - 24 - \frac{5}{1 - 2\lambda_k}\eta_N\right] \tag{132}$$

where the anomalous dimension of Newton's constant $\eta_N = G_k^{-1} k\partial_k G_k$ is given by

$$\eta_N = \frac{g_k B_1}{1 - g_k B_2} \tag{133}$$

where

$$B_1 = \frac{1}{3\pi}\left[\frac{5}{1 - 2\lambda_k} - \frac{9}{(1 - 2\lambda_k)^2} - 7\right], \qquad B_2 = -\frac{1}{12\pi}\left[\frac{5}{1 - 2\lambda_k} - \frac{6}{(1 - 2\lambda_k)^2}\right]. \tag{134}$$

The β-functions contain the information on fixed points $g_\star$ and $\lambda_\star$ of the RG flow where the beta functions vanish simultaneously. They give rise to two fixed points: the Gaussian (G) UV fixed point situated at $(g_\star, \lambda_\star) = (0, 0)$ and the non-Gaussian (NG) UV fixed point located at $(g_\star, \lambda_\star) = (0.707, 0.193)$. In addition one can discuss the existence of the IR convexity fixed point. The existence of the non-Gaussian UV fixed point can solve important problems of quantum gravity. In order to find cosmological applications the running RG cutoff k is identified with a typical length scale of the system [100]. There are several types of cutoff identifications [100], for example, $k \sim t^{-1}$ where t is the cosmic time or $k \sim H(t)$ where $H(t)$ is the Hubble parameter or $k \sim T$ where T is the temperature of the cosmic plasma. The idea is to use RG running to connect the physics of various energy scales. For example, one should find the NG fixed point above the Planck scale $k \gg m_p = 2.4 \times 10^{27}$ eV, cosmic inflation takes place below the Planck scale $k = k_{\text{inf}} = 10^{22}$ eV, the well-known value of Newton's constant is fixed by laboratory experiments $G_k = G = 6.67 \times 10^{-57}$ eV^{-1} at low-energies $k = k_{\text{lab}} = 10^{-5}$ eV and finally one should mention the accelerated expansion of the Universe at present which requires $\Lambda_k = \Lambda = 4 \times 10^{-66}$ eV^2 at the scale $k = k_{\text{Hub}} = 10^{-33}$ eV. The nonperturbative RG (using various extension of the Einstein-Hilbert truncation, see for example [101]) is capable to build up connection between these scales and cover many orders of magnitude in change of couplings, like the Newton and the cosmological constants.

If one would like to extend the Einstein-Hilbert (130) truncation of the effective average action, one possible way is to incorporate matter fields where the simplest choice is the scalar field. Let me consider the following gravity-scalar model

$$\Gamma_k[\phi] = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_k} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V_k(\phi) \right] , \tag{135}$$

where the scalar potential is usually expanded in terms of the field. If this expansion is terminated at the quadratic order, it has the following form [27]

$$V_k(\phi) = V_k(0) + \frac{1}{2} M_k^2 \phi^2, \qquad V_k(0) \equiv \frac{2\Lambda_k}{16\pi G_k} , \tag{136}$$

which is one of the simplest scenarios when a single real scalar field is coupled to gravity. For a detailed study of scalar fields coupled to asymptotically safe quantum gravity see for example [112]. The fixed points of the RG flow for a scalar field in curved space with non-minimal coupling is discussed in [113] where the RG equation for the scalar potential in the so-called Local Potential Approximation of the Wetterich RG equation with the Litim cutoff reads as

$$k\partial_k V_k(\phi) = \mu_d k^d \frac{k^2}{k^2 + \partial_\phi^2 V_k(\phi)} , \tag{137}$$

with $\mu_d = 1/[(4\pi)^{d/2} \Gamma(d/2+1)]$. The existence of the Gaussian (G) and non-Gaussian (NG) fixed points and the RG flow of the full (dimensionless) potential is discussed, for example, in Refs. [112] and [113] but no β function is given for the field-independent term.

Let me discuss the RG flow of of the cosmological constant in the absence of quantum gravity effects, but take into account the RG equation (137) for the scalar potential (136). I use the relation $\tilde{V}_k(0) \equiv \frac{2\lambda_k}{16\pi g_k}$ which connects dimensionless couplings and gives [27]

$$k\partial_k \lambda_k = 8\pi \left(g_k k\partial_k \tilde{V}_k(0) + \tilde{V}_k(0) k\partial_k g_k \right) \tag{138}$$

In the absence of quantum gravity effects, i.e., assuming a scale-independent dimensionful Newton's constant, $G_k = G$, the anomalous dimension vanishes because $\eta_N = G^{-1} k\partial_k G = 0$, and one finds $k\partial_k g_k = 2g_k$ which results in a trivial RG scaling $g_k \sim k^2$. The RG flow equation for the dimensionless field-independent term obtained from (137) reads as

$$k\partial_k \tilde{V}_k(0) = \frac{1}{32\pi^2} \left(\frac{1}{1 + \tilde{M}_k^2} \right) - 4\tilde{V}_k(0) \tag{139}$$

which can be used to obtain the RG flow equation for the dimensionless cosmological constant [27],

$$k\partial_k \lambda_k = \frac{1}{4\pi} g_k \left(\frac{1}{1 + \tilde{M}_k^2} \right) - 2\lambda_k . \tag{140}$$

In this approximation, g_k has a trivial scaling, $g_k \sim k^2$, it tends to infinity in the UV limit. The UV scaling of the dimensionless mass term is $\tilde{M}_k^2 \sim k^{-2}$, so, it tends to zero in the UV limit. Although the UV Gaussian fixed point formally exists, it cannot be reached because the corresponding β-function diverges in the UV limit, $g_k \left(\frac{1}{1+\tilde{M}_k^2}\right) \to \infty$ if $k \to \infty$. Thus, it is a relevant question to ask [27] whether one can apply additional subtraction terms in order to restore the Gaussian fixed point in (140).

5.2 Periodic Effective Branon Action

In the framework of models of large extra dimensions, in particular in the so-called Brane World Scenario (BWS), elementary particles except for the graviton are localized on (3+1)-dimensional branes. Although experimental tests from the Large Hadron Collider severely constrain theories of large extra dimensions, the BWS served as one of the simplest extensions of the Standard Model. The brane fluctuations of the BWS in a 5th dimensional bulk lead to a low energy effective four-dimensional theory, where branons (representing quanta of the brane fluctuations) are described by a scalar field living on a flat brane, thus, possible applications of periodic scalar models can be discussed. For references on branon studies see e.g., [114, 115, 116, 117, 118, 119, 120].

In order to show the construction of an effective description of brane fluctuations, let me start with a general setup where a single brane model in large extra dimensions is considered [21]. The four-dimensional space-time is embedded in the $D = 4 + N$ dimensional bulk space. In what follows, the brane coordinates are denoted with the indices μ, ν and the bulk coordinates with M, N. In this general framework the coordinates parametrizing the points of the bulk are denoted by $X^M = (x^\mu, y^m)$ and the position of the brane in the bulk is given by $X^M = (x^\mu, Y^m(x))$, so thus $y^m = Y^m(x)$. Let me now switch back to the simplest case, i.e., for $N = 1$ and consider a 5-dimensional Universe with generic coordinates $X^M = (x^\mu, y)$, where x are the coordinates on the brane, which is defined by the equation $y = Y(x)$. Motivated by scenarios involving confinement on the brane, let me consider the following block-diagonal bulk metric [21],

$$g_{MN} = \begin{pmatrix} e^{2\sigma(y)}\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix}, \tag{141}$$

with the Randall-Sundrum warp factor, $\sigma(y)$ which is assumed to be even in y. If the brane is centered on a Randall-Sundrum warp factor, a typical choice is the absolute value function which is non-differentiable [21], or the differentiable quadratic one [118],

$$\sigma(y) = -a|y|, \qquad \sigma(y) = -\frac{M^2}{2}y^2, \tag{142}$$

where a and M^2 are constants. Here I suggest SG and MSG type Randall-Sundrum warp factors,

$$\sigma(y) = -u\cos(y), \qquad \sigma(y) = -\frac{M^2}{2}y^2 - u\cos(y). \tag{143}$$

The induced metric $h_{\mu\nu}$ on the brane is

$$h_{\mu\nu}(x) = \partial_\mu X^M \partial_\nu X^N g_{MN}(X) = e^{2\sigma(Y)}\eta_{\mu\nu} - \partial_\mu Y \partial_\nu Y,$$

and, if f^4 is the brane tension, the brane action is then

$$S_{brane} = -f^4 \int d^4x\sqrt{-h} = -f^4 \int d^4x\; e^{4\sigma(Y)} \left(1 - \frac{1}{2}e^{-2\sigma(Y)}\eta^{\mu\nu}\partial_\mu Y \partial_\nu Y + \cdots\right),$$

where dots represent higher orders in derivatives of Y, which will be disregarded in the framework of the gradient expansion. The dynamical variable is the canonically normalized branon field $\phi = f^2 Y$, with mass dimension 1, and the classical brane ground state is $Y = 0$. The resulting effective action for branons is then

$$S_{branon} = \int d^4x \left(\frac{e^{2\sigma(\phi)}}{2}\partial^\mu\phi\partial_\mu\phi - f^4 e^{4\sigma(\phi)}\right). \tag{144}$$

For the details of the derivation see [118]. If one takes the choice of the absolute value function for the Randall-Sundrum warp factor then the Liouville-type terms in the potential and in the wave function renormalization depend on the absolute value of the scalar field, i.e., it is non-differentiable. Thus, it is a natural question to ask whether the non-analitycal behavior of the potential conflicts with its quantisation and renormalization [21].

The Randall-Sundrum warp factor with an SG type choice results in a periodic, and with an MSG type one leads to a polynomial scalar field theory. However, in the latter case if the mass term and the Fourier amplitude are small, the exponential of the Liouville term can be expanded in Taylor series and keeping the linear term only, one finds an MSG type scalar model. This motivates the use of SG and MSG type models in gravity-scalar systems (135), thus the field-dependent part of the potential in (136) can be replaced by SG and MSG type models.

5.3 Periodic Axion potential

My next example for applications of sine-Gordon type scalar field theory in higher dimensions is related to the so called axion. Constraints from symmetry and renormalizability on the standard model QCD action allows to extend it by a CP violating term. However, experimental data do not favour such an extension although the standard model Lagrangian is not CP symmetric, so, QCD could be CP violating as well. Peccei and Quinn proposed a mechanism and introduced a new hypothetical scalar field with $U(1)$ symmetry in order to build up a CP conserving theory from a model with massive fermions coupled to a non-Abelian gauge field [121]. The axion appears as a phase of a Goldstone mode for a complex scalar Φ with a vacuum expectation value $<\Phi>= fe^{i\theta}$ which corresponds to the spontaneous break down of the $U(1)$ symmetry at the scale f. Integrating over the QCD degrees of freedom one arrives at the following effective action

$$S = \int d^4x \left(\frac{f^2}{2} \partial_\mu \theta \partial^\mu \theta + u[1-\cos(\theta)] \right) = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + u[1-\cos(\beta\phi)] \right), \quad \phi = f\theta, \quad \beta = 1/f \tag{145}$$

where a periodic potential appears naturally and the rescaling of the field has been done by using the assumption that f is independent of the spacetime. Thus, axion physics motivates further the use of periodic scalar models in gravity-scalar systems (135). In addition, it was shown in Ref. [122] that the axion potential flattens out under RG transformations which were taken in the Local Potential Approximation. It is, however, important to clarify whether the axion potential flattens out if the RG study is performed beyond the Local Potential Approximation [18].

5.4 Periodic inflationary potential

Another possible application of the higher dimensional periodic, i.e. sine-Gordon type scalar field theory is inflationary cosmology [18, 19, 20]. The standard model of cosmology explains the exponentially fast expansion [123, 124, 125] of the early Universe, and describes the observed accelerated expansion of the Universe at present [126, 127].

As a first step, let me discuss the latter case, i.e., the accelerated expansion of the Universe at present. While, in ordinary particle physics, a constant, field-independent term of the potential carries no physical meaning, it has great importance in the case of gravity as it was shown in (130). For example, in order to describe the observed accelerated expansion of the Universe at present a possible solution is the inclusion of a constant term into the Einstein-Hilbert action (130). In this case Einstein's equation reads as (in the absence of matter),

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{146}$$

where R is the scalar curvature and Λ is the cosmological constant which is assumed to be related to dark energy [128, 129, 130, 131, 132] and is expected to cause the accelerated expansion of the universe observed today. Indeed, one can use the Friedmann–Lemaître–Robertson–Walker (FLRW, see Ref. [133, 134, 135, 136, 137]) metric (in our units, the speed of light is $c \equiv 1$ and $\hbar \equiv 1$),

$$g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2), \qquad a = a(t), \qquad g^{\mu\nu} = \mathrm{diag}(-1, a^{-2}, a^{-2}, a^{-2}), \tag{147}$$

where the scale factor $a(t)$ of the expanding homogeneous and isotropic Universe can be calculated which results in an exponentially fast expansion. Indeed, substituting the FLRW metric into the Einstein equation (146) and assuming a flat Universe, one finds the Friedmann equation which results in an exponentially fast expansion

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\Lambda}{3} = 0 \quad \Longrightarrow \quad a(t) \sim \exp(\sqrt{\Lambda/3}\, t). \tag{148}$$

Exponential expansion of the early universe can be explained by cosmic inflation, a theory which is developed to explain major issues such as the origin of the large-scale structure, the flatness of the universe, the horizon problem, the absence of magnetic monopoles and in general properties of Cosmic Microwave Background Radiation (CMBR) [123]. A very comprehensively studied work hypothesis is that a hypothetical scalar field, i.e., the inflaton particle, is responsible for inflation which is caused by the slow-roll motion starting from a metastable false vacuum towards the real vacuum [125]. Indeed, the key observation is that scalar fields can mimic the equation of state for negative pressure, thus they represent an excellent model for inflation. Various types of scalar potentials have been proposed in inflationary cosmology. Let me consider the simplest of these scenarios which is provided by the slow-roll single-field models with minimal kinetic terms [137, 138, 139] which is the action (135) discussed for gravity-scalar systems,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \mathcal{L}_\phi\right], \qquad \mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi)\,, \tag{149}$$

and I assume an expanding homogeneous and isotropic Universe (with flat curvature), so, one can rely on the FLRW metric, $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})} = a^3$ and the time-dependence of the scalar factor $a(t)$ is given by the Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{150}$$

where the stress energy tensor is calculated from the matter (scalar) fields and reads as

$$\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi, \\
T^\mu_\nu &= \mathrm{diag}(-\rho, p, p, p), \qquad\qquad T_{\mu\nu} = g_{\mu\alpha} T^\alpha_\nu = \mathrm{diag}(\rho, a^2 p, a^2 p, a^2 p)\,, \\
T_{00} &= \rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{1}{a^2}(\nabla\phi)^2 + V, \qquad T_{ii} = a^2 p = a^2 \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\frac{1}{a^2}(\nabla\phi)^2 - V\right)
\end{aligned} \tag{151}$$

Substituting the FLRW metric and the stress-energy tensor into the Einstein equations gives only two independent equations, the Friedmann and the Raychaudhuri equations,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} \quad \text{(Friedman)}, \qquad \frac{\ddot{a}}{a} = -\frac{8\pi G}{6}(\rho + 3p) \quad \text{(Raychaudhuri)} \tag{152}$$

where the Hubble constant H is introduced. Now I turn to the discussion of the solutions of the Friedmann equation (in case of flat curvature). Let's assume that $\rho \sim a^{-n}$. This is equivalent with an assumption that the equation of state is $p = \omega\rho$. Let me first discuss the case of negative pressure, $p = -\rho$ which results in exponential expansion

$$\begin{gathered}
n = 0, \omega = -1 \qquad \Longrightarrow \qquad \rho \sim a^0 = \mathrm{const} \;\Longrightarrow \\
H^2 = \frac{8\pi G}{3}\rho = \mathrm{const} \;\Longrightarrow\; \dot{a} = Ha \;\Longrightarrow\; a \sim \exp(Ht).
\end{gathered} \tag{153}$$

Although the cosmological constant and the special equation of state ($\rho = -p$) both results in the same rate of expansion but the former cannot be used for inflation since it has to end. Thus, one has to rely on the special equation of state ($\rho = -p$) which can be obtained for the scalar field under some conditions. If these conditions are fulfilled one finds exponential inflation. If these conditions are not fulfilled the inflation is over.

Let me discuss these requirements. The first observation is that over inflation the field can be considered to be homogeneous ($\nabla\phi/a = 0$). Than the relation between the density and pressure reads

$$\omega = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \quad \text{if} \quad \frac{1}{2}\dot{\phi}^2 \ll V \implies \omega = -1. \tag{154}$$

There is another condition which is coming from the "slow roll" inflation which assures a prolonged inflation

$$|\ddot{\phi}| \ll |3H\dot{\phi}|. \tag{155}$$

These condition can be written as

$$\epsilon \equiv \frac{1}{2}\frac{1}{8\pi G}\frac{V'^2}{V^2} \ll 1, \qquad \eta \equiv \frac{1}{8\pi G}\frac{V''}{V} \ll 1 \tag{156}$$

Thus, if $\epsilon(\phi_f) \approx 1$ or $\eta(\phi_f) \approx 1$ then the inflation ends. Further constraints coming from experimental data can be drawn by using the formal solution of the Friedman equation and the so-called e-fold number which has to be in the range $N = 50 - 60$,

$$a(t) = \exp\left\{\int_{t_0}^{t} dt' H(t')\right\}, \qquad N = \ln\frac{a(t_f)}{a(t_i)} = -8\pi G \int_{\phi_i}^{\phi_f} d\phi \frac{V}{V'}$$

which is needed in order to let the Universe to expand by a factor of at least $10^{26} \approx e^{60}$ during inflation.

At this stage it is useful to overview very briefly the inflationary mechanism. The original idea of Alan Guth for inflation was based on the existence of a relatively stable false vacuum which can be long-lived or metastable (see Ref. [123]). The system moves to the true vacuum through a bubble nucleation caused by instanton effects via quantum tunneling; this induces inflation. However, this old scenario for inflation (see Fig. 4) suffers from problems. For example, it is required to

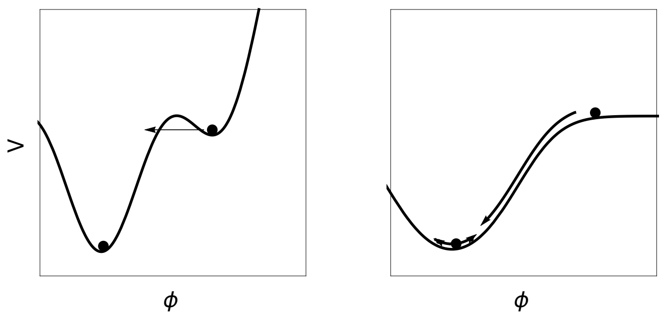

Figure 4: Inflation from false vacuum (left) and by slow-roll (right) where oscillations of the field produce reheating

heat up the Universe after the inflationary period and it is not clear how to define a proper reheating mechanism. In order to produce the present observable Universe, exponential expansion should continue long enough to eliminate magnetic monopoles, but then bubbles become very rare and in addition, they never merge. This generates two major problems: *(i)* the decay process is never complete, *(ii)* radiation cannot be generated by collisions between bubble walls (which was the proposed mechanism for radiation). A possible solution for problems of bubble nucleations is provided by a scenario proposed in Ref. [125], where the ground state starts from a metastable position and rolls down very slowly to the true minimum (see Fig. 4). Thus, inflation is thought to be caused by a scalar field rolling down a potential energy hill instead of tunneling out of a false vacuum. The inflation is over when the hill becomes steeper than its height. Although this, in particular, solves the problem of formulating a "graceful exit" from the inflation period but leaves the initial condition problem to be addressed. It is a natural question to ask whether the RG evolution of the potential (which can be identified as the time evolution of the Universe) can solve the initial value problem [20].

Various types of inflationary potentials can be considered and the goal is always to determine ϵ, η and N. In order to compare them by measured data one has to study the temperature fluctuations of the cosmic microwave background radiation (CMBR). These fluctuations are the relic of the physical properties of the universe at the inflationary period after which the universe has to be reheated and the quantum fluctuations (of the scalar field and of the metric) should be the seeds for structure formation. Fluctuations can be described by the power spectrum $P_g(k)$ which has the following definition for a generic quantity $g(x,t)$

$$g(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ikx} g_k(t), \qquad \langle g^*_{k_1}, g_{k_2} \rangle = \delta^{(3)}(k_1 - k_2) \frac{2\pi^2}{k^3} P_g(k), \qquad \langle g^2(x,t) \rangle = \int \frac{dk}{k} P_g(k) \tag{157}$$

which finds application to describe scalar (P_s) and tensor (P_T) fluctuations. The power spectra (both for the scalar and the tensor cases) is expanded around a chosen pivot scale ($k_\star$),

$$\frac{P(k)}{P_0} = a_0 + a_1 \ln\left(\frac{k}{k_\star}\right) + \frac{a_2}{2} \ln^2\left(\frac{k}{k_\star}\right) + ... \tag{158}$$

where the coefficients can be expressed by the slow-roll parameters, e.g., for the scalar case $a_1^{(s)} = -2\epsilon_1 - \epsilon_2$ where $\epsilon_0 = H_{\text{ini}}/H$ and $\epsilon_{n+1} = d \ln |\epsilon_n| / dN$ which gives $\epsilon_1 = \epsilon$, $\epsilon_2 = 4\epsilon - 2\eta$ and results in $a_1^{(s)} = 2\eta - 6\epsilon$. The scale dependence of the power spectra $P_s \propto k^{n_s - 1}$ and $P_T \propto k^{n_T - 1}$ can be described by spectral indices n_s and n_T. Thus one can define the spectral index n_s for scalar fluctuations and in addition the ratio r of tensor and scalar fluctuations in the following way

$$n_s - 1 = \frac{d\ln P_s}{d\ln k}|_{k=k_\star} = a_1^{(s)}, \qquad \boxed{n_s - 1 \approx 2\eta - 6\epsilon}, \qquad r \equiv \frac{P_T}{P_s}, \qquad \boxed{r \approx 16\epsilon} \tag{159}$$

where P_s is the scalar and P_T is the tensor power spectrum. The spectral index (n_s) and the ratio (r) can be related to each other and can be determined by experimental data.

5.4.1 Quadratic Inflation

The simplest example for inflationary potential is the so-called Chaotic (monomial) or in other words, the quadratic, large field inflationary (LFI) scalar potential, $V(\phi) = \frac{1}{2} m^2 \phi^2$. Its most general form is

$$V = \lambda m_p^{4-\alpha} \phi^\alpha, \qquad m_p = \frac{1}{\sqrt{8\pi G}} \approx 2.4 \times 10^{18} \text{GeV} \tag{160}$$

where m_p is the (reduced) Planck mass. Calculating ϵ, η and N parameters one finds

$$\epsilon = \frac{1}{2} m_p^2 \left(\frac{V'}{V}\right)^2 = \frac{1}{2} m_p^2 \left(\frac{\lambda \alpha m_p^{4-\alpha} \phi^{\alpha-1}}{\lambda m_p^{4-\alpha} \phi^\alpha}\right)^2 = \frac{m_p^2}{2} \frac{\alpha^2}{\phi^2} \tag{161}$$

$$\eta = m_p^2 \frac{V''}{V} = m_p^2 \left(\frac{\lambda \alpha(\alpha-1) m_p^{4-\alpha} \phi^{\alpha-2}}{\lambda m_p^{4-\alpha} \phi^\alpha}\right) = m_p^2 \frac{\alpha(\alpha-1)}{\phi^2} \tag{162}$$

$$N = -\frac{1}{m_p^2} \int_{\phi_i}^{\phi_f} d\phi \frac{V}{V'} = -\frac{1}{m_p^2} \int_{\phi_i}^{\phi_f} d\phi \frac{\phi}{\alpha} = \frac{1}{2\alpha m_p^2} (\phi_i^2 - \phi_f^2) = 50 - 60 \tag{163}$$

Inflation ends when $\epsilon \approx 1$ or $\eta \approx 1$ therefore one has to choose

$$\phi_f = \alpha m_p \implies \epsilon = \frac{1}{2} \qquad \eta = \frac{\alpha - 1}{\alpha}, \qquad \phi_i = \sqrt{2\alpha m_p^2 N + \phi_f^2} \approx \sqrt{2\alpha N} m_p \tag{164}$$

where at the last approximation we used that $\phi_i \gg \phi_f$. Let me determine the spectral index and the ratio for the quadratic case ($\alpha = 2$) where $\phi_i = \sqrt{4N} m_p$,

$$n_s - 1 \approx 2\eta(\phi_i) - 6\epsilon(\phi_i) = \frac{4m_p^2}{\phi_i^2} - \frac{12 m_p^2}{\phi_i^2} = -\frac{2}{N}, \qquad r \approx \frac{32 m_p^2}{\phi_i^2} = \frac{8}{N} \qquad \Longrightarrow \qquad \boxed{(n_s - 1) + \frac{r}{4} = 0} \tag{165}$$

where the latter relation is independent of N, valid up to order $(1/N)$ and can be compared to measured data.

5.4.2 Natural Inflation

I will show that recent Planck data excludes the LFI or more general, the monomial inflationary potentials which have a single minimum. One can add more minima through higher-order powers of the form ϕ^{2n}. In this logic, one can explore a periodic potential of a form having infinitely many minima, which is known as the Natural Inflation (NI) or pseudo-Nambu-Goldstone boson model, i.e., the sine-Gordon scalar theory,

$$V_{\rm NI}(\phi) = u\,[1-\cos(\beta\phi)] \qquad \text{or} \qquad V_{\rm NI}(\phi) = u\,[\cos(\beta\phi)-1] \tag{166}$$

where u, β are dimensionful parameters. It has also been proposed and studied as a viable inflationary scenario [140, 141, 142] and to construct a convenient scalar sector by incorporating the periodic scalar axion potential too [18]. I will show that the NI potential is able to produce agreement with PLANCK results [143, 144, 145] on the thermal fluctuations of the CMBR with a better agreement than the simplest LFI potential.

In $d = 4$ dimensions the scalar field carries a dimension, $\phi = k^{(d-2)/2}\tilde{\phi}$ where $\tilde{\phi}$ is dimensionless and k is an arbitrarily chosen momentum scale convenient to take at the planck mass $k = m_p$. Thus, the corresponding dimensionless parameters are $\beta = m_p^{-1}\tilde{\beta}$ and $u = m_p^4\tilde{u}$. Calculating ϵ, η and N parameters one finds [18]

$$\epsilon = \frac{m_p^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{m_p^2}{2}\left(\frac{u\beta\sin(\beta\phi)}{u\,[1-\cos(\beta\phi)]}\right)^2 = \frac{m_p^2\beta^2}{2}\frac{\sin^2(\beta\phi)}{[1-\cos(\beta\phi)]^2} = \frac{\tilde{\beta}^2}{2}\cot^2\left(\frac{\tilde{\beta}\tilde{\phi}}{2}\right), \tag{167}$$

$$\eta = m_p^2\frac{V''}{V} = m_p^2\frac{u\beta^2\cos(\beta\phi)}{u\,[1-\cos(\beta\phi)]} = m_p^2\beta^2\frac{\cos(\beta\phi)}{1-\cos(\beta\phi)} = \tilde{\beta}^2\frac{\cos(\tilde{\beta}\tilde{\phi})}{1-\cos(\tilde{\beta}\tilde{\phi})} = \frac{\tilde{\beta}^2}{2}\frac{\cos(\tilde{\beta}\tilde{\phi})}{\sin^2\left(\frac{\tilde{\beta}\tilde{\phi}}{2}\right)}, \tag{168}$$

$$\begin{aligned} N = &-\frac{1}{m_p^2}\int_{\phi_i}^{\phi_f} d\phi\frac{V}{V'} = -\int_{\phi_i}^{\phi_f} d\phi\frac{u\,[1-\cos(\beta\phi)]}{m_p^2 u\beta\sin(\beta\phi)} = \int_{\tilde{\beta}\tilde{\phi}_f}^{\tilde{\beta}\tilde{\phi}_i} d(\tilde{\beta}\tilde{\phi})\frac{1}{\tilde{\beta}^2}\frac{1-\cos(\tilde{\beta}\tilde{\phi})}{\sin(\tilde{\beta}\tilde{\phi})} \\ = &-\frac{2}{\tilde{\beta}^2}\log\cos\left(\frac{\tilde{\beta}\tilde{\phi}}{2}\right)\Big|_{\tilde{\phi}_f}^{\tilde{\phi}_i} \end{aligned} \tag{169}$$

which imply the relations

$$n_s - 1 \approx \tilde{\beta}^2\left[1 - 2\sin^{-2}\left(\frac{\tilde{\beta}\tilde{\phi}}{2}\right)\right], \qquad r \approx 8\tilde{\beta}^2\cot^2\left(\frac{\tilde{\beta}\tilde{\phi}}{2}\right) \qquad \Longrightarrow \qquad \boxed{(n_s-1) + \frac{r}{4} = -\tilde{\beta}^2} \tag{170}$$

where the last equation can be compared to the that of the quadratic potential (165). Precise measurements on n_s and r can be used to distinguish between the two scenarios or in other words to put constraints on the dimensionless frequency $\tilde{\beta}$, see Fig. 5 taken from [19]. Indeed, the figure clearly shows that the quadratic inflationary potential is excluded and there is an optimal choice for the frequency, $\tilde{\beta} = 0.15$, of the periodic inflationary model which gives the best agreement with the observed Planck data.

5.4.3 Massive Natural Inflation

Let me extend the Natural Inflation, i.e., the periodic (SG type) potential in such a way that: *(i)* the potential has definite lower bounds, *(ii)* the model has Z_2 symmetry, and *(iii)* the model has more than one non-degenerate minima, separated in energy by a tunable amount [19]. The construction of the potential is based on the MSG model where a term sinusoidal in the field is added to the standard quadratic mass term; this potential has already received significant attention in statistical field theory [4, 9, 22]. I denote my proposal as the Massive Natural Inflation (MNI)

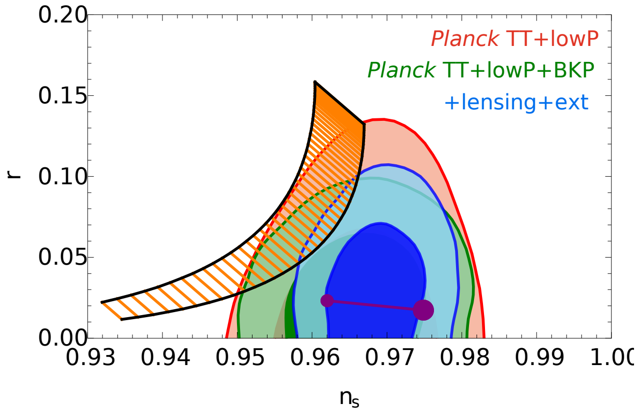

Figure 5: CMBR parameters, i.e., scalar tilt n_s and tensor-to-scalar ratio r derived *(i)* for the Large Field Inflation (i.e., quadratic type) model (165) (black line segment) *(ii)* for the Natural Inflation (i.e. SG type) model (166) for various frequencies (orange line segments) and *(iii)* for the Massive Natural Inflation (i.e., MSG type) model (171) with fixed ratio $\tilde{u}/\tilde{M}^2 \sim 1/(0.22)^2$ and fixed frequency $\tilde{\beta} \sim 0.3$ (purple line segment). The starting and end points of each line segments are calculated for $N = 50$ and $N = 60$, respectively. Predictions (line segments) are compared to results of the Planck mission [143, 144, 145] where dark color regions stand for 95% CL and light color regions correspond to 68% CL. The figure clearly shows that the quadratic inflationary potential – which gives the same prediction as the first, black line segment of the SG model – is excluded, the SG type one is almost excluded but the predictions of the MSG type inflationary model are in perfect agreement with the Planck data.

model and I consider the following variants of the model [19],

$$V_{\mathrm{MNI}_1}(\phi) = \frac{1}{2}M^2\phi^2 + u\left[1 - \cos(\beta\phi)\right], \tag{171}$$

$$V_{\mathrm{MNI}_2}(\phi) = \frac{1}{2}M^2\phi^2 + u\left[\cos(\beta\phi) - 1\right], \tag{172}$$

$$V_{\mathrm{MNI}_3}(\phi) = \frac{1}{2}M^2\phi^2 + u\left[\cos(\beta\phi) - 1\right] - V_0, \tag{173}$$

where the second version differs from the first only by the sign of the u. The third version has an additional constant V_0 to keep the minima of the potential at zero. In general, the three variants of the MNI model require different slow-roll analysis. Indeed, an essential difference between the MSG variants (171) and (172) is the position of the global minima. For the model (171), the potential has a global minimum at zero, while for (172), it has two degenerate global minima, and a local maximum at zero. Therefore, the position of the expectation value of the field (ϕ_f) after the inflation, and other results of the slow-roll analysis, are also different for the two variants.

I here focus on the first variant of the MNI model, given in Eq. (171). Using dimensionless quantities ($\tilde{\beta}$, $\tilde{u}$, $\tilde{\phi}$) where the dimension is taken off by the Planck mass, i.e., $k = m_p$, the parameters ϵ, η and N can be expressed as follows for the first version of the MNI model [19],

$$\epsilon = \frac{1}{2}\left(\frac{\frac{\tilde{u}}{\tilde{M}^2}\tilde{\beta}\sin(\tilde{\beta}\tilde{\phi}) + \tilde{\phi}}{\frac{\tilde{u}}{\tilde{M}^2}\left[1 - \cos(\tilde{\beta}\tilde{\phi})\right] + \frac{1}{2}\tilde{\phi}^2}\right)^2,$$

$$\eta = \frac{\frac{\tilde{u}}{\tilde{M}^2}\tilde{\beta}^2\cos(\tilde{\beta}\tilde{\phi}) + 1}{\frac{\tilde{u}}{\tilde{M}^2}\left[1 - \cos(\tilde{\beta}\tilde{\phi})\right] + \frac{1}{2}\tilde{\phi}^2},$$

$$N = -\int_{\tilde{\phi}_i}^{\tilde{\phi}_f} d\tilde{\phi}\, \frac{\frac{\tilde{u}}{\tilde{M}^2}\left[1 - \cos(\tilde{\beta}\tilde{\phi})\right] + \frac{1}{2}\tilde{\phi}^2}{\frac{\tilde{u}}{\tilde{M}^2}\tilde{\beta}\sin(\tilde{\beta}\tilde{\phi}) + \tilde{\phi}}. \tag{174}$$

One finds similar results for the other two variants of the MNI model. These quantities only depend on the equally dimensionless ratio $\tilde{u}/\tilde{M}^2$ and the dimensionless frequency $\tilde{\beta}$. If the mass term is negligible compared to the periodic one, then one obtains back the natural inflation (i.e., sine-Gordon) model. In the limit of a negligible periodic term (compared to the mass), one obtains the quadratic monomial inflationary model.

Let me first compare results obtained from NI (SG type) and MNI (MSG type) models. As shown in Fig. 5, the MNI model provides much more reliable results. Moreover, the ratio $\tilde{u}/\tilde{M}^2$ and the frequency $\tilde{\beta}$ can be fixed by choosing the best fit to observations (see Fig. 5).

For small values of the frequency, numerical results give $\tilde{u}/\tilde{M}^2 \approx 1/(0.22)^2$, $\tilde{\beta} \approx 0.3$ and $r \approx 0.05$ so, in this case, the scale of inflation, $k_{\mathrm{inf}} = 1.5 \times 10^{16}$ GeV, is around the GUT scale.

For large values of the frequency, one can find agreement with the Planck data but in this case the tensor-to-scalar ratio r is very small, so, the scale of inflation is smaller than the GUT scale. For example, for $\tilde{\beta} \sim 30$, one obtains $k_{\mathrm{inf}} = 2.5 \times 10^{13}$ GeV. In the following I do not use these "large frequency slow-roll" results except in Section IX where the visual representation of the so called "RG running induced inflation" is done for the MSG model with $\tilde{\beta} \sim 30$.

The potential is determined by the slow-roll conditions up to an overall multiplicative factor, but this factor is fixed by the absolute normalisation. According to Eq. (23) of Ref. [139] and Eq. (218) of Ref. [137], the normalisation condition is

$$V(\phi_i) \equiv \frac{r}{0.01}(10^{16}\,\mathrm{GeV})^4 \,. \tag{175}$$

The tensor-to-scalar ratio r is given by the slow-roll parameters which are fixed at the scale of inflation (k_{inf}), according to remarks preceding Eq. (218) of Ref. [137]. Thus, the scale of inflation is given by the following relation

$$V(\phi_i) \equiv k_{\mathrm{inf}}^4 \,, \qquad k_{\mathrm{inf}} = \left(\frac{r}{0.01}\right)^{\frac{1}{4}} 10^{16}\ \mathrm{GeV}, \tag{176}$$

which entirely fixes the inflationary potential including the constant term. Therefore, the field-independent term is fixed at the scale of inflation, too. Important to note that if the tensor-to-scalar ratio is small but not too small ($r \approx 0.01$) the scale of inflation can be commensurate with the GUT scale, i.e., $k_{\mathrm{inf}} \sim k_{\mathrm{GUT}} = 2 \times 10^{16}$GeV.

The results of this subsection have several important consequences. It was shown that the simplest possible quadratic inflationary model, i.e., the LFI potential (165) is excluded by Planck data as a viable scenario for inflation. It was also shown that SG model, i.e., the NI potential (166) gives a moderately good agreement with the Planck data if the frequency is chosen to be small, i.e., $\tilde{\beta} = 0.15$. It turned out, that the MSG model, more precisely, the MNI potential (171) and their variants, (172) and (173) serve as excellent inflationary potentials. The slow-roll analysis fixes the ratio $\tilde{u}/\tilde{M}^2 \approx 1/(0.22)^2$ and the frequency $\tilde{\beta} \approx 0.3$. The potential is determined by the slow-roll conditions up to an overall multiplicative factor, but this factor is fixed by the absolute normalisation, (176). Thus, the theoretical predictions obtained from the MSG model are in an excellent agreement with observations and one can use the results of the slow-roll study in order to fix all parameters of the MSG model at the scale of inflation. Therefore, one can argue that the scalar sector of the gravity-scalar system (135) has to be chosen to be equal to an SG or which is even better, an MSG type model because then the scalar part can be used to initiate inflation. In addition, as I argued, SG and MSG type models can also be used as effective branon potentials, so, the hypothetical branon particle can be identified with the inflaton. Both are represented by scalar fields. The recent detection of the Higgs boson renewed research activity where the inflaton is identified with the Higgs field. Thus, it is a natural question to ask whether the MSG type branon-inflaton model can serve as an UV completion of the usual quartic Higgs potential. If the answer is affirmative, it is also important to clarify whether one can use RG methods (either perturbative or non-perturbative ones) to connect the parameters of the same MSG type theory at various energy scales [19]. Thus, as a final step of this section, I will briefly overview possible application of the SG and MSG type models for Higgs physics.

5.5 Periodic Higgs potential

There is a strong interest to find a link between the scalar fields of the Higgs and inflationary physics [146, 147, 148, 149, 150, 151, 152, 153, 154, 156, 155]. The Standard Model (SM) Higgs field is an SU(2) complex scalar doublet with four real components, and the underlying symmetry of the electroweak sector is $SU(2)_L \times U(1)_Y$, thus, the SM Higgs Lagrangian reads as

$$\mathcal{L} = (D_\mu \phi)^\star (D^\mu \phi) - V(\phi) - \frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{177}$$

with

$$V = \mu^2 \phi^\star \phi + \lambda(\phi^\star \phi)^2 \tag{178}$$

and

$$D_\mu = \partial_\mu + ig\mathbf{T}\cdot \mathbf{W}_\mu + ig' y_j B_\mu, \tag{179}$$

where the vacuum expectation of the Higgs field is either at zero field for $\mu^2 > 0$ or at $\sqrt{\phi^\star\phi} = \sqrt{-\mu^2/(2\lambda)} = v/\sqrt{2}$ for $\mu^2 < 0$ with $v = 246$ GeV known from low-energy experiments. The field can be parametrized around its ground state, where the unitary phase can be dropped by choosing an appropriate gauge. As a consequence of the Brout-Englert-Higgs mechanism [157, 158], three degrees of freedom of the Higgs scalar field (out of the four) mix with weak gauge bosons. The remaining degree of freedom becomes the Higgs boson discovered at CERN's Large Hadron Collider [159, 160]. The complete Lagrangian for the Higgs sector of the SM with the single real scalar field h reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}M_h^2 h^2 - \frac{M_h^2}{2v}h^3 - \frac{M_h^2}{8v^2}h^4 + \left(M_W^2 W_\mu^+ W^{-\,\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu\right)\left(1 + 2\frac{h}{v} + \frac{h^2}{v^2}\right), \tag{180}$$

where $M_h = \sqrt{-2\mu^2} = \sqrt{2\lambda v^2}$. The measured value for the Higgs mass $M_h = 125.6\,\mathrm{GeV}$ implies $\lambda = 0.13$. Incidentally, I note that the latter value is close to the predicted value based on an assumption of the absence of new physics between the Fermi and Planck scales and the asymptotic safety of gravity [161].

Extrapolating the SM of particle physics up to very high energies leads to an interpretation of the Higgs boson as the inflaton. Therefore, the most "economical" choice would be to use the same scalar field for Higgs and inflationary physics. The action can be defined either in the Jordan frame in which some function of the scalar field multiplies the Ricci scalar R, or in the Einstein frame in which the Ricci scalar is not multiplied by a scalar field [162]. To perform the slow-roll study, the action is usually rewritten in the Einstein frame and it takes the form for the case of minimal coupling to gravity,

$$S = \int d^4x\sqrt{-g}\left[\frac{m_p^2 R}{2} - \frac{1}{2}g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right], \qquad V \equiv \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 = \frac{M_h^2}{8v^2}\left(\phi^2 - v^2\right)^2, \tag{181}$$

where the metric tensor being denoted by $g^{\mu\nu}$, $\sqrt{-g} \equiv \sqrt{-\det g}$ while $\phi \equiv h$ and V is the quartic-type double-well scalar potential of (180),

$$V(\phi) = \frac{\lambda}{4}v^4 - \frac{1}{2}\lambda v^2\phi^2 + \frac{\lambda}{4}\phi^4\,. \tag{182}$$

where the field variable is shifted as $h \to h + v$.

Another proposal to build up the scalar sector is the Higgs inflation from false vacuum with minimal coupling to gravity where the SM Higgs potential is extended and assumed to develop a second (or more) minimum [146, 147, 148, 149]. The difficulty is to achieve an exit from the inflationary phase: one may introduce new fields, but then the attractive minimality of the model would be lost.

Another possible drawback of the Higgs-inflaton potential to its applicability is that the measured Higgs mass is close to the lower limit, 126 GeV, ensuring absolute vacuum stability within the

SM [163]. However, it was also shown [164] that traditional Higgs inflation can be possible within a minimalistic framework even if the SM vacuum is not completely stable. Various polynomial Higgs potentials have been studied by functional RG [165, 166] and reported no stability problems.

Let me try to consider the previously introduced MNI (Massive Natural Inflation) model which is an MSG type scalar theory and I used three variants, (171), (172) and (173). All versions of the MNI model can be written in a general form

$$V_{\rm MNI}(\phi) = V_0 + \frac{1}{2}M^2\phi^2 + u\cos(\beta\phi), \tag{183}$$

which contains two adjustable parameters (the ratio u/M^2 and the frequency β) and a normalization (the field-independent terms has been fixed by me). The Taylor expansion of the MNI model (183) recovers the SM Higgs potential (182) up to quartic terms and the parameters can be related [19],

$$V_{\rm MNI} \approx V_0 + \frac{1}{2}(M^2 - u\beta^2)\phi^2 + \frac{1}{24}u\beta^4\phi^4 + \mathcal{O}(\phi^6)\,, \tag{184}$$

so that

$$\lambda v^2 \equiv (u\beta^2 - M^2), \quad \lambda \equiv \frac{1}{6}u\beta^4. \tag{185}$$

Thus the MNI model (183) can be considered as an UV extension of the SM Higgs potential. The measurable quantities are related to the parameters of the model according to the following relations [19],

$$M_h \equiv M\sqrt{2\left(\frac{u\beta^2}{M^2} - 1\right)}, \quad v \equiv \frac{1}{\beta}\sqrt{\frac{6(u\beta^2/M^2 - 1)}{u\beta^2/M^2}}. \tag{186}$$

Their low-energy/IR values are given at the electroweak scale by

$$M_{h,\rm IR} = 125 \text{ GeV}, \quad v_{\rm IR} = 245 \text{ GeV}\,, \tag{187}$$

at the scale $k_{\rm IR} \sim 250$GeV.

Let me note that the Higgs mass and vacuum expectation value (VeV) defined by (186) can be calculated also at the cosmological scales. For example, the slow-roll study produces values for the Higgs mass which serves as a high-energy/UV scale. In the previous subsection I discussed that for small frequencies (small β) one finds the following slow-roll results for the MNI potential, [19],

$$\frac{\tilde u\tilde\beta^2}{\tilde M^2} \approx \frac{0.3^2}{0.22^2} > 1\,. \tag{188}$$

which, together with the normalization condition leads to the following numerical values for the dimensionless and dimensionful parameters,

$$\tilde M \approx 5.92\times 10^{-6} \;\Rightarrow\; M \approx 1.42\times 10^{13}\,\text{GeV}\,, \tag{189a}$$
$$\tilde u \approx 7.24\times 10^{-10} \;\Rightarrow\; u \approx 2.4\times 10^{64}\,\text{GeV}^4\,, \tag{189b}$$
$$\tilde\beta \approx 0.3 \;\Rightarrow\; \beta \approx 1.25\times 10^{-19}\,\text{GeV}^{-1}\,. \tag{189c}$$

which results in

$$M_{h,\rm UV} \sim 10^{15}\,\text{GeV} \tag{190a}$$

at the scale $k_{\rm UV} \sim 10^{15}$GeV which needs to be scaled down (by orders of magnitude) to its measured value at the electroweak scale (187). In this lecture note I will show how to do this in the framework of the FRG method.

6 Functional RG – Wegner-Houghton and Polchinski equations

To describe an arbitrary physical system at a certain observational (i.e., energy) scale one has to find the *relevant* (important) interactions, and degrees of freedom of the system at this energy. If we change the scale, new interactions could become relevant. This is a consequence of the fact, that the description of a physical system strongly depends on the observational scale. For example, at the macroscopic length scale we can use the Newton equation to evaluate the motion of a macroscopic physical system. At the microscopic length scale, e.g. at the atomic one, the atoms inside the system are the relevant degrees of freedom, and new interactions become important therefore one has to use e.g. the Schrödinger equation in order to describe the components of that system. It is important, that usually the relevant interactions and degrees of freedom of a system at a certain energy scale are relatively independent of what is the behavior of the system at lower or at higher energy scales. Therefore, instead of using a kind of theory which contains all the interactions that could be relevant in any length scale, one can use a chain of effective theories valid in different energy domains. Generally, during the renormalization one relates a low-energy effective theory to a model defined at the high-energy scale. In order to obtain the low-energy effective theory, a possible method is the usage of the renormalization group (RG) transformations, which relate the effective theories at different energy scales.

6.1 Scale invariance and the Wilson-Kadanoff blocking

Since the cornerstone of the RG method is scale invariance, let me first discuss it in details. In the thermodynamic limit, the statistical systems are determined by their thermodynamic potential Φ. The absolute minima of Φ corresponds to the state of equilibrium of the system. The thermodynamic potential is a continuous function of the parameters of the system (e.g. $\Phi = \Phi(T, p)$ with the temperature T, the pressure p). Therefore, if Φ_1 and Φ_2 are the thermodynamic potentials of the two different phases of the model, the following relation holds

$$\Phi_1(T, p) = \Phi_2(T, p). \tag{191}$$

This implies that the curve $p = p(T)$ on the (p − T) plane separates the two phases of the model. Crossing this separator $p(T)$ the system undergoes a phase transition. Due to the theorem of Ehrenfest, the phase transitions can be classified according to the partial derivatives of the thermodynamic potential Φ. The phase transition is of the first, second, third etc. order if the first, second, third etc, partial derivatives of Φ are discontinuous, respectively.

It is argued that in case of second order phase transitions various physical quantities (e.g. heat capacity, susceptibility etc.) have an asymptotic scaling behavior near the phase transition point T_c. They are power-law functions of the reduced temperature $t = (T - T_c)/T_c$ and the external field h. This is the critical behavior. The correlation function $G(r)$, the correlation length ξ and the order parameter Δ have also critical scaling behavior near T_c. This asymptotic scaling behavior can be found in physically different systems. The exponents of the power-law functions (the critical exponents) can be different in various models but there are relations between these exponents (scaling laws) which are found to be universal. One can classify the models into different universality classes according to their critical exponents. Systems belonging to the same universality class, have the same critical exponents.

What is the physical reason for the critical behavior? The answer is scale invariance. The systems at their phase transition points are scale invariant. Changing the observational scale (e.g. changing the lattice site $a \to a'$), the functional form of the thermodynamic potential of the model remains unchanged:

$$\Phi'(t', h') = \Phi(t', h'), \tag{192}$$

where Φ and Φ' are the thermodynamic potentials of the original and the rescaled systems, respectively. Both of them depends on the rescaled reduced temperature t' and the rescaled external field h'. Therefore, the functional form of the partition function of the model remains unchanged.

One can find the analogy to QFT where the generating functional is invariant under changing the observational scale (moving the momentum cut-off). Due to scale invariance (192), the thermodynamic potential is a homogeneous function of the reduced temperature t and the external field h,

$$\Phi(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda \Phi(t, h), \tag{193}$$

where a_t, a_h and λ are given. This relation can be understood by the Wilson-Kadanoff blocking construction [167]. Let me consider a classical spin system on the lattice. In one blocking step, the lattice size of the original system is rescaled $a' = b\,a$, see Fig. 6.

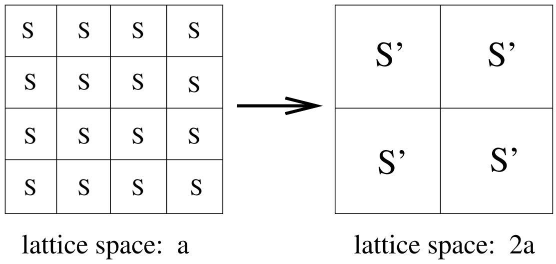

Figure 6: Schematic picture of the Kadanoff-Wilson blocking [167]. In one blocking step, the lattice size of the original system is rescaled $a' = b\,a$ where in the figure $b = 2$ has been used.

In the blocked system, one can define a block of spins with the new lattice size a', which contains b^d lattice points of the original system. In every block the spins of the original system are replaced by the 'average' of the spins. Every physical quantity is rescaled according to the new lattice size. Since the thermodynamic potential is an extensive quantity, Φ' of one block, is equal to the original Φ multiplied with the number of the lattice points in the block,

$$\Phi'(t', h') = b^d\, \Phi(t, h). \tag{194}$$

Due to scale invariance close to the scale invariant fixed point, the following relations hold for the blocked reduced temperature and the blocked external field

$$t' = b^{a_t d} t, \qquad h' = b^{a_h d} h. \tag{195}$$

Inserting Eq. (195) into the (194), the homogenity (193) of the thermodynamic potential is obtained (with $\lambda = b^d$).

The homogenity of the thermodynamic potential (193) implies critical scaling properties. The critical exponents can be derived from the relation (193) by derivation with respect to the external field or the reduced temperature. For example, the critical exponent δ is obtained in a following way. The equation (193) is differentiated with respect to the external field h

$$\frac{\partial \Phi(\lambda^{a_t} t, \lambda^{a_h} h)}{\partial h} = \lambda^{a_h} \Delta(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda \Delta(t, h), \tag{196}$$

where $\Delta(t, h) \equiv \partial \Phi(t, h)/\partial h$ is the order parameter of the phase transition. Since λ is arbitrary, it can be defined via $1 = \lambda^{a_h} h$. Introducing $\lambda = h^{-1/a_h}$ into the equation (196) and setting $t = 0$, the equation reduces to

$$\Delta(0, 1) h^{(1-a_h)/a_h} = \Delta(0, h) \tag{197}$$

where $\Delta(0, 1)$ is constant. The critical exponent δ is obtained from (197) $\delta = a_h/(1 - a_h)$. In this manner, other critical exponents can be derived from the relation (193) which explains the relations between the critical exponents, since they can depend on a_t, and a_h only. Therefore, I conclude the discussion of phase transitions with the important statement, that the scale invariance of the thermodynamic potential implies the critical behavior of the systems near their phase transition point.

6.2 RG transformation

During the application of the RG transformations, the degrees of freedom which are responsible for the high-energy (microscopic) behavior of the system are integrated out and their impact are taken into account by influencing the scaling of the parameters (coupling constants) of the system.

Let me first consider a schematic interpretation of the RG transformation. A system near the phase transition point is scale invariant, thus in the framework of Kadanoff blocking construction [167], its partition function is assumed to be invariant under a blocking step

$$Z \equiv Tr \exp[-\beta J_a \sum S_i S_j] = Tr' \exp[-\beta J_{2a} \sum S'_i S'_j]$$

where $\beta \equiv 1/(k_b T)$ is the inverse temperature. What happens if the system is far from the transition point or if it does not undergo any phase transitions? In this case new interaction terms are generated by the blocking step and the functional form of the partition function is not preserved

$$H_a = J_a \sum S_i S_j \quad \rightarrow \quad H_{2a} = J_{2a} \sum S'_i S'_j + G_{2a} \sum S'_i S'_j S'_k.$$

The solution for this problem is Wilson's idea [168]: let us start with a general ansatz which contains all the interaction terms generated by the blocking transformations

$$H_a = J_a \sum S_i S_j + G_a \sum S_i S_j S_k, \qquad G_a = 0$$

then the functional form is preserved

$$\begin{aligned} H_a &= J_a \sum S_i S_j + G_a \sum S_i S_j S_k \\ H_{2a} &= J_{2a} \sum S'_i S'_j + G_{2a} \sum S'_i S'_j S'_k \\ H_{3a} &= J_{3a} \sum S''_i S''_j + G_{3a} \sum S''_i S''_j S''_k \end{aligned}$$

and one can read off RG flow equations for the couplings

$$\frac{d}{da} J(a) = f_1(J, G, a), \qquad \frac{d}{da} G(a) = f_2(J, G, a).$$

The solution of the RG flow equations provides us with the scaling of the coupling constants of the theory [168].

Let me consider a blocking in the momentum space where the scale parameter is a running momentum cut-off, $k \sim 1/a$. The fixed-point of the RG transformation ($\mathcal{R}_k$) is defined as

$$\mathcal{R}_k(H^*) = H^* \tag{198}$$

where H^* is the (strictly speaking dimensionless) fixed-point Hamiltonian of the system. Around H^*, one can classify the coupling constants of the model in the following way. Using the assumption that the RG transformation $\mathcal{R}_k$ is an analytical function of the coupling constants of the theory, one can expand the transformation $\mathcal{R}_k$ around the fixed-point Hamiltonian H^*. Considering the action of $\mathcal{R}_k$ on the Hamiltonian $H = H^* + \epsilon O$ with ϵ infinitesimal,

$$\mathcal{R}_k(H^* + \epsilon O) = \mathcal{R}_k(H^*) + \epsilon L_k(O) = H^* + \epsilon L_k(O) \tag{199}$$

the linearized RG transformation L_k is defined around the fixed-point H^*. Then one has to find the eigenvectors (scaling operators) O_i of the linearized RG transformation L_k,

$$L_k(O_i) = \lambda_i(k)\, O_i \tag{200}$$

with the eigenvalues $\lambda_i(k)$ depending on the parameter k of the RG transformation (e.g. in case of differential RG transformation performed in momentum space, k is the momentum cut-off). The Hamiltonian of the model can be written

$$H = H^* + \sum_i g_{i,0} O_i \tag{201}$$

with the coupling constants g_i. The eigenvalues $\lambda_i(k)$ are the power-law function of the scale parameter of the RG transformation,

$$\lambda_i(k) = k^{y_i}. \tag{202}$$

Then performing the RG transformation on the Hamiltonian of the system,

$$\mathcal{R}_k(H(g_{i,0})) = \mathcal{R}_k(H^* + \sum_i g_{i,0}O_i) = H^* + \sum_i g_{i,0}k^{y_i}O_i, \tag{203}$$

the RG equation for the coupling constants g_i is obtained:

$$g_i(k) = g_{i,0}\, k^{y_i}. \tag{204}$$

Around the fixed point, the exponents y_i determine the scaling of the coupling constants. There are relevant, irrelevant or marginal scaling operators (coupling constants) corresponding to negative, positive or zero exponent of the eigenvalues, see Fig. 7.

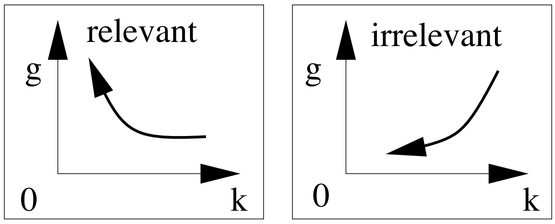

Figure 7: Changing the scale parameter of the RG transformation (e.g. decreasing the momentum scale k) around the fixed point, the relevant (irrelevant) coupling constant is increasing (decreasing).

It is possible to perform the renormalization group transformations in the coordinate or equivalently in the momentum space. In this chapter I discuss the RG transformations in the momentum space based on Wilson's RG approach, where the scale parameter of the RG transformation is the moving momentum cut-off k. During the differential RG transformation the momentum cut-off k is decreased with infinitesimal steps from the UV cut-off Λ towards $k = 0$ and the high-frequency quantum fluctuations of the field ($\phi_p, \quad p > k$) are integrated out in every infinitesimal steps. The infinitesimal changes of the cut-off k provide infinitesimal changes of the action, that is, infinitesimal changes of the coupling constants.

Changing the scale parameter, which implies moving the momentum cut-off k to the IR (low-energy) limit, the relevant coupling constants increase, the irrelevant coupling constants decrease. Since the coupling of the irrelevant scaling operators go to zero in the IR limit, they do not play any role in the low-energy behavior of the theory. Therefore, theories which are originally defined at an UV scale in a different way, but differ from each other only in irrelevant interactions, should have the same low-energy effective theory. This is called universality. The irrelevant interaction terms do not influence the critical behavior of the system. Therefore, theories belonging to the same universality class have the same critical exponents, they differs from each other only up to irrelevant terms.

In the framework of the usual, perturbative renormalization the theories which contain irrelevant interactions cannot be considered consistently, since the irrelevant coupling constants increase towards the high energies, and the corresponding vertices V have positive canonical dimensions $\delta(V) > 0$, so that the UV momentum cut-off cannot be removed to infinity. In this classification the irrelevant interactions corresponds to non-renormalizable theories. On the other hand, the irrelevant interactions are unimportant, because due to universality they do not influence the low-energy effective theory of the model. Therefore, the view was held for a long time that renormalizable theories describe the real physical world containing only relevant and marginal interactions.

Universality may be lost, however if there is more than one fixed point in the theory [169]. In this case, around every different fixed points one may find different classifications of the coupling constants into relevant, irrelevant or marginal coupling constants [169]. It can happen that an irrelevant interaction at a high energy fixed point becomes relevant at a low energy fixed-point Fig. 8.

If there is no universality, the irrelevant interactions at the UV fixed-point may become important at low energy, so one has to use a renormalization method which can follow the scaling of the irrelevant coupling constants, as well. The RG method has several advantages. (i) It is one of

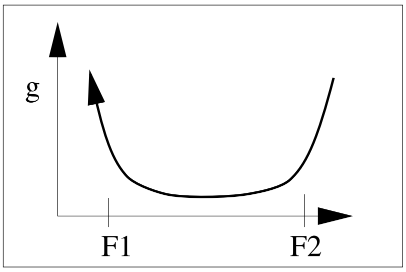

Figure 8: Decreasing the momentum scale k it may happen that the irrelevant coupling constants near the fixed-point $F2$ become relevant around another fixed-point $F1$.

the most important features of the RG method that it can handle models which contain irrelevant interactions. (ii) The RG method provides a scheme for summing up all the quantum contributions non-perturbatively. (iii) It has a differential formulation. In the next subsections, I discuss two possible functional RG methods, the Wegner–Houghton [170] and the Polchinski [171] equations which are based on Wilson's RG approach [168] applied in the momentum space.

6.3 Gradient expansion

The functional RG method is applied in momentum space with sharp cut-off to the one-component scalar field theory, in order to obtain the scale dependence of the couplings. It is generally assumed that the blocked action $S_k[\phi]$ contains only local interactions, and that it can be expanded in the gradient of the field [172, 173]

$$S_k[\phi] = \int \mathrm{d}^d x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)\ (\partial_\mu \phi)(\partial^\mu \phi) + Y_k(\phi)\ ((\partial_\mu \phi)(\partial^\mu \phi))^2 + ... \right]. \tag{205}$$

where the leading order term of the gradient expansion is called the Local Potential Approximation (LPA) and the wavefunction renormalization is taken into account beyond LPA. Let us note, LPA' denotes the case when the field dependence of the wavefunction renormalization is neglected, i.e., $Z_k(\phi) = z(k)$.

6.4 Wegner–Houghton RG equation

In the framework of Wilson's renormalization group [168] approach the differential RG transformations are realized via a blocking construction [167], the successive elimination of the degrees of freedom which lie above the running ultraviolet momentum cut-off k. Consequently, the effective theory defined with the action $S_k[\phi]$ contains quantum fluctuations whose frequencies are smaller than the momentum cut-off k. The separation of the modes of the field according to their length scale is not a gauge invariant method, therefore the gauge symmetry is lost during the blocking. One possible solution for this problem could be the usage of the smooth cut-off, where the higher frequency modes of the field are suppressed partially, but not eliminated.

6.4.1 Exact Wegner–Houghton RG equation

The generating functional Z, expressed in terms of the action $S_k[\phi]$ defined at the momentum cut-off k, reads as follows

$$Z = \left(\prod_{|q|<k} \int \mathrm{d}\Phi_q \right) \exp\left[-\frac{1}{\hbar} S_k[\Phi_q] \right]. \tag{206}$$

The action $S_k[\Phi_q]$ depends on the field decomposed in Fourier series, contains modes ϕ_q with momenta less than the moving momentum cut-off $q < k$. Applying an infinitesimal RG transformation, the field is expanded in Fourier series and separated into a slow and a fast fluctuating part, which obey low-frequency and high-frequency Fourier modes, respectively,

$$\Phi(x) = \phi(x) + \tilde{\phi}(x) = \sum_{|q|<k-\Delta k} \phi_q e^{iqx} + \sum_{k-\Delta k<|q|<k} \tilde{\phi}_q e^{iqx}. \tag{207}$$

Then the momentum cut-off k is moved to $k - \Delta k$, and the high-frequency fluctuations of the field are integrated out in the momentum space. On the one hand, the expression (206) can be rewritten

$$Z = \left(\prod_{|q|<k-\Delta k} \int \mathrm{d}\phi_q \right) \left(\prod_{k-\Delta k<|q|<k} \int \mathrm{d}\tilde{\phi}_q \right) \exp\left[-\frac{1}{\hbar} S_k[\Phi_q] \right]. \tag{208}$$

On the other hand, requiring the invariance of the generating functional Z:

$$Z = \left(\prod_{|q|<k-\Delta k} \int \mathrm{d}\phi_q \right) \exp\left[-\frac{1}{\hbar} S_{k-\Delta k}[\phi_q] \right], \tag{209}$$

one can read off the transformation for the blocked action when the cut-off is moved from the UV cut-off k to $k - \Delta k$,

$$\exp\left[-\frac{1}{\hbar} S_{k-\Delta k}[\phi] \right] = \int \mathcal{D}[\tilde{\phi}] \ \ \exp\left[-\frac{1}{\hbar} S_k[\phi + \tilde{\phi}] \right], \tag{210}$$

where the field variables ϕ and $\tilde{\phi}$ contain Fourier components with momenta $|p| < k - \Delta k$, and $k - \Delta k < |p| < k$, respectively. In every infinitesimal step, the path integration in Eq. (210) is evaluated with the help of the saddle point approximation. There are two cases. The action has a saddle point either at $\tilde{\phi} = 0$ (any constant saddle point can be transformed to zero by a constant shift of the field variable), or at $\tilde{\phi} = \tilde{\phi}_{cl} \neq 0$. In both cases, I keep only the linear and the quadratic terms, in the Taylor expansion at the saddle point therefore one can perform the Gaussian integration. The expansion around a general saddle point $\tilde{\phi}_{cl}$ reads:

$$S_k[\phi + \tilde{\phi}_{cl} + \tilde{\phi}'] = S_k[\phi + \tilde{\phi}_{cl}] + \sum_{k-\Delta k<|p|<k} F_p \, \tilde{\phi}'_p + \frac{1}{2} \sum_{k-\Delta k<|p|<k} \tilde{\phi}'_p \, K_{p,-p} \, \tilde{\phi}'_{-p} + \mathcal{O}(\tilde{\phi}'^3) \tag{211}$$

with the saddle point equations

$$F_p = \frac{\delta S_k[\phi + \tilde{\phi}_{cl}]}{\delta \phi_p} = 0, \qquad K_{p,-p} = \frac{\delta^2 S_k[\phi + \tilde{\phi}_{cl}]}{\delta \phi_p \delta \phi_{-p}}. \tag{212}$$

Then one can evaluate the Gaussian integral in (210) and the result is

$$\exp\left[-\frac{1}{\hbar} S_{k-\Delta k}[\phi + \tilde{\phi}_{cl}] \right] = \left(\det K_{p,-p}[\phi + \tilde{\phi}_{cl}] \right)^{-1/2} \exp\left[-\frac{1}{\hbar} S_k[\phi + \tilde{\phi}_{cl}] \right]. \tag{213}$$

After taking the logarithm of both sides of (213) and using the identity $\ln \det(K) = \mathrm{Tr} \ln(K)$, it becomes

$$-\frac{1}{\hbar} S_{k-\Delta k}[\phi + \tilde{\phi}_{cl}] = -\frac{1}{\hbar} S_k[\phi + \tilde{\phi}_{cl}] - \frac{1}{2} \mathrm{Tr} \ln(K_{p,-p}[\phi + \tilde{\phi}_{cl}]) + \mathcal{O}(\hbar^2). \tag{214}$$

If Δk is infinitesimal, the trace in Eq. (214) can be written as $\mathrm{Tr} = k^{d-1}\Delta k(2\pi)^{-d}\int \mathrm{d}\omega$ where $\int \mathrm{d}\omega$ is the integral over the solid angle in dimension d. Thus, the second term on the r.h.s. of (214) is $\mathcal{O}(\Delta k)$ and one can see that the higher-loop contributions neglected in (214) can give only $\mathcal{O}(\Delta k^2)$ contributions for the blocked action. Therefore taking the limit $\Delta k \to 0$ the equation (214) becomes an exact, integro-differential equation for the blocked action, which contains all the loop contributions. This is called the Wegner–Houghton renormalization group equation [170]

$$k\partial_k S_k[\phi+\tilde{\phi}_{cl}] = -\frac{k^d}{2}\int \frac{\mathrm{d}\omega}{(2\pi)^d}\,\hbar\,\ln(K_{p,-p}[\phi+\tilde{\phi}_{cl}]), \tag{215}$$

where $\tilde{\phi}_{cl} = \tilde{\phi}_{cl}[\phi]$ as the functional of the background field ϕ is given by $\delta S_k[\phi+\tilde{\phi}_{cl}]/\delta\phi_p = 0$. If the saddle point is trivial $\tilde{\phi}_{cl} = 0$, then the Wegner–Houghton RG equation [170] reduces to

$$\boxed{k\partial_k S_k[\phi] = -\frac{k^d}{2}\hbar\int \frac{\mathrm{d}\omega}{(2\pi)^d}\ln\left(\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right).} \quad \Leftrightarrow \quad \boxed{\partial_k S_k[\phi] = -\frac{\hbar}{2}\,\mathrm{Tr}\,\ln\left(\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right)}. \tag{216}$$

When the saddle point is non-trivial $\tilde{\phi}_{cl} \neq 0$, then the inverse propagator $K_{p,p'} = \delta^2 S_k[\phi+\tilde{\phi}_{cl}]/\delta\phi_p\delta\phi_{-p}$ has at least one negative eigenvalue. The system becomes unstable against developing an inhomogeneous classical field configuration $\tilde{\phi}_{cl}$, and the equation (216) loses its validity. In this case, one should expand the action $S_k[\phi+\tilde{\phi}]$ around its true saddle point $\tilde{\phi}_{cl}$, and one arrives at the system of equations (212) and (215). This is, however, of small practical use. Instead, the non-trivial saddle point can be determined by minimizing the action in Eq. (210) directly [169, 174]

$$S_{k-\Delta k}[\phi] = \min_{\tilde{\phi}_{cl}}\left[S_k[\phi+\tilde{\phi}_{cl}]\right]. \tag{217}$$

For this one has to restrict the search of the minimum to a subspace of functions, e. g. to plane-wave like saddle points. In order to follow the evaluation of the action one can use the RG equation (216) until the eigenvalues of $K_{p,p'}$ are positive, $k_\mathrm{c}[\phi] < k$. If $k < k_\mathrm{c}[\phi]$ then one should use the tree-level blocking relation (217), see [169]. When the saddle point becomes non-trivial, the argument of the logarithm in equation (216) becomes zero, which determines the critical scale k_c implicitly. The computation of the higher-order quantum corrections is difficult since the propagator is non-diagonal in momentum space when the saddle point differs from zero.

6.4.2 Wegner–Houghton RG equation and the loop expansion

The solution of equation (216) is the scale dependent blocked action $S_k[\phi]$, which tends to the full quantum effective action $\Gamma_\mathrm{eff}[\phi]$ in the limit $k \to 0$, i.e. $S_{k\to 0}[\phi] = \Gamma_\mathrm{eff}[\phi]$. One can read off the 1-loop contribution from (216) using the independent mode approximation where the k-dependence of the action is ignored [175]

$$\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi} \quad \Rightarrow \quad \frac{\delta^2 S_\Lambda[\phi]}{\delta\phi\delta\phi} \tag{218}$$

inside the argument of the logarithm

$$\partial_k S_k[\phi] = -\frac{k^{d-1}}{2}\hbar\int \frac{\mathrm{d}\omega}{(2\pi)^d}\ln\left(\frac{\delta^2 S_\Lambda[\phi]}{\delta\phi\delta\phi}\right) \tag{219}$$

where the action $S_\Lambda[\phi]$ is scale-independent. One can compare equation (219) with the perturbative form of the effective action up to one-loop order, which reads as follows

$$\Gamma_\mathrm{eff} = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \mathcal{O}(\hbar^3), \tag{220}$$

with the tree-level action $\Gamma_0 = S_\Lambda[\phi]$ and with the 1-loop correction

$$\Gamma_1 = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\ln\left(\frac{\delta^2 S_\Lambda[\phi]}{\delta\phi\delta\phi}\right) \tag{221}$$

Integrating out both sides of equation (219) with respect to k between zero and Λ, one finds Eqs. (220) and (221). In this sense, the Wegner-Houghton RG equation (216) can be understood as a 1-loop improved RG equation [175].

It is interesting to consider the case when one integrates both sides of equation (219) with respect to k between $k - \Delta k$ and Λ. In this case the result is a one-loop expression at the scale $k - \Delta k$ which can be substituted back into the logarithm of (219) and the momentum integration can be performed again between $k - 2\Delta k$ and $k - \Delta k$. The new result is $\mathcal{O}(\hbar^2)$. Repeating this procedure and keeping $\Delta k/k$ to be a small number it is obvious that infinitely many RG steps are required to reach $k \to 0$ [169]. Therefore, the exactness of the Wegner-Houghton RG equation (216) is just the consequence of these infinitely many RG steps required for the IR limit, see [169].

6.4.3 Wegner–Houghton RG equation and the gradient expansion

In the LPA one uses the leading order expression for the action in the gradient expansion (205),

$$S_k = \int \mathrm{d}^d x \ \left[\frac{1}{2}\ (\partial_\mu \phi)(\partial^\mu \phi) + V_k(\phi)\right], \tag{222}$$

and the Wegner–Houghton equation (216) reduces to a differential equation for the scale dependent potential $V_k(\phi_0)$ and ($\phi(x) = \phi_0$ with $\phi_0 = $ constant).

$$\boxed{k\partial_k V_k(\phi_0) = \ -k^d \hbar\, \alpha_d\ \ln\left(\frac{k^2 + \partial^2_{\phi_0} V_k(\phi_0)}{k^2}\right)}, \tag{223}$$

with $\alpha_d = \frac{1}{2}\Omega_d (2\pi)^{-d}$, and the solid angle Ω_d in dimension d. Let us note that the equation (223) is exact in that sense that it contains all the quantum corrections.

There are dimensionful quantities in equation (223). In order to look for fixed-point solutions of (223) or to follow the scaling of an arbitrary potential one should remove the trivial scaling of the dimensionful coupling constants and rewrite equation (223). Therefore, one has to introduce dimensionless quantities via the following reparametrization

$$\tilde{\phi} = k^{-\frac{d-2}{2}}\phi, \qquad\qquad \tilde{x}_\mu = k x_\mu, \tag{224}$$

with the changes in the derivate of the potential with respect to the field

$$V_k(\phi) = k^d \tilde{V}_k(\tilde{\phi}), \qquad\qquad \partial^2_\phi V_k(\phi) = k^d k^{-(d-2)} \partial^2_{\tilde{\phi}} \tilde{V}_k(\tilde{\phi}). \tag{225}$$

One can rewrite equation (223) for dimensionless quantities

$$\boxed{\left(d - \frac{d-2}{2}\tilde{\phi}_0 \partial_{\tilde{\phi}_0} + k\partial_k\right) \tilde{V}_k(\tilde{\phi}_0) = \ -\hbar\, \alpha_d\ \ln\left(1 + \partial^2_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0)\right)}. \tag{226}$$

Notice that in dimension $d = 2$ the field has no trivial scale dependence, therefore the second term on the left-hand side of (226) does not appear in the dimensionless equation. it is useful to reduce the dimensionless equation (226) for dimension $d = 2$

$$(2 + k\partial_k)\tilde{V}_k(\tilde{\phi}_0) = \ -\hbar\, \alpha_2\ \ln\left(1 + \partial^2_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0)\right) \tag{227}$$

with $\alpha_2 = (4\pi)^{-1}$. Notice that the argument of the logarithm in (223) or in (226) must be non-negative for the expansion made around a stable saddle point. If the argument changes sign at a critical value $k_c > 0$, given by $k_c^2 = -\partial^2_\phi V_{k_c}(\phi_0)$ then the Wegner–Houghton equation (223) loses its validity for $k < k_c$. The saddle point becomes non-zero and the tree-level blocking relation (217) has to be used. Restricting the search for the minimum to plane waves propagating in a given direction n_μ we find from Eq. (217)

$$V_{k-\Delta k}(\phi_0) = \min_\rho \left[k^2\rho^2 + \frac{1}{2}\int_{-1}^{1} \mathrm{d}u\ \ V_k(\phi_0 + 2\rho\cos(\pi u))\right], \tag{228}$$

where ρ is the amplitude of the plane wave.

In order to clarify the supposed role of the wavefunction renormalization ($Z_k(\phi)$) I let the next-to-leading term in the gradient expansion to evolve, but neglect its field dependence (that is using the ansatz $Z_k(\phi) = z(k)$)

$$S_k = \int \mathrm{d}^d x \left[z(k) \, \frac{1}{2} \, (\partial_\mu \phi)(\partial^\mu \phi) + V_k(\phi) \right] . \tag{229}$$

Inserting $\phi(x) = \phi_0 + \epsilon(x)$ with $\phi_0 =$ const. and $\epsilon(x)$ infinitesimal inhomogeneous (therefore, $\epsilon(x)$ depends on the space-time) into both sides of Eq. (215), expanding them in powers of $\epsilon(x)$, and keeping the terms up to the second order, the Wegner–Houghton equation can be reduced to [2]

$$\begin{aligned} k\partial_k V_k(\phi_0) &= -k^d \hbar \, \alpha \, \ln \left(\frac{z(k)k^2 + \partial^2_{\phi_0} V_k(\phi_0)}{k^2} \right) , \\ k\partial_k z(k) &= k^d \hbar \, \alpha \, [\partial^3_{\phi_0} V_k(\phi_0)]^2 \left[\frac{4 \, [z(k)]^2 \, k^2}{d \; A^4} - \frac{z(k)}{A^3} \right] , \end{aligned} \tag{230}$$

with $A = (z(k)k^2 + \partial^2_{\phi_0} V_k(\phi_0))$. The last equation should hold only up to $\mathcal{O}(\phi_0^0)$ since the ansatz for the action contains only the field independent wavefunction renormalization. Equation (230) can be rewritten for dimensionless coupling constants [2]

$$\begin{aligned} \left(d - \frac{d-2}{2} \tilde{\phi}_0 \partial_{\tilde{\phi}_0} + k\partial_k \right) \tilde{V}_k(\tilde{\phi}_0) &= -\hbar \, \alpha \, \ln \left(z(k) + \partial^2_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0) \right) , \\ k\partial_k z(k) &= \hbar \, \alpha \, [\partial^3_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0)]^2 \left[\frac{4 \, [z(k)]^2}{d \; \tilde{A}^4} - \frac{z(k)}{\tilde{A}^3} \right] , \end{aligned} \tag{231}$$

with $\tilde{A} = (z(k) + \partial^2_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0))$.

Unfortunately the gradient expansion contradicts the usage of the sharp momentum cut-off. The higher order terms in the gradient expansion which correspond to the higher derivatives of the field, cannot be considered consistently due to the sharp momentum cut-off used in the Wegner–Houghton RG method [169]. Therefore it is not possible to obtain reliable RG equations for the field dependent wavefunction renormalization in the framework of the Wegner–Houghton method. One possibility to avoid the problems with the gradient expansion is the usage of a smooth cut-off. In order to consider the renormalization of $Z_k(\phi)$, in the next subsection, I discuss Polchinski's renormalization group method [171] which realizes the RG transformations in successive infinitesimal steps in momentum space using a smooth cut-off function.

6.5 Polchinski RG equation

In Polchinski's renormalization group method [171] the realization of the differential RG transformations is based on a non-linear generalization of the blocking procedure using a smooth momentum cut-off. In the infinitesimal blocking step, the field variable $\Phi(x)$ is split into an IR (slow oscillating) and an UV (fast oscillating) part, but both fields contain low- and high-frequency modes due to the smoothness of the cut-off. The propagator for the IR component is suppressed by a properly chosen regulator function $K(p^2/k^2)$ at high frequency above the moving momentum scale k. The introduction of the regulator function generates infinitely many vertices with higher derivatives of the field. But these vertices are considered irrelevant and their flow is completely neglected (because the regulator function is not evolved under the RG transformations). One of the most important advantages of Polchinski's RG method is the usage of the smooth momentum cut-off which does not contradict with the gradient expansion. Therefore it is possible to consider the evolution of the field dependent wavefunction renormalization ($Z_k(\phi)$). In order to determine Polchinski's RG equation for the one-component scalar field theory, one can follow the method explained in Ref. [169]. Here, I do not discuss the details.

Let me write the partition function for the scalar field Φ at the scale k in the following form

$$Z = \int \mathcal{D}[\Phi] \;\; \exp\left[-\hbar S_k[\Phi]\right] = \int \mathcal{D}[\Phi] \;\; \exp\left[-\frac{1}{2\hbar} \, \Phi \, G_k^{-1} \, \Phi - \hbar \, S^I_k[\Phi] \right] , \tag{232}$$

where the blocked action is split into an interaction part: $S_k^I[\phi]$ and a quadratic part: $\frac{1}{2}\Phi\, G_k^{-1}\, \Phi \;=\; (2\pi)^{-d}\int \mathrm{d}^d p\, \frac{1}{2}\Phi_{-p}\, G_k^{-1}(p^2)\, \Phi_p$ containing the regularized inverse propagator $G_k^{-1}(p^2) = p^2 K^{-1}(p^2/k^2)$. The regulator function $K(z)$ suppresses the high-frequency modes ($|p| \gg k$) and keeps the low-frequency ones ($|p| \ll k$) unchanged due to the limiting behaviors $K(z) \to 0$ for $z \gg 1$ and $K(z) \to 1$ for $z \ll 1$, respectively. Using the above definition, one can derive the Polchinski RG equation for the complete action [171, 169, 3]

$$\boxed{\partial_k S_k[\phi] \;=\; \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \partial_k G_k(p^2)\left[\frac{\delta S_k[\phi]}{\delta\phi_{-p}}\,\frac{\delta S_k[\phi]}{\delta\phi_p} - \hbar\,\frac{\delta^2 S_k[\phi]}{\delta\phi_{-p}\delta\phi_p} - 2\,\phi_p\, G_k^{-1}(p^2)\,\frac{\delta S_k[\phi]}{\delta\phi_p}\right]} \tag{233}$$

which is valid for arbitrary one-component scalar field theory.

In LPA one uses the leading order expression for the action in the gradient expansion, that means the wavefunction renormalization is set to one $Z_k(\phi) = 1$ and the field is set to a constant field: $\phi(x) = \phi_0$. Then the Polchinski equation reduces to a differential equation for the scale-dependent potential $V_k(\phi_0)$ [169, 3]

$$\boxed{k\partial_k V_k(\phi_0) = \hbar V_k^{(2)}(\phi_0)\left(\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, K'(p^2/k^2)\right) - [V_k^{(1)}(\phi_0)]^2 K_0'} \tag{234}$$

with $K' = \partial_{p^2} K(p^2/k^2)$ and the derivatives of the potential $V_k^{(n)}(\phi_0) = \partial_{\phi_0}^n V_k(\phi_0)$. Equation (234) can be rewritten in terms of dimensionless quantities [176, 177]

$$\boxed{\left(d - \frac{d-2}{2}\tilde\phi_0\partial_{\tilde\phi_0} + k\partial_k\right)\tilde V_k(\tilde\phi_0) = \hbar\tilde V_k^{(2)}(\tilde\phi_0)\left(\int \frac{\mathrm{d}^d \tilde p}{(2\pi)^d}\, K'(\tilde p^2)\right) - [\tilde V_k^{(1)}(\tilde\phi_0)]^2 K_0',} \tag{235}$$

with the dimensionless potential $\tilde V_k(\tilde\phi_0)$, dimensionless smooth regulator function $K'(\tilde p^2) = \partial_{\tilde p^2} K(\tilde p^2)$ and the dimensionless momentum $\tilde p^2 = p^2/k^2$.

The Polchinski RG equation in LPA can be rewritten by the redefinition of the field and the potential in such a way that the smooth regulator function does not appear explicitly in the equation. However, beyond LPA it is not possible and one has to specify the specific form for the regulator in order to obtain quantitative predictions. Moreover, beyond LPA the Polchinski RG equation has difficulties in producing reliable results for a periodic scalar field theory [3], thus, I do not discuss it beyond LPA.

7 Functional RG – Wetterich equation

Scale-invariance is the key issue behind Wilson's RG approach [168] where the successive elimination of degrees of freedom is realised by a Kadanoff blocking construction [167] in the momentum space. The Wegner-Houghton [170] and the Polchinski [171] RG equations were the first direct implementations of Wilson's RG idea which are exact equations, i.e., they provide us a framework to perform the renormalization non-perturbatively but valid for functionals thus systematic approximations are required to find their solutions. The gradient (derivative) expansion is one of the commonly used approximation schemes and LPA stands for the leading order term. The Wegner-Houghton (216) and the Polchinski (233) equations are realised by a sharp and a smooth momentum cutoff, respectively. Both has advantages and disadvantages: the sharp momentum cutoff is well defined but confronts to the gradient expansion (beyond LPA); the smooth cutoff can be used at any order of the gradient expansion but the flow equations depend on the particular choice of the cutoff function and (in some cases) it produces unphysical results beyond LPA. In this section I derive the modern form of RG equations, i.e, the Wetterich [178, 179] RG equation and show that it can be considered as a kind of unified RG equation in a sense that by an appropriate choice of the so called regulator function it recovers the Wegner-Houghton [170] and the Polcinski [171] RG equations (at least in LPA).

7.1 The Wetterich RG equation

Instead of going through the standard derivation of the Wetterich RG equation [178, 179] discussed in the literature in details, here I choose a different rout based on the idea of the improved 1-loop expression. (In the previous section it was shown that the Wegner-Houghton equation can be considered as a 1-loop improved equation.) In other words, it is illustrative to discuss its connection to the effective action, which has the following form at the one-loop level, [175]

$$\Gamma_{\text{eff}}[\varphi] = S_\Lambda[\varphi] + \frac{\hbar}{2}\int \frac{d^d p}{(2\pi)^d} \ln\left[S^{(2)}_\Lambda[\varphi]\right] + \mathcal{O}(\hbar^2), \tag{236}$$

where S_Λ is the classical (bare) action. A Pauli-Villars approach is used to regularise the momentum integral which can be divergent at its upper (UV) and lower (IR) bounds. This can be achieved by adding a momentum dependent mass term $\frac{1}{2}\int R_k(p)\varphi^2$ to the bare action, and introduce a scale-dependent action

$$\Gamma_k[\varphi] \equiv S_\Lambda[\varphi] + \frac{\hbar}{2}\int \frac{d^d p}{(2\pi)^d} \ln\left[R_k(p) + S^{(2)}_\Lambda[\varphi]\right], \tag{237}$$

which recovers the effective action (at one-loop) in the IR limit if the regulator function $R_k(p)$ fulfils the requirements, $R_{k\to 0}(p) = 0$, $R_k(p \to 0) > 0$ [see Eqs. (13)—(15) of Ref. [180]]. The latter condition is important to avoid IR divergences. However, one canonically also imposes the condition

$$R_{k\to\Lambda}(p) = \infty \tag{238}$$

(see Ref. [180]), and thus, in the UV limit, the scale-dependent action reproduces the classical (bare) action only up to a field-independent, constant term. If one can differentiate Eq. (237) with respect to the running scale k (and multiplies both sides by k), then one finds

$$k\partial_k \Gamma_k[\varphi] = \frac{\hbar}{2}\int \frac{d^d p}{(2\pi)^d} \frac{k\partial_k R_k(p)}{R_k(p) + S^{(2)}_\Lambda[\varphi]}, \tag{239}$$

which recovers the "exact" Wetterich RG equation [178] up to the replacement $S^{(2)}_\Lambda \to \Gamma^{(2)}_k$,

$$k\,\partial_k \Gamma_k[\varphi] = \frac{\hbar}{2}\int \frac{d^d p}{(2\pi)^d} \frac{k\,\partial_k R_k(p)}{R_k(p) + \Gamma^{(2)}_k[\varphi]}, \quad \to \quad \boxed{k\,\partial_k \Gamma_k[\varphi] = \frac{\hbar}{2}\mathrm{Tr}\left(\frac{k\,\partial_k R_k(p)}{R_k(p) + \Gamma^{(2)}_k[\varphi]}\right)}, \tag{240}$$

which is called the Wetterich RG equation [178, 179] derived for the one component scalar field theory where k is the RG scale, $\Gamma_k[\varphi]$ is the running effective action with its Hessian $\Gamma_k^{(2)}[\varphi]$, and $R_k(p)$ is the regulator function.

Let me come back to various limits of the scale-dependent action Eq. (237). It recovers the effective action in the limit $k \to 0$ and the bare action for $k \to \Lambda$, up to a field-independent but k-dependent term, which I will denote as $V_k(0)$ for reasons which will become obvious immediately,

$$\Gamma_{k\to\Lambda}[\varphi] = \Gamma_\Lambda[\varphi] = S_\Lambda[\varphi] + \text{const.} = S_\Lambda[\varphi] + \int d^d x\, V_{k\to\Lambda}(0)\,. \tag{241}$$

This clearly signals that the formulation of the RG evolution of the constant, field-independent term $V_k(0)$ requires special care within the nonperturbative approach implied by the Wetterich equation (see also Sec. 2.3 of Ref. [180]). Moreover, if one implements the condition $R_{k\to\Lambda}(p) = \infty$ on the regulator, then it turns out that in many cases, the "constant term" $V_k(0)$ in Eq. (241), actually is given by a divergent integral.

Therefore, the constant term $V_k(0)$ needs a special treatment in the framework of the nonperturbative RG method. In the following sections, I will consider cases where $V_k(0)$ can naturally be identified with the zeroth-order term (in φ) obtained from the scale-dependent potential $V_k(\varphi)$. One might argue that, for many purposes, the precise form of the function $V_k(0)$ is physically irrelevant as it constitutes a field-independent constant. However, there are special cases where the RG evolution of a constant (field-independent) part of the potential has physical meaning. For example, if one aims at a determination of the free energy in a flat background or of the cosmological constant in a general non-flat background, then the problem of unambiguously determining $V_k(0)$ has to be seriously considered.

7.2 Optimization and regulator functions

The physical results obtained by the exact RG equation are independent of the particular choice of the regulator [175, 182, 183] which means that the UV and IR limits of the scale-dependent effective action is well-defined, i.e., $\Gamma_{k\to 0} = \Gamma_{\text{eff}}$ and $\Gamma_{k\to\Lambda} = S_\Lambda$. This is guaranteed by the properties of the regulator function. The RG flow in the parameter space depends on the actual choice of the regulator but the initial and final value does not, see Fig. 9.

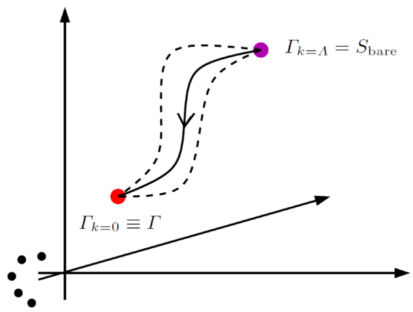

Figure 9: Exact RG flow in the parameter space for the effective action which depends on the particular choice of the regulator. The initial and final values of the running effective action are regulator independent [175, 182, 183].

The RG equation constitutes a functional partial differential equation and it is therefore not possible to indicate general solutions. Hence, approximations are required. One of the commonly used systematic approximations is the truncated gradient expansion,

$$\Gamma_k[\varphi] = \int d^d x \left[V_k(\varphi) + Z_k(\varphi)\frac{1}{2}(\partial_\mu\varphi)^2 + ...\right]. \tag{242}$$

The solution of the RG equations sometimes requires further approximations, e.g., the Taylor or

Fourier series of the potential $V_k(\varphi)$ in terms of the field variable (with a truncation $N_{\rm cut}$)

$$V_k(\varphi) = \sum_{n=1}^{N_{\rm cut}} \frac{g_{2n}(k)}{(2n)!}\,\varphi^{2n}, \qquad\qquad V_k(\varphi) = \sum_{n=1}^{N_{\rm cut}} u_n(k)\,\cos(n\beta\varphi), \tag{243}$$

where the scale-dependence is encoded in the coupling constants $g_{2n}(k)$ or $u_n(k)$.

The necessity of approximations implies that the RG flow depends on the choice of the regulator function, i.e., on the renormalization scheme. In principle, therefore, physical results could become scheme-dependent. Therefore, a general issue is the comparison of results obtained by various RG schemes [184, 185, 181, 188, 186, 187, 22, 4, 23, 182, 183, 189, 190].

7.2.1 Optimization

In order to increase the predicting power of the RG method, an optimization of the scheme-dependence is required. A rather general optimization procedure (Litim–Pawlowski method) [184, 181] leads to Litim's regulator [184] which is a function of class C^0 with compact support thus it is a continuous function and it has a finite range but it is not differentiable. In typical cases, it agrees well with experimental data, and furthermore, the corresponding RG equation can be mapped onto the Polchinski RG at least in the leading order of the gradient expansion [186]. Its disadvantage is that it is non-differentiable and thus incompatible with the gradient expansion, thus, beyond LPA a solution to the general criterion for optimization has to meet the necessary condition of differentiability to the given order [188, 186, 184, 181].

When using the Litim–Pawlowski optimization, the optimal choice for the parameters of the regulator functions can be determined in such a way to provides us the most favorable convergence of the amplitude expansion. Among the regulators, Litim's optimized one is seen to lead to the fastest convergence of the amplitude expansion. Here and in the following sections of this lecture note, the "optimum" parameters are always to be understood in terms of the additional approximations employed in the optimization process, e.g., the LPA. The caveat is that parameters which are determined as optimal within a specific, leading-order approximation to the RG flow, are implicitly assumed to approximate the optimum parameters within different and more detailed approximation schemes. Without this assumption, or a variation of this assumption, the determination of "optimum" parameters within any approximation to the RG flow would not be meaningful. I adopt the implicit assumption and proceed accordingly. The most general definition of an "optimized RG flow" would otherwise encompass the "shortest" RG trajectory in theory space, where "short" trajectories are quantified in terms of criteria for the gap in the flow equation. This sense of optimization is independent of any approximation scheme.

Another optimization scenario is based on the principle of minimal sensitivity (PMS) and discussed in [187], where the optimal parameters of a given regulator are chosen such as to make the physical quantities as insensitive as possible to any conceivable changes of the parameters entering the regulator. Its advantage is that it can be used at any order of the gradient expansion, its disadvantage is that regulators of different functional form cannot easily be compared to each other based on the PMS alone.

Solution for the above problems of differentiability (in case of the Litim–Pawlowski optimization) and comparability (in case of the PMS method) could be the so called compactly supported smooth (CSS) regulator [23]. This is a function of class C^∞ with compact support which encompasses all major types of regulator functions discussed so far in the literature, in appropriate limits. Thus, it can be used to compare various regulator functions to each other in the framework of the PMS optimization method. Moreover, it is a smooth, infinitely differentiable function, and it has a compact support (it is non-zero only in a finite range). Therefore, it can be applied to consider the "Litim limit" at any order of the gradient expansion.

7.2.2 Regulator functions

A large variety of regulator functions has already been discussed in the literature by introducing its dimensionless form

$$R_k(p) = p^2 r(y), \qquad y = p^2/k^2 \tag{244}$$

where $r(y)$ is dimensionless. For example, one of the simplest regulator function is the sharp-cutoff regulator

$$r_{\mathrm{sharp}}(y) = \frac{1}{\theta(y-1)} - 1 \tag{245}$$

where $\theta(y)$ is the Heaviside step function. The sharp-cutoff regulator has the advantage that the momentum integral in (240) can be performed analytically in the LPA. The corresponding RG equation is the Wegner-Houghton [170] RG. Its disadvantage is that it confronts to the derivative expansion, i.e. higher order terms (beyond LPA) cannot be evaluated unambiguously.

The compatibility with the derivative expansion can be fulfilled by e.g. using an exponential type regulator function [178]

$$r_{\mathrm{exp}}(y) = \frac{a}{\exp\left(c_2 y^b\right) - 1} \tag{246}$$

Within the LPA, a favorable choice for the parameters has been determined as $a = 1$, $c_2 = \ln(2)$ and $b = 1.44$, based on the Litim–Pawlowski method. Their disadvantage is that no analytic form can be derived for RG equations neither in LPA nor beyond. Thus, the momentum integral in (240) has to be performed numerically, and consequently, the dependence of the results on the upper bound of the numerical integration has to be considered.

The momentum integral of Eq. (240) can be performed analytically using the power-law type regulator [179]

$$r_{\mathrm{pow}}(y) = \frac{a}{y^b}\,, \tag{247}$$

at least for $b = 1$ and $b = 2$ in LPA. Again $a = 1$ and $b = 2$ are the optimal choices. The power-law regulator is compatible with the derivative expansion (for any $b \geq 1$) but its disadvantage is that it is not ultraviolet (UV) safe for $b = 1$ (at least not in all dimensions). One has to note that analyticity is lost beyond LPA. Therefore, similarly to the exponential type regulators, the dependence of the results on the upper bound of the numerical integration has to be considered.

Problems related to UV safety and the upper bound of the momentum integral can be handled by the (general) optimized regulator function [184]

$$r_{\mathrm{opt}}^{\mathrm{gen}}(y) = a\left(\frac{1}{y^b} - 1\right)\Theta(1 - y^b) \tag{248}$$

which is a continuous (but not differentiable) function with compact support [the Heaviside step function is denoted as $\Theta(y)$]. The parameters $b = 1$ and $a = 1$ are obtained as a result of the Litim–Pawlowski optimization method in LPA. Furthermore, the momentum integral can be performed analytically in all dimensions in LPA and also if the wave function renormalization is included. Moreover, it was also shown that in LPA, the optimized regulator and the Polchinski RG equation provides us the best results (closest to the exact ones) for the critical exponents of the $\mathrm{O}(N)$ symmetric scalar field theory in $d = 3$ dimensions [185]. This equivalence between the optimized and the Polchinski flows in LPA is the consequence of the fact that the optimized functional RG can be mapped by a suitable Legendre transformation to the Polchinski one in LPA [186] but this mapping does not hold beyond LPA. It was also shown that the regulator (248) is a simple solution of the general criterion for optimization in LPA. Although, the regulator (248) is a continuous function but it is not differentiable and it was shown that it does not support the derivative expansion beyond second order. Indeed, it was argued that optimization has to meet the necessary condition of differentiability.

The so called CSS (Compactly Supported Smooth) regulator [23] with exponential norm is defined as

$$\boxed{r_{\rm css}^{\rm norm}(y) = \frac{\exp[\ln(2)c]-1}{\exp\left[\frac{\ln(2)cy^b}{1-hy^b}\right]-1}\Theta(1-hy^b) = \frac{2^c-1}{2^{\frac{c\,y^b}{1-hy^b}}-1}\,\Theta(1-hy^b),} \tag{249}$$

which has the following limits

$$\lim_{c\to 0, h\to 1} r_{\rm css}^{\rm norm} = \left(\frac{1}{y^b}-1\right)\Theta(1-y^b), \tag{250a}$$

$$\lim_{c\to 0, h\to 0} r_{\rm css}^{\rm norm} = \frac{1}{y^b}, \tag{250b}$$

$$\lim_{c\to 1, h\to 0} r_{\rm css}^{\rm norm} = \frac{1}{\exp[\ln(2)y^b]-1}. \tag{250c}$$

Its advantage is that the form (249) reproduces all the major types of regulators with optimal parameters thus various regulators can be compared to each other in the PMS optimiziation method and the CSS regulator is a smooth, infinitely differentiable function (with a compact support), thus, it can be applied to consider the "Litim limit" at any order of the gradient expansion.

7.3 Wetterich RG Equation and the gradient expansion

7.3.1 Leading order (LPA)

In the following I set $\hbar = 1$ and take the leading order expression for the action in the gradient expansion (242),

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x\, \left[\frac{1}{2}\,(\partial_\mu\varphi)^2 + V_k(\varphi)\right]. \tag{251}$$

Than, the Wetterich equation (240) reduces to a differential equation for the scale dependent potential $V_k(\varphi)$ (for a constant field configuration $\varphi(x) = \varphi$)

$$\boxed{k\partial_k V_k(\varphi) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d^dp}{(2\pi)^d}\,\frac{k\partial_k R_k}{R_k + p^2 + V_k''}}, \tag{252}$$

with $V_k'' = \partial_\varphi^2 V_k$ which can be further simplified as

$$k\partial_k V_k(\varphi) = -\alpha_d k^d \int_0^\infty dy\,\frac{r'\;y^{\frac{d}{2}+1}}{[1+r]\,y\,+\frac{V_k''}{k^2}}\,, \tag{253}$$

with $\alpha_d = \Omega_d/(2(2\pi)^d)$ where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ and $r(y)$ is the dimensionless regulator with $y = p^2/k^2$ while $r' = dr/dy$. The corresponding dimensionless form reads

$$\boxed{\left(d - \frac{d-2}{2}\tilde\varphi\partial_{\tilde\varphi} + k\partial_k\right)\tilde V_k(\tilde\varphi) = -\alpha_d\int_0^\infty dy\,\frac{r'\;y^{\frac{d}{2}+1}}{[1+r]\,y\,+\tilde V_k''}}. \tag{254}$$

which is valid for the scale-dependent dimensionless potential with arbitrary regulator functions. The integral in Eq. (254) is usually performed numerically, however, analytic forms are available for a few types of regulators in arbitrary dimensions,

$$r_{\rm sharp}(y): \qquad \left(d - \frac{d-2}{2}\tilde\varphi\partial_{\tilde\varphi} + k\partial_k\right)\tilde V_k(\tilde\varphi) \;=\; -\alpha_d \ln\left(1+\tilde V_k''(\tilde\varphi)\right), \tag{255}$$

$$r_{\rm opt}^{\rm gen}(y), a=1, b=1: \qquad \left(d - \frac{d-2}{2}\tilde\varphi\partial_{\tilde\varphi} + k\partial_k\right)\tilde V_k(\tilde\varphi) \;=\; \alpha_d\frac{2}{d}\frac{1}{1+\tilde V_k''(\tilde\varphi)}. \tag{256}$$

Further examples are for dimensions $d = 2$ (since the field is dimensionless one can write $\tilde{\varphi} = \varphi$)

$$\begin{array}{r} \mathrm{d} = 2, r_{\mathrm{pow}}(y), \\ a = 1, b = 1 \end{array} : \quad (2 + k\partial_k)\, \tilde{V}_k(\varphi) = -\alpha_2 \ln\left(1 + \tilde{V}_k''(\varphi)\right), \tag{257}$$

$$\begin{array}{r} \mathrm{d} = 2, r_{\mathrm{pow}}(y), \\ a = 1, b = 2 \end{array} : \quad (2 + k\partial_k)\, \tilde{V}_k(\varphi) = -\frac{2\alpha_2}{\sqrt{[\tilde{V}_k''(\varphi)]^2 - 4}} \ln\left(\frac{|\tilde{V}_k''(\varphi) - \sqrt{[\tilde{V}_k''(\varphi)]^2 - 4}|}{|\tilde{V}_k''(\varphi) + \sqrt{[\tilde{V}_k''(\varphi)]^2 - 4}|}\right), \tag{258}$$

$$\begin{array}{r} \mathrm{d} = 2, r_{\mathrm{opt}}^{\mathrm{gen}}(y), \\ a \neq 1, b = 1 \end{array} : \quad (2 + k\partial_k)\, \tilde{V}_k(\varphi) = -\alpha_2 \frac{a}{a-1} \ln\left(\frac{1 + \tilde{V}_k''(\varphi)}{a + \tilde{V}_k''(\varphi)}\right) \tag{259}$$

where the latter stands for the generalised optimised regulator (248) which tends to the sharp one (245), thus, Eq. (259) recovers Eq. (255) for $d = 2$ in the limit $a \to \infty$ (after neglecting a field independent term on the right hand side). This property of the sharp cutoff limit holds in arbitrary dimensions. The RG equation (257) obtained by the power-law regulator with $b = 1$ (with the so-called mass cutoff) is derived by introducing an upper limit Λ^2/k^2 in the integration which sent to infinity after performing the integral while dropping a field independent term. Notice, that the RG equation (255) is identical to the Wegner-Houghton equation (226), and for $d = 2$ they are equivalent to Eq. (257). Let us note that Eq. (256) can be mapped onto the Polchinski equation (235) via an appropriate Legendre transformation [186], thus, it is expected to produce the same results in LPA.

In order to be able to perform the FRG study of layered sine-Gordon models introduced in previous sections, one has to generalise the RG equation for multi-component scalar fields. Let me chose the sharp cutoff form of the Wetterich equation (255) which is identical to the Wegner-Houghton equation (226) and to the mass cutoff RG equation (257) (for $d = 2$) and reads as

$$\left(d - \frac{d-2}{2} \underline{\tilde{\varphi}} \partial_{\underline{\tilde{\varphi}}} + k\partial_k\right)\ \tilde{V}_k(\underline{\tilde{\varphi}}) = -\alpha_d \ln\left[\det\left(\delta_{ij} + \tilde{V}_k^{ij}(\underline{\tilde{\varphi}})\right)\right], \tag{260}$$

where $\tilde{V}_k^{ij}(\underline{\tilde{\varphi}})$ denotes the second derivatives of the potential with respect to $\tilde{\varphi}_i$, $\tilde{\varphi}_j$. For dimensions $d = 2$ it has the form

$$(2 + k\partial_k)\ \tilde{V}_k(\underline{\varphi}) = -\frac{1}{4\pi} \ln\left[\det\left(\delta_{ij} + \tilde{V}_k^{ij}(\underline{\varphi})\right)\right], \tag{261}$$

where the multi-component field $\underline{\varphi}$ has no dimensions, thus, the tilde superscript is omitted. Similarly the RG equation for the multi-component field in $d = 2$ dimensions by means of the optimised cutoff reads as

$$(2 + k\partial_k)\ \tilde{V}_k(\underline{\varphi}) = \frac{1}{4\pi} \mathrm{Tr} \frac{1}{\left(\delta_{ij} + \tilde{V}_k^{ij}(\underline{\varphi})\right)}\ . \tag{262}$$

7.3.2 Next-to-leading order (LPA')

In order to include the field independent wavefunction renormalization I let the next-to-leading term in the gradient expansion evolve which is know as the LPA',

$$\Gamma_k[\varphi] = \int d^d x \left[\frac{1}{2} z_k (\partial_\mu \varphi)^2 + V_k(\varphi)\right], \tag{263}$$

where the local potential stands for SG type models. For example, if this is the pure SG model one can use the approximation where the potential contains a single Fourier mode only $V_k(\varphi) = u_k \cos(\beta\varphi)$ where the Fourier amplitude u_k and the field independent wavefunction renormalization z_k depend on the RG scale k.

If one considers periodic interactions, the Wetterich equation (240) reduces to the following coupled differential equations for the scale dependent (periodic) potential V_k and the field-independent wavefunction renormalization z_k,

$$k\partial_k V_k = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\mathcal{D}_k\, k\partial_k R_k, \tag{264}$$

$$k\partial_k z_k = \left(\frac{\beta}{2\pi}\int_0^{2\pi/\beta} d\varphi\right)(V_k''')^2 \int \frac{d^d p}{(2\pi)^d}\mathcal{D}_k^2\, k\partial_k R_k \left(\frac{2}{d}\frac{\partial^2 \mathcal{D}_k}{\partial p^2 \partial p^2}p^2 + \frac{\partial \mathcal{D}_k}{\partial p^2}\right), \tag{265}$$

with $\mathcal{D}_k = (z_k p^2 + R_k + V_k'')^{-1}$ where $V_k'' \equiv \partial_\varphi^2 V_k$ and $V_k''' \equiv \partial_\varphi^3 V_k$. Since the l.h.s of (265) is independent of the field, a projection onto the field-independent subspace has been introduced on the r.h.s of (265). The scale k covers the momentum interval from the high-energy/ultraviolet (UV) cutoff Λ to zero.

In this lecture note I am interested in the FRG study of SG type models with periodic self-interactions where the ansatz for the effective action in LPA and in LPA' reads as,

$$\text{LPA}: \qquad \Gamma_k = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \sum_n^\infty u_n(k)\cos(n\beta\varphi)\right].$$

$$\text{LPA}': \qquad \Gamma_k = \int d^d x \left[\frac{1}{2}z_k(\partial_\mu \varphi_x)^2 + \sum_n^\infty u_n(k)\cos(n\beta\varphi)\right]. \tag{266}$$

The dimensionful frequency β, is scale independent, i.e., it remains constant over the RG flow because the RG transformation retains the periodicity of the dimensionful model with an unchanged period length. Thus, β is a free parameter of the model which can be chosen arbitrarily. Since the dimensionful frequency is scale-independent, it is convenient to merge it with the scale-dependent wave function renormalization z_k which can be done by rescaling the field as $\theta = \beta\varphi$ where θ is dimensionless. Thus, it can be shown that actually, one finds only two independent couplings at LPA' level, too. In the very last section of the lecture note, RG equations corresponding to this rescaling and the appropriate choice for the regulator function beyond LPA are discussed.

Indeed, the regulator function beyond LPA should be given by the inclusion (multiplicative approach) or the exclusion (additive approach) of the field independent wavefunction renormalization z_k. Important to note, that the additive approach requires the use of the power-law regulator function. Of course, the phase structure should be independent whether one uses the multiplicative or additive approaches.

7.4 Periodicity and the FRG equation

The challenge in developing an RG method for the periodic scalar field theory is that the essential symmetry of the theory, namely the periodicity, should not be violated during the RG transformations. Indeed, one of the important general properties of the renormalization group methods is that they retain the symmetries of the action. Therefore, if the bare action $S_\Lambda(\phi)$ defined at the UV cut-off Λ has the symmetry under the transformation

$$S_\Lambda \to S_\Lambda, \qquad \text{for} \quad \varphi(x) \to \varphi(x) + \Delta \tag{267}$$

then, the blocked (effective) action $\Gamma_k(\varphi)$ which is the solution of the RG equations must be periodic with the same length of period Δ. In the local-potential approximation the blocked action reduces to the blocked potential $V_k(\varphi)$ which is tending to the effective potential $V_{\text{eff}}(\varphi)$ in the limit $k \to 0$. If the bare action is invariant under the symmetry transformation (267), then the effective potential should be periodic with the same length of period Δ.

Let me show that FRG equations preserve periodicity [1]. The demonstration is for the Wegner-Houghton case but can be generalised for any RG equations. It is actually obvious that the blocking, the transformation [1]

$$kV_{k-\Delta k}(\varphi) = kV_k(\varphi) + \left[k^d \alpha_d \ln\left(z(k)k^2 + \partial_\varphi^2 V_k(\varphi)\right)\right]\Delta k \tag{268}$$

preserves the periodicity of the potential if the wavefunction renormalization $z(k)$ is independent of the field. Therefore, one can look for solutions of the Wegner–Houghton equation obtained in the LPA among periodic functions, if the initial condition for the action contains a periodic self-interaction at the UV cut-off Λ. The inclusion of the field independent wavefunction renormalization does not change the situation, since it is independent of the field.

7.5 Differentiability and the FRG equation

Previously, the various forms of the Branon effective potential has been discussed where the simplest case was a non-differentiable function, i.e., $V(\varphi) \equiv V(|\varphi|)$. It is illustrative to show that the FRG equations are not sensitive to the case when the potential is non-differentiable in a single point [21]. To demonstrate this, let me consider the Wetterich FRG equation in LPA which reduces to a differential equation for the scale dependent potential $V_k(\phi)$

$$k\partial_k V_k(\phi) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d^d p}{(2\pi)^d} \, \frac{k\partial_k R_k}{R_k + p^2 + V_k''} \ , \tag{269}$$

where $V_k'' = \partial_\phi^2 V_k$. An important comment is the following: the latter equation depends only on the second field derivative of the running potential, which allows to consider the singularity $|\phi|$. Indeed,

$$\begin{aligned} \frac{\partial}{\partial\phi}\Big(V_k(|\phi|)\Big) &= \mathrm{sign}(\phi) V_k'(|\phi|) \\ \frac{\partial^2}{\partial\phi^2}\Big(V_k(|\phi|)\Big) &= V_k''(|\phi|) \ , \end{aligned} \tag{270}$$

and the equation (252) is not sensitive to the absolute value of the field [21]. An alternative argument is based on the regularisation

$$V_{\mathrm{reg},k}(\phi) \equiv V_k\big(\sqrt{r^2+\phi^2}\ \big) \ , \tag{271}$$

from which one can note that

$$\lim_{r\to 0} V''_{\mathrm{reg},k}(\phi) = V_k''(|\phi|) \ . \tag{272}$$

Let me demonstrate this general feature on a specific example. I consider the second derivative of the regularised and original dimensionful exponential potentials

$$\begin{aligned} V_{\mathrm{EXP}}(\phi) &= -u_k \exp(-a|\phi|) \\ V_{\mathrm{EXP,reg}}(\phi) &= -u_k \exp(-a\sqrt{r^2+\phi^2}) \end{aligned} \tag{273}$$

where the scale dependence is encoded in the amplitude u_k. Important to note that the dimensionful parameter a is scale-independent in LPA since in this case the wavefunction renormalization z is kept constant and by an appropriate rescaling of the field $\phi' \to a\phi$ these two couplings can be related to each other $z = 1/a^2$. The second derivative of the regularised potential can be taken in the limit $r \to 0$

$$\begin{aligned} & V''_{\mathrm{EXP,reg}}(\phi) = -u_k \frac{a^2\phi^2\sqrt{r^2+\phi^2} - r^2}{(r^2+\phi^2)^{3/2}} \exp(-a\sqrt{r^2+\phi^2}) \\ & \lim_{r\to 0} V''_{\mathrm{EXP,reg}}(\phi) = -u_k a^2 \exp(-a|\phi|), \end{aligned} \tag{274}$$

which is equivalent to the second derivative of the original non-analytic potential. This indicates that the functional RG equation (253) is not sensitive to the non-analytic nature of the potential [21].

8 Linearised FRG equation in the single Fourier mode approximation in LPA

Before going into the details of the solution of the exact FRG equation, in this section I take the linearised form (around the Gaussian fixed point) of RG equations obtained in the LPA level which is the most "drastic" approximation and reads as [21]

$$\left(d - \frac{d-2}{2}\tilde{\varphi}\partial_{\tilde{\varphi}} + k\partial_k\right)\tilde{V}_k(\tilde{\varphi}) = -\alpha_d\; C\; \tilde{V}''_k(\tilde{\varphi}) + \mathcal{O}(\tilde{V}''^2_k), \tag{275}$$

where the constant C is usually regulator-dependent except for $d = 2$ where $C = 1$ for any choice of the regulator function $r(y)$. For example, the Litim regulator gives $C = 2/d$, so it depends on the dimension d but the sharp-cutoff gives $C = 1$ in arbitrary dimension. Since I would like to apply it to sine-Gordon type models (SG, MSG, LSG, ShG and to the two interpolation cases, the Shine-Gordon and the SnG models) which have important physical realization in $d = 2$ dimensions, it is convenient to write out the two-dimensional linearised RG equation [1]

$$(2 + k\partial_k)\tilde{V}_k(\varphi) = -\frac{1}{4\pi}\tilde{V}''_k(\varphi) + \mathcal{O}(\tilde{V}''^2_k) \tag{276}$$

where the field carries no dimension thus $\tilde{\varphi} = \varphi$. Similarly, the linearised RG equation for the multi-component field can be written [9]

$$(2 + k\partial_k)\,\tilde{V}_k(\underline{\varphi}) = -\frac{1}{4\pi}\sum_{n=1}^{N}\tilde{V}^{nn}_k(\underline{\varphi}) + \mathcal{O}([\tilde{V}^{nn}_k]^2). \tag{277}$$

However, if apart from the periodic self-interaction, a non-periodic term is present in the action, i.e. an explicit mass or mass-matrix appears then it is more reliable if one linearises the RG equation (261) in the periodic piece of the blocked potential. This is the so-called, mass-corrected linearised RG equation which depends on the choice of the regulator function. In order to demonstrate this, an explicit form is given for the simplest case, the MSG model (41), where the dimensionless potential $\tilde{V}_k = \frac{1}{2}\tilde{M}^2_k\varphi^2 + \tilde{U}_k(\varphi)$ contains a periodic piece $\tilde{U}_k(\varphi)$ and the RG equation is written for the mass-cutoff regulator in $d = 2$ dimensions [9, 8, 7, 12]

$$\begin{aligned}
(2 + k\partial_k)\tilde{V}_k(\varphi) &= -\frac{1}{4\pi}\log(1 + \tilde{V}''_k(\varphi)) = -\frac{1}{4\pi}\log(1 + \tilde{M}^2_k + \tilde{U}''_k(\varphi)) \\
&\approx -\frac{1}{4\pi}\left[\log(1 + \tilde{M}^2_k) + \frac{\tilde{U}''_k(\varphi)}{1 + \tilde{M}^2_k}\right] + \mathcal{O}([\tilde{U}''_k)]^2 \\
\Longrightarrow \quad (2 + k\partial_k)\tilde{U}_k(\varphi) &\approx -\frac{1}{4\pi}\frac{\tilde{U}''_k(\varphi)}{1 + \tilde{M}^2_k}, \qquad (2 + k\partial_k)\tilde{M}^2_k = 0
\end{aligned} \tag{278}$$

where the field independent term $\log(1 + \tilde{M}^2_k)$ is dropped and the original FRG equation is split into two separate equations where latter gives a trivial scaling for the dimensionless mass (thus the dimensionful mass remains unchanged over the flow). Please observe that the right hand side of the RG equation (both the original and the approximated) is *periodic*, so, the non-periodic term of the left hand side must vanish which results in a trivial scaling for the mass. It can be generalised for the N-component case [13, 14, 7, 12, 9, 8]

$$(2 + k\,\partial_k)\tilde{U}_k(\underline{\varphi}) = -\frac{1}{4\pi}\,\frac{F_1(\tilde{U}_k)}{C} + \mathcal{O}([\tilde{U}^{nn}_k]^2), \qquad (2 + k\partial_k)\underline{\underline{\tilde{M}}}^2(k) = 0 \tag{279}$$

where the periodic piece is $\tilde{U}_k(\underline{\varphi}) = \sum_{n=1}^{N}\tilde{u}_n(k)\cos(\beta\,\varphi_n)$, so the full potential reads $\tilde{V}_k = \frac{1}{2}\underline{\varphi}\;\underline{\underline{\tilde{M}}}^2(k)\,\underline{\varphi}^T + \tilde{U}_k(\underline{\varphi})$ and C and $F_1(\tilde{U}_k)$ stand for the constant and linear pieces of the determinant $\det[\delta_{ij} + \tilde{V}^{ij}_k] \approx C + F_1(\tilde{U}_k)$. The mass-corrected linearised RG equation (279) depends on the choice of the regulator function.

8.1 The Ising model

Let us first consider the linearised RG flow for the Ising model by substituting

$$\tilde{V}_{\text{Ising}}(\varphi) = \sum_{n=1}^{\mathrm{N_{CUT}}} \frac{\tilde{g}_{2n}(k)}{(2n)!}\varphi^{2n}, \tag{280}$$

into Eq. (276). Then one can read off the RG flow equations for the scale dependent dimensionless couplings $\tilde{g}_{2n}(k)$. For any finite $\mathrm{N_{CUT}}$, the linearised FRG equation does not preserve the functional form of the bare theory (280), i.e., the l.h.s of (276) contains polynomial terms ϕ^{2n} of order $n = \mathrm{N_{CUT}}$ but the r.h.s of (276) has terms of order $n < \mathrm{N_{CUT}}$. Thus, I conclude that the linearised RG is not sufficient to signal a second (finite) order phase transition.

8.2 The SG model

The situation is different for the SG model where the potential is defined by (for the sake of simplicity keeping only the fundamental Fourier mode)

$$\tilde{V}_{\text{SG}}(\varphi) = \tilde{u}_k \cos(\beta\varphi) \tag{281}$$

where the dimensionless Fourier amplitude carries the scale-dependence since in LPA the frequency β does not depend on the running momentum cutoff k. The linearised FRG equations (276) and (275) preserve the functional form of the bare potential (no higher harmonics are generated). Considering the $d = 2$ case one finds [1]

$$(2 + k\partial_k)\tilde{u}_k \cos(\beta\varphi) = \frac{1}{4\pi}\beta^2 \tilde{u}_k \cos(\beta\varphi) \tag{282}$$

and the RG flow equation for the Fourier amplitude reads [1]

$$k\partial_k \tilde{u}_k = \tilde{u}_k \left(-2 + \frac{1}{4\pi}\beta^2\right) \tag{283}$$

exhibiting the solution [1]

$$\tilde{u}_k = \tilde{u}_\Lambda \left(\frac{k}{\Lambda}\right)^{-2+\frac{\beta^2}{4\pi}} \qquad \to \qquad \beta_c^2 = 8\pi \tag{284}$$

where $\tilde{u}_\Lambda$ is the initial (bare) value of the Fourier amplitude at the high energy ultra-violet (UV) cutoff Λ. Eq. (284) determines the critical frequency $\beta_c^2 = 8\pi$ where the model undergoes a KTB-type phase transition, see Fig. 10. The coupling $\tilde{u}$ is irrelevant (decreasing) for $\beta^2 > \beta_c^2$ and relevant (increasing) for $\beta^2 < \beta_c^2$. It is important to note that even if the bare theory of the SG model

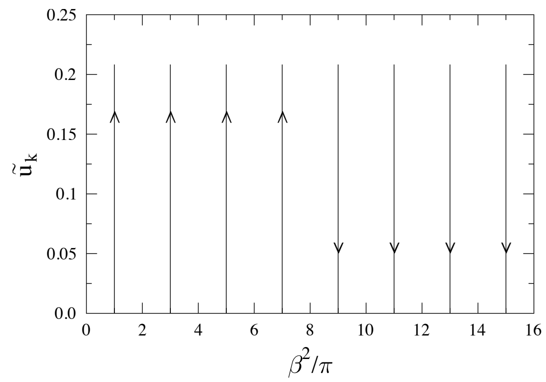

Figure 10: Linearised RG flow obtained in LPA for the SG model in $d = 2$ dimensions.

contains higher harmonics, the linearised FRG equation (276) reduces to decoupled flow equations for the Fourier amplitudes of various modes.

Let me consider now the $d \neq 2$ case where the linearised RG equation (for the sharp cutoff with $C = 1$) reads as [17]

$$\left(d - \frac{d-2}{2}\tilde\varphi\partial_{\tilde\varphi} + k\partial_k\right)\tilde u_k \cos(\tilde\beta_k\tilde\varphi) = \alpha_d \tilde\beta_k^2 \tilde u_k \cos(\tilde\beta_k\tilde\varphi) \tag{285}$$

where a scale-dependent (dimensionless) frequency $\tilde\beta_k$ should be introduced in order to keep the argument in the cos term dimensionless for the dimensionful potential. Thus, one expects a trivial scaling for the frequency, i.e., $\tilde\beta_k \sim k^{(d-2)/2}$. The RG flow equations for the periodic part and the non-periodic part are separated and read as [17]

$$k\partial_k \tilde u_k = \tilde u_k\left(-d + \alpha_d\tilde\beta_k^2\right), \qquad k\partial_k\tilde\beta_k = \frac{d-2}{2}\tilde\beta_k \quad \to \quad \tilde\beta_k^2 = \tilde\beta_\Lambda^2\left(\frac{k}{\Lambda}\right)^{d-2} \tag{286}$$

where the second RG flow equation gives back exactly the trivial scaling for the dimensionless frequency. It is illustrative to show the solution of the flow equation (286), see Eq.(27) of [17],

$$\tilde u_k = \tilde u_\Lambda\left(\frac{k}{\Lambda}\right)^{-d}\exp\left\{\frac{\alpha_d\tilde\beta_\Lambda^2}{d-2}\left[\left(\frac{k}{\Lambda}\right)^{d-2} - 1\right]\right\} \tag{287}$$

where $\tilde\beta_\Lambda$ and $\tilde u_\Lambda$ are the bare values of the couplings. Since the dimensionless frequency becomes scale-dependent, the solution of the flow equation for the Fourier amplitude changes compared to the $d = 2$ case. If $d > 2$, then in the IR limit, when $k \to 0$, the coupling constant $\tilde u(k)$ always becomes a relevant parameter ($\tilde u \to \infty$) independently of $\tilde\beta^2$, see Fig. 11. Therefore, the 3D-SG

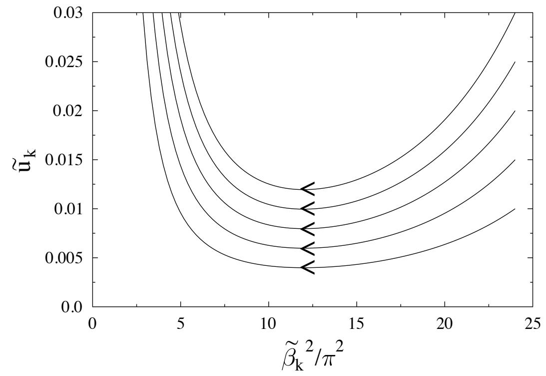

Figure 11: Linearised RG flow obtained in LPA for the SG model in $d = 3$ dimensions.

model has only a single phase within the LPA, *no* KTB transition is observed.

8.3 The MSG and LSG models

The dimensionless potential of the LSG model (considered as a generalised form of the MSG model with magnetic type interlayer coupling) is defined by the following potential [14]

$$\tilde V_{\mathrm{LSG}}(\varphi) = \frac{1}{2}\underline{\varphi}\,\underline{\underline{\tilde M}}^2(k)\,\underline{\varphi}^T + \tilde U_k(\underline{\varphi}), \qquad \tilde U_k(\underline{\varphi}) = \sum_{n=1}^{N}\tilde u_n(k)\,\cos(\beta\,\varphi_n) \tag{288}$$

Regarding the LSG model one has two options. The ansatz (288) can be inserted (i) either into the linearised RG equation (277), (ii) or into the mass-corrected linearised RG equation (279). The scaling laws obtained by (277) valid at the asymptotically large UV scales ($k \sim \Lambda$), and being independent of the interlayer coupling (i.e. mass terms) predicting exactly the same phase structure as that of the massless 2D-SG model with $\beta_c^2 = 8\pi$. The simplest way to go beyond the linearized approximation and to improve the extrapolating power of the UV scaling laws towards the IR regime, is to take corrections into account of the order $\mathcal{O}(J/k^2)$ or $\mathcal{O}(G/k^2)$, which results in the mass-corrected linearised RG scaling laws derived for the M-LSG (3.1.3) and J-LSG (3.1.3) models. This is achieved by using the equation (279) for the multi-component case which provides the trivial scaling for the mass matrix, i.e., for dimensionless interlayer couplings [13, 14, 6, 7, 12, 9, 8]

$$\tilde{J}_k = k^{-2} J, \qquad \tilde{G}_k = k^{-2} G, \tag{289}$$

where the corresponding dimensionful parameters J, G remain constant during the blocking.

8.3.1 The J-LSG model

Let me first determine the mass-corrected linearised RG scaling laws for the J-LSG model (3.1.3) for $N = 2$. In this case the two layers are assumed to be equivalent, so, the Fourier amplitudes are the same ($\tilde{u}_1 \equiv \tilde{u}_2$) and the solution of Eq. (279) reads [9, 13, 7]

$$\tilde{u}_k = \tilde{u}_\Lambda \left(\frac{k}{\Lambda}\right)^{\frac{\beta^2}{8\pi}-2} \left(\frac{k^2+2J}{\Lambda^2+2J}\right)^{\frac{\beta^2}{16\pi}} \qquad \to \qquad \beta_c^2 = 16\pi \tag{290}$$

with the initial value $\tilde{u}_\Lambda$ at the UV cutoff $k = \Lambda$. From the extrapolation of the scaling law Eq. (290) to the IR limit, one can read off the critical values $\beta_c^2 = 16\pi$, for $N = 2$ (because when $k \to 0$ the second term becomes constant and the critical value is determined by the first term of (290)). The coupling $\tilde{u}$ is irrelevant for $\beta^2 > \beta_c^2$ and relevant for $\beta^2 < \beta_c^2$, see the schematic flow diagram Fig. 12. The general expressions for the critical frequency and the corresponding critical

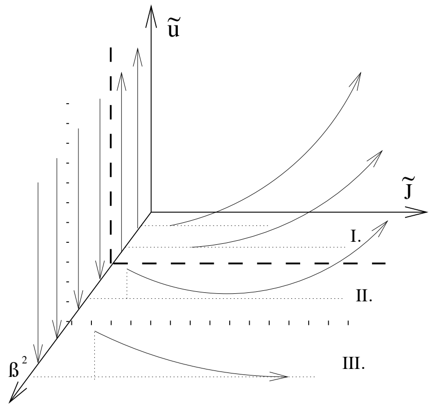

Figure 12: Schematic RG trajectories of the J-LSG model with $N = 2$ layers [7]. The critical frequency $\beta_c^2 = 16\pi$ is denoted by the dotted line which separates the two phases. The coupling $\tilde{u}$ is irrelevant for $\beta^2 > 16\pi$ and relevant for $\beta^2 < 16\pi$ in the IR limit. The dashes line stands for $\beta^2 = 8\pi$ which is the critical frequency of the massless 2D-SG model. The coupling $\tilde{u}$ is irrelevant for $\beta^2 > 8\pi$ and relevant for $\beta^2 < 8\pi$ in the UV limit.

temperature read [9, 13]

$$\beta_c^2(N) = 8\pi N, \quad \to \quad T^{(N)}_{\text{J-LSG}} = \frac{2\pi}{\beta_c^2(N)} = T^{\star}_{\text{KTB}} \frac{1}{N}. \tag{291}$$

The presence of the coupling J between the layers modifies the critical parameter β_c^2 of the J-LSG model as compared to the massless 2D-SG model, see Fig. 13. This important modification can only be deduced if one goes beyond the linearized (i.e. dilute gas) approximation, e.g. by the usage of the mass-corrected linearised RG scaling laws.

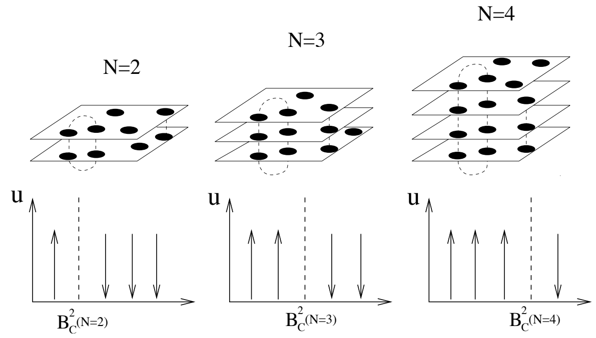

Figure 13: Schematic RG trajectories of the J-LSG model with $N = 2, 3, 4$ layers in the plane $(B^2 \equiv \beta^2, u \equiv \tilde{u}_k)$ and the shift of the critical value $B_c^2(N) \equiv \beta_c^2(N) = 8N\pi$ [8]. Each layer corresponds to an SG model which are coupled by the coupling J. The solid discs represent the topological excitation of the layered system.

8.3.2 The M-LSG model

Similar consideration can be done for the M-LSG model (3.1.3) which has real application for the vortex dynamics of magnetically coupled layered superconductors. Since the layers are assumed to be equivalent for the M-LSG model, the RG flow equations for the Fourier amplitudes (i.e. fugacities) of different layers should be the same ($\tilde{u}_n(k) \equiv \tilde{u}_k$) and the solution can be obtained analytically [9, 14]

$$\tilde{u}_k = \tilde{u}_\Lambda \left(\frac{k}{\Lambda}\right)^{\frac{(N-1)\beta^2}{N4\pi}-2} \left(\frac{k^2+NG}{\Lambda^2+NG}\right)^{\frac{\beta^2}{N8\pi}} \quad \to \quad \beta_c^2 = \frac{8\pi N}{N-1} \tag{292}$$

where $\tilde{u}_\Lambda$ is the initial value for the fugacity at the UV cutoff Λ and G, β^2 are scale-independent parameters. The critical frequency and the corresponding critical temperature which separates the two phases of the model can be read directly [9, 14]

$$\beta_c^2(N) = \frac{8\pi N}{N-1}, \to T^{(N)}_{\mathrm{M-LSG}} = \frac{2\pi}{\beta_c^2(N)} = T^\star_{\mathrm{KTB}} \frac{N-1}{N}. \tag{293}$$

In Fig. 14, I show the UV (linearised RG) and the IR (mass-corrected linearised RG) scaling of the coupling $\tilde{u}_k$ where the panels to the left contains RG running of $\tilde{u}_k$ for various initial values of β_c^2. The mass-corrected scaling appears below the mass scale $k < M_N = \sqrt{NG}$. For $N \to \infty$ the M-LSG behaves like a massless 2D-SG model with the critical frequency $\beta_c^2 = 8\pi$. For $N = 1$ the M-LSG models reduces to the MSG model which is known to have *no* KTB phase transitions, see Fig. 15. This is signalled by the layer-dependence of the critical frequency of the KTB transition of the M-LSG model since $\beta_c^2(N = 1) = \infty$. The layer number dependence of the critical frequencies of the J-LSG and M-LSG models are summarised in Fig. 16.

Finally let me consider the co-called rotated LSG model (50) where $O(N)$ rotations are performed which diagonalise the mass matrices (52) both for the J-LSG and M-LSG cases [8, 9, 12]. The rotated models do not have interlayer interactions via the mass matrix (since it is diagonal) but they do have in the periodic parts, see Eq. (53) for $N = 2$, and Eqs. (54), (55) for $N = 3$. Depending on the number of the non-trivial mass eigenvalues, some of the rotated fields have explicit mass terms (massive modes) and the other ones are massless, SG-type fields. At low energies, below the mass-scale the quantum fluctuations are suppressed by the mass terms producing a trivial scaling for the massive modes, so, the massive modes can be considered perturbatively and they do not influence the phase structure of the rotated models. At the lowest order of the perturbation theory, all the massive modes are set to be equal to zero. In this case the effective potential for the rotated J-LSG model reads as [8, 9, 12],

$$\tilde{V}_{\mathrm{J-rot}}(\alpha_1) = \sum_{\sigma_1} \tilde{w}_{\sigma_1} \exp\left[i\,\sigma_1\, b_1\, \alpha_1\right], \qquad b_1^2 = \frac{\beta^2}{N}, \tag{294}$$

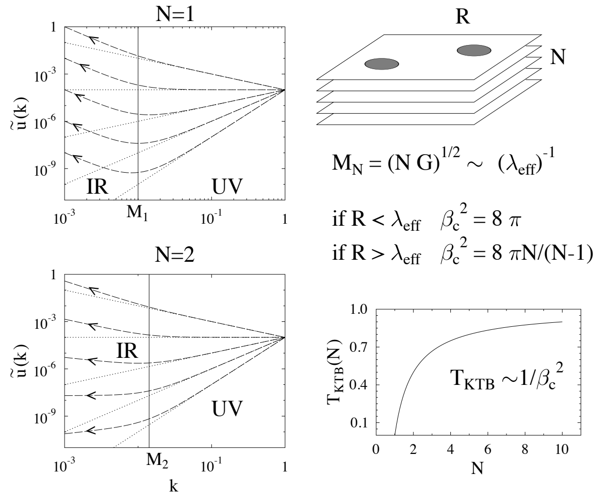

Figure 14: RG trajectories of the M-LSG model with $N = 1, 2$ layers [14]. The shift of the critical value is given by $\beta_c^2(N) = \frac{8\pi N}{N-1}$. Each layer corresponds to an SG model which are coupled by the coupling G. The solid discs represent the topological excitation of the layered system. The UV and IR scaling are separated by the mass term $M_N = \lambda_{\text{eff}}^{-1}$ which serves as an effective length scale for the corresponding layered superconductor where the vortices of different layers are coupled magnetically. The critical temperature T_{KTB} depends on the number of layers (N) if the system size (R) is larger than this effective screening length.

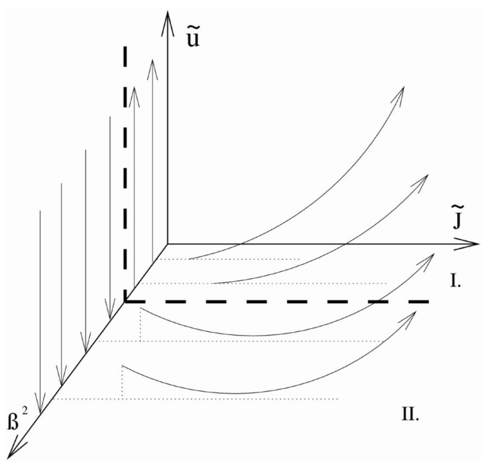

Figure 15: RG trajectories of the M-LSG model with $N = 1$ layer, i.e. the MSG model which has an explicit mass term ($\tilde{J} \equiv \tilde{M}^2$) in addition to the periodic part [7]. Due to the presence of the explicit mass, the Fourier amplitude $\tilde{u}_k$ always increases in the IR limit independently of the choice of the initial value for the frequency β. This results in *no* KTB phase transition (i.e. $\beta_c^2 = \infty$).

and for the M-LSG model the effective potential is [8, 9, 12]

$$\tilde{V}_{\text{M-rot}}(\alpha_2, ..., \alpha_N) = \sum_{\sigma_2,...,\sigma_N} \tilde{w}_{\sigma_2,...,\sigma_N} \prod_{n=2}^{N} \exp\left[i\,\sigma_n\, b_n\, \alpha_n\right], \qquad b_{n>1}^2 = \frac{\beta^2}{(n(n-1))}. \quad (295)$$

One can consider the remaining massless SG fields in order to determine the phases of the layered

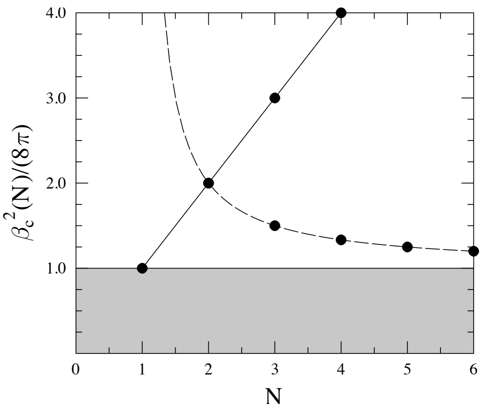

Figure 16: The critical frequency $\beta_c^2(N)$ versus the layer-number N is shown for the J-LSG (solid line) and the M-LSG (dashed line) models, respectively. The critical frequencies lie outside of the shaded area, irrespectively of N.

system (with $\sigma_n = \pm 1$) which can be done by using the simple linearised form of the FRG equation

$$(2 + k\partial_k)\,\tilde{V}_k(\alpha_1, ..., \alpha_N) \approx -\frac{1}{4\pi}\sum_{n=1}^{N} \tilde{V}_k^{nn}(\alpha_1, ..., \alpha_N), \tag{296}$$

with $\tilde{V}_k^{nn} = \partial^2_{\alpha_n}\tilde{V}_k$ which leads to the linearized flow equations [8, 9, 12]

$$(2 + k\partial_k)\tilde{w}_{\mathrm{J}}(k) = \frac{b_1^2}{4\pi}\tilde{w}_{\mathrm{J}}(k), \qquad (2 + k\partial_k)\tilde{w}_{\mathrm{M}}(k) = \frac{1}{4\pi}\left(\sum_{n=2}^{N} b_n^2\right)\tilde{w}_{\mathrm{M}}(k), \tag{297}$$

exhibiting the solutions [8, 9, 12]

$$\tilde{w}_J(k) = \tilde{w}_J(\Lambda)\left(\frac{k}{\Lambda}\right)^{-2+\frac{\beta^2}{N(4\pi)}}, \qquad \tilde{w}_M(k) = \tilde{w}_M(\Lambda)\left(\frac{k}{\Lambda}\right)^{-2+\frac{(N-1)\beta^2}{N(4\pi)}} \tag{298}$$

where $\tilde{w}_J(\Lambda)$ and $\tilde{w}_M(\Lambda)$ are the initial values for the Fourier amplitudes at the high energy UV cutoff Λ. The solutions determine the critical value of the frequency parameter which separates the two phases of the LSG-type models and the corresponding critical temperatures are [8, 9, 12]

$$T^{(N)}_{\mathrm{J-LSG}} = T^{\star}_{\mathrm{KTB}}\frac{1}{N}, \qquad T^{(N)}_{\mathrm{M-LSG}} = T^{\star}_{\mathrm{KTB}}\frac{N-1}{N}. \tag{299}$$

which results in the same layer-dependent critical frequency as obtained for the above analysis.

8.4 The ShG model

By using the replacement $\beta \to i\beta$ in Eq. (281) one finds the potential for the ShG model

$$\tilde{V}_{\mathrm{ShG}}(\phi) = \tilde{u}_k \cos(i\beta\varphi) = \tilde{u}_k \cosh(\beta\phi) \tag{300}$$

which is inserted into (276) preserving again the functional form of the bare potential [11],

$$(2 + k\partial_k)\tilde{u}_k \cosh(\beta\varphi) = -\frac{1}{4\pi}\beta^2\tilde{u}_k \cosh(\beta\varphi) \tag{301}$$

and the RG flow equation and the solution for the Fourier amplitude reads [11]

$$k\partial_k\tilde{u}_k = \tilde{u}_k\left(-2 - \frac{1}{4\pi}\beta^2\right) \quad \to \quad \tilde{u}_k = \tilde{u}_\Lambda\left(\frac{k}{\Lambda}\right)^{-2-\frac{\beta^2}{4\pi}} \tag{302}$$

which shows that in case of $\beta^2 = 8\pi$ the exponent does not change sign, hence, the ShG model has *no* KTB-type phase transition. In other words, the linearised FRG of the ShG model can be derived from the the SG model by using the replacement $\beta \to i\beta$ which results in a sign change of β^2 and no KTB-type phase transition.

8.5 The Shine-Gordon model

Let me now study the first class of interpolating models, i.e. the dimensionless Shine-Gordon potential

$$\tilde{V}_{\text{Shine}}(\phi) = \frac{\tilde{u}_k}{2} \left(\cos[(\beta_1 + i\beta_2)\phi] + \cos[(\beta_1 - i\beta_2)\phi]\right) = \tilde{u}_k \cos(\beta_1 \phi) \cosh(\beta_2 \phi), \tag{303}$$

where β_1 and β_2 are real value frequencies. Inserting (303) into (276) one finds [11]

$$\begin{aligned}(2 + k\partial_k)\tilde{u}_k \cos(\beta_1\phi)\cosh(\beta_2\phi) =& \frac{1}{4\pi}\tilde{u}_k \left[(\beta_1^2 - \beta_2^2)\cos(\beta_1\phi)\cosh(\beta_2\phi)\right. \\ &\left. +2\beta_1\beta_2 \sin(\beta_1\phi)\sinh(\beta_2\phi)\right]\end{aligned} \tag{304}$$

which indicates that the functional form of the bare potential is not preserved similarly to the Ising model. One may try adding a $\sin(\beta_1\phi)\sinh(\beta_2\phi)$ to to bare action (303) and then the second derivative of the potential has the same form [11],

$$\begin{aligned}\tilde{V}_{\text{Shine}} &= \tilde{u}_k \left[\cos(\beta_1\phi)\cosh(\beta_2\phi) + \sin(\beta_1\phi)\sinh(\beta_2\phi)\right] \\ \tilde{V}''_{\text{Shine}} &= -\tilde{u}_k \left[(\beta_1^2 - \beta_2^2 - 2\beta_1\beta_2)\cos(\beta_1\phi)\cosh(\beta_2\phi) + (\beta_1^2 - \beta_2^2 + 2\beta_1\beta_2)\sin(\beta_1\phi)\sinh(\beta_2\phi)\right]\end{aligned}$$

however, the pre-factors of the $\cos(\beta_1\phi)\cosh(\beta_2\phi)$ and $\sin(\beta_1\phi)\sinh(\beta_2\phi)$ are different which requires the introduction of different amplitudes $\tilde{u}_1(k)$, $\tilde{u}_2(k)$ for each of these terms. Thus, the functional form of the bare model has not been preserved by the linearised FRG equation (opposite to the SG and ShG models) and one cannot read off a single flow equation for the coupling $\tilde{u}_k$ unless one of the frequency is set to be zero, i.e. in the two limiting cases. This signals no KTB-type phase transition for the Shine-Gordon model if $\beta_2 \neq 0$ [11].

8.6 The SnG model

The second class of interpolating theories are represented by the SnG model which is written in terms of Jacobi functions and its dimensionless bare potential reads [11],

$$\tilde{V}_{\text{SnG}}(\phi) = \tilde{A}_k \,\text{cd}(\beta\phi, m)\,\text{nd}(\beta\phi, m), \tag{305}$$

where the amplitude $\tilde{A}_k$ is scale-dependent and by using the properties of the Jacobi functions $\text{cd}(u,m) = \text{cn}(u,m)/\text{dn}(u,m)$ and $\text{nd}(u,m) = 1/\text{dn}(u,m)$ it can also be written as [11]

$$\tilde{V}_{\text{SnG}}(\phi) = \tilde{A}_k \,\text{cn}(\beta\phi, m)\,[\text{nd}(\beta\phi, m)]^2. \tag{306}$$

Inserting Eq. (305) or Eq. (306) into the linearised FRG equation (276) one observes that the functional form is not preserved since the second derivatives of the potential has the following form [11]

$$\tilde{V}''_{\text{SnG}}(\phi) = \beta^2 \tilde{A}_k \frac{\text{cn}(\beta\phi, m)}{\text{dn}(\beta\phi, m)^4}\left(6(m-1) + (5-4m)\,\text{dn}(\beta\phi, m)^2\right).$$

However, it is important to note that (305) is a periodic function, so, it can be expanded in Fourier (Lambert) series (see the corresponding section) which results in [11]

$$\tilde{V}_{\text{SnG}}(\phi) = \sum_{n=1}^{\infty} \tilde{u}_n(k)\cos(n\,b\,\phi), \qquad b = \frac{\beta}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, m\right)}. \tag{307}$$

Inserting (307) into the linearised FRG equation (276) one can derive a set of uncoupled differential equations for the Fourier modes [11],

$$k\partial_k \tilde{u}_n(k) = \tilde{u}_n(k)\left(-2 + \frac{1}{4\pi}n^2 b^2\right). \tag{308}$$

Similarly to the SG model the critical frequency corresponds to the fundamental mode, i.e., for $n = 1$ where one finds $b_c^2 = 8\pi$ and the higher harmonics do not modify it. Thus, one can read the m-dependence of the original frequency

$$\beta_c^2(m) = 8\pi \left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, m\right)\right]^2 \tag{309}$$

which clearly signals the existence of a KTB-type phase transition if $m \neq 1$. In the limit $m \to 1$ the original frequency blows up thus the system is always in the, so called massive (ionised) phase, see Fig. 17, where the fundamental Fourier amplitude is increasing in the IR limit. Thus, in the

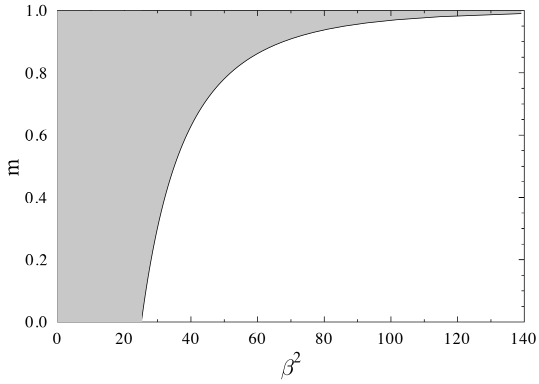

Figure 17: Phase structure of the SnG model in the m, β^2 plane based on Eq. (309) indicates a KTB-type phase transition where the grey area stands for the massive (ionised) phase [11].

$m = 1$ case the SnG model reduces to the ShG model which has no KTB-type phase transition.

8.7 Non-differentiable potentials

As a final step, I give here a simple illustration of how quantum fluctuations smoothen a non-differentiable microscopic (bare) potential and leads to a differentiable Wilsonian effective potential [21]. I start with the bare non-differentiable potential

$$V_\infty(\phi) = \mu^{d/2+1}|\phi| , \tag{310}$$

where $\mu > 0$ is the only bare parameter of the model. Let me use the following ansatz for the running (blocked) potential [21]

$$V_k(\phi) = \mu^{d/2+1}|\phi| + u_k \exp(-|\phi|/\mu^{(d-2)/2}) , \tag{311}$$

where u_k is to be determined and c_k corresponds to a redefinition of the origin of energies, for each value of k. Plugging the ansatz in the dimensionful form of the linearised FRG equation (275) with $C = 2/d$ one finds,

$$k\partial_k V_k = -\frac{2\alpha_d}{d} k^{d-2} V_k'' \qquad \to \qquad k\partial_k u_k = -\frac{2\alpha_d}{d}\left(\frac{k}{\mu}\right)^{d-2} u_k . \tag{312}$$

In $d > 2$ dimensions the solution of the latter equation reads

$$u_k = A \exp\left(-\frac{2\alpha_d (k/\mu)^{d-2}}{d(d-2)}\right) , \tag{313}$$

and the constant A can be determined by imposing the IR potential $V_{k=0}(\phi)$ to be differentiable and minimum at $\phi = 0$,

$$V'_{k=0}(0) = \pm\left(\mu^{d/2+1} - \frac{A}{\mu^{d/2-1}}\right) = 0 , \tag{314}$$

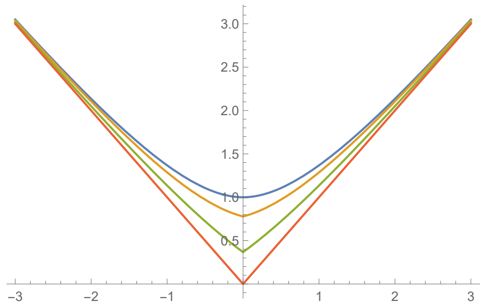

Figure 18: Quantum corrections turn the singularity into a smooth IR potential ($d = 4$): the V-shaped bare potential corresponds to $k = \infty$ and the smooth Wilsonian effective potential to $k = 0$. The intermediate running potentials correspond to finite values of k. Note that the latter potentials are not differentiable, and only the deep IR effective potential is differentiable.

where the sign $\pm$ depends on which side of 0 the derivative is taken from. Hence the running average effective potential is

$$V_k(\phi) = \mu^{d/2+1}|\phi| + \mu^d \exp\left(-\frac{2\alpha_d(k/\mu)^{d-2}}{d(d-2)} - \frac{|\phi|}{\mu^{d/2-1}}\right) , \tag{315}$$

and indeed corresponds to what is expected - see Fig.(18) for the case $d = 4$: (i) the ultraviolet limit $k \to \infty$ reproduces the bare potential; (ii) the IR limit $k \to 0$ leads to a differentiable effective potential [21]. The summary of the results of this section is the following. It was shown that FRG equations taken at the level of the most "drastic" approximation, (i.e., linearised flow equations in LPA) are able to determine the KTB-type phase transitions of SG-type models providing also the exact value of the critical frequency but they are unable to determine whether the model undergoes an Ising-type phase transition or not.

9 Exact FRG equation in the single Fourier mode approximation in LPA

In this section I discuss the FRG equations obtained in LPA for SG and MSG models which contains a single Fourier mode only. Thus, I use the ansatz for the running dimensionful (and dimensionless) potential for the SG model

$$\text{SG}: \qquad V_k(\varphi) = u_k \cos(\beta\varphi), \qquad \tilde{V}_k(\tilde{\varphi}) = \tilde{u}_k \cos(\tilde{\beta}_k \tilde{\varphi}), \tag{316}$$

where the dimensionful frequency β is scale-independent. Similarly, the running dimensionful (and dimensionless) potential of the MSG model reads as

$$\text{MSG}: \qquad V_k(\varphi) = \frac{1}{2} M^2 \varphi^2 + u_k \cos(\beta\varphi),, \qquad \tilde{V}_k(\tilde{\varphi}) = \frac{1}{2} \tilde{M}_k^2 \tilde{\varphi}^2 + \tilde{u}_k \cos(\tilde{\beta}_k \tilde{\varphi}), \tag{317}$$

where the dimensionful frequency β and the dimensionful mass M^2 are scale-independent. In LPA the wave-function renormalization is set equal to constant, i.e. $Z_k \equiv 1$ and the Wetterich FRG equation reduces to the partial differential equation for the dimensionful (252) and the dimensionless (254) potentials. By inserting the ansatz for the SG (316) and MSG (317) models into the FRG equation which splits into two parts, one finds the non-periodic part which results in a trivial scaling for the mass and the frequency [20],

$$(d - (d-2) + k\partial_k)\, \tilde{M}_k^2 = 0 \;\; \to \;\; (2 + k\partial_k)\tilde{M}_k^2 = 0 \;\; \to \;\; \tilde{M}_k^2 = \tilde{M}_\Lambda^2 \left(\frac{k}{\Lambda}\right)^{-2} \;\; \to \;\; M^2 = \text{const},$$

$$\left(-\frac{d-2}{2} + k\partial_k\right) \tilde{\beta}_k = 0 \;\to\; \left(\frac{2-d}{2} + k\partial_k\right) \tilde{\beta}_k = 0 \;\to\; \tilde{\beta}_k = \tilde{\beta}_\Lambda \left(\frac{k}{\Lambda}\right)^{(d-2)/2} \;\to\; \beta = \text{const}, \tag{318}$$

and the (dimensionful) periodic part has been expanded in Fourier series and only the fundamental mode is kept,

$$k\partial_k u_k = \int_p \frac{k\partial_k R_k}{\beta^2 u_k} \left(\frac{P - \sqrt{P^2 - (\beta^2 u_k)^2}}{\sqrt{P^2 - (\beta^2 u_k)^2}} \right), \tag{319}$$

where for the SG model one finds $P = p^2 + R_k$ [4, 17] and for the MSG model $P = p^2 + M^2 + R_k$ [20, 9] and $\int_p = \int dp\, p^{d-1} \Omega_d/(2\pi)^d$ with the d-dimensional solid angle Ω_d.

9.1 The Ising model

Before the detailed study of the SG and MSG models let me first discuss very briefly the Ising model where apart from the trivial mass term $M^2\varphi^2/2$, a quartic $g_4\varphi^4/4!$ self-interaction is taken into account. The FRG equations are taken in the LPA level for the sharp cutoff (which gives identical results in $d = 2$ with the power-law regulator with $b = 1$), for the dimensionless couplings, $\tilde{M}_k^2$ and $\tilde{g}_{4,k}$, reading in $d = 2$ and $d = 4$ dimensions as [194],

$$\begin{aligned} \text{d} = 2: \qquad k\partial_k \tilde{M}_k^2 &= -2\tilde{M}_k^2 - \frac{1}{4\pi} \frac{\tilde{g}_{4,k}}{(1 + \tilde{M}_k^2)}, \\ k\partial_k \tilde{g}_{4,k} &= -2\tilde{g}_{4,k} + \frac{3}{4\pi} \frac{\tilde{g}_{4,k}^2}{(1 + \tilde{M}_k^2)^2}. \end{aligned} \tag{320}$$

$$\begin{aligned} \text{d} = 4: \qquad k\partial_k \tilde{M}_k^2 &= -2\tilde{M}_k^2 - \frac{1}{16\pi^2} \frac{\tilde{g}_{4,k}}{(1 + \tilde{M}_k^2)}, \\ k\partial_k g_{4,k} &= \frac{6}{32\pi^2} \frac{g_{4,k}^2}{(1 + \tilde{M}_k^2)^2} \approx \frac{3}{16\pi^2} g_{4,k}^2 + \mathcal{O}(\tilde{M}_k^2, g_{4,k}^3) \end{aligned} \tag{321}$$

The above equations have a trivial Gaussian and in $d = 2$ a non-trivial (cutoff-dependent) Wilson-Fisher (WF) fixed point, where the latter indicates the existence of two phases [194].

Please observe that $g_{4,k}$ is dimensionless in $d = 4$ and the leading order term of its FRG flow equation reproduces the perturbative result (29). Moreover, it is known that this leading order result is the same for all types of regulators. In general, the scheme-dependence of perturbative RG is connected to the regulator-dependence of FRG method, see [191, 192, 193]. The universality of the FRG method is discussed in [192] and it was proven in [193] that the FRG flow equation admits a perturbative solution and a scheme transformation was given which was used to obtain the β function of the FRG method with a special choice of the regulator function from the perturbative β function obtained in the modified minimal subtraction scheme. The β functions of the FRG approach are not universal because the FRG method is a mass-dependent scheme which manifests through the nontrivial coupling of mass. In other words, the explicit mass makes the relation between the FRG and perturbative RG β functions nontrivial, i.e., if the quantum field theory has no explicit mass, this scheme transformation is simple and trivial. Thus, one expect identical results for the perturbative RG and FRG β functions in $d = 4$ dimensions for vanishing mass. In [193] it is argued that a similar relation exists for all regulator types.

It is illustrative to recap known conformal properties of the Ising model in $d = 2$, see e.g., [10]. On Fig. 19 I plot the RG flow diagram of the Ising model with central charges at its fixed points. The c-function along the trajectory starting at the Gaussian and terminating at the WF

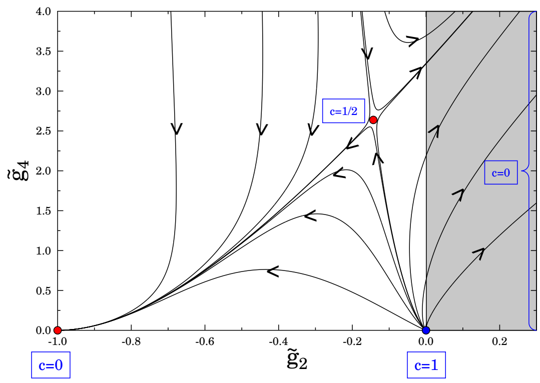

Figure 19: Representation of the RG flow diagram of the Ising model in the $\tilde{g}_2 \equiv \tilde{M}^2$, $\tilde{g}_4$ plane [10]. The shaded area stands for initial conditions where $\tilde{M}^2 > 0$ is positive which has important consequences on the FRG study of the ShG model.

fixed points is known to decrease by $\Delta c = 1/2$. However, if one considers the massive deformation of the Gaussian fixed point which is the case in the shaded area of Fig. 19 then one finds $\Delta c = 1$ [65, 195].

It is also useful to consider what happens with the RG flow diagram and its fixed points if one include the constant, i.e., field-independent term into the FRG study of the Ising model [27]. As it was shown in Section IV, the inclusion of the constant term could be problematic. Although it does not modify the scaling of $\tilde{M}_k^2$ and $\tilde{g}_{4,k}$ but its RG flow equation suffers from the absence of the Gaussian fixed point. Indeed, in $d = 4$ dimensions, for the optimised (i.e., Litim) cutoff the RG flow equation for the constant term $\tilde{V}_k(0)$ of the potential reads as [27],

$$k\partial_k \tilde{V}_k(0) = \frac{1}{32\pi^2}\left(\frac{1}{1+\tilde{M}_k^2}\right) - 4\tilde{V}_k(0) \tag{322}$$

where the dimensionless coupling (i.e" mass term) $\tilde{M}_k^2$ has a trivial scaling $\tilde{M}_k^2 \sim k^{-2}$ if one neglects the quartic coupling. Thus, the β-function $1/(1+\tilde{M}_k^2)$ tends to a non-vanishing constant in the UV limit, i.e., for $k \to \infty$.

It has already been mentioned that, within the nonperturbative RG equations extra care is needed in the analysis of the field-independent terms. The essence of the problem of the RG scaling of the constant term is the (possible) absence of the Gaussian fixed point which, otherwise, is present, if the constant term is not considered. In other words, the β-function of the constant term should vanish if all couplings are set to zero. So, if the Gaussian fixed point is missing once the field-independent coupling is included, then one finds problematic UV divergences which requires a subtraction method. It was shown that such divergences occur for the quantum anharmonic oscillator [180] for $d = 1$ dimensions and if one considers it in higher dimensions one has to generalise the subtraction method for $d = 4$ dimensions [27].

As suggested by the above general considerations, I observe that a single subtraction, based on the replacement

$$k\partial_k V_k(0) = \frac{k^4}{32\pi^2}\frac{k^2}{k^2+\partial_\phi^2 V_k(0)} \to \frac{k^4}{32\pi^2}\left[\frac{k^2}{k^2+\partial_\phi^2 V_k(0)} - 1\right] \to -\frac{k^2}{32\pi^2}\,\partial_\phi^2 V_k(0)\,, \qquad k\to\infty\,, \tag{323}$$

leaves a quadratic term (in k) in the UV limit [27]. This subtraction term modifies the RG evolution of the cosmological constant. Thus, Eq. (140) is changed as [27],

$$k\partial_k \lambda_k = \frac{1}{4\pi} g_k \left(\frac{1}{1+\tilde{M}_k^2} - 1\right) - 2\lambda_k, \tag{324}$$

from which one observes that the corresponding β-function does not diverge in the UV limit but tends to a non-vanishing finite value, i.e., the expression $g_k\left(\frac{1}{1+\tilde{M}_k^2} - 1\right)$ approaches a negative constant, where one keeps in mind that $g_k \sim k^2$ and $\tilde{M}_k^2 \sim k^{-2}$ in the UV limit. Let me therefore investigate the doubly-subtracted RG evolution [27],

$$k\partial_k V_k(0) = \frac{k^4}{32\pi^2}\left(\frac{k^2}{k^2+\partial_\phi^2 V_k(0)} - 1 + \frac{\partial_\phi^2 V_k(0)}{k^2}\right), \qquad k\to\infty\,, \tag{325}$$

which leads to modification of the RG flow equation (140) of the dimensionless cosmological constant [27],

$$k\partial_k \lambda_k = \frac{1}{4\pi} g_k \left(\frac{1}{1+\tilde{M}_k^2} - 1 + \tilde{M}_k^2\right) - 2\lambda_k, \tag{326}$$

from which one observes that the β-function tends to zero in the UV limit, since we have the asymptotic behavior that $g_k\left(\frac{1}{1+\tilde{M}_k^2} - 1 + \tilde{M}_k^2\right) \to 0$ if $k\to\infty$, where one keeps in mind that $g_k \sim k^2$ and $\tilde{M}_k^2 \sim k^{-2}$ in the UV limit. Thus, it was shown that the rampant divergent terms k^2 and k^4 which naturally appear in the FRG equation for $d = 4$ dimensions for the scalar (matter) field (in the absence of the Gaussian fixed point) can be removed by a suitable subtraction method. These divergent terms are the consequence of the construction of FRG method and considered as unphysical.

9.2 The SG model in $d = 2$

In order to obtain the RG flow equation for the (dimensionless) Fourier amplitude $\tilde{u}_k$ of the SG model (316) one has to perform the momentum integral in (319) which can be done analytically for $d = 2$ dimensions with a specific choice of the regulator. Indeed, in the mass cutoff case, i.e. for the power law regulator with $b = 1$, one can derive the RG flow equation from (319) which reads as,

$$(2 + k\partial_k)\tilde{u}_k = \frac{1}{2\pi\beta^2\tilde{u}_k}\left[1 - \sqrt{1-\beta^4\tilde{u}_k^2}\right] \tag{327}$$

see [10] or Eq. (21) of [22] for vanishing mass. Similarly, using the optimized (Litim) regulator one finds

$$(2 + k\partial_k)\tilde{u}_k = \frac{1}{2\pi\beta^2\tilde{u}_k}\left[\frac{1}{\sqrt{1-\beta^4\tilde{u}_k^2}} - 1\right], \tag{328}$$

see [10] or Eq. (19) of [22] for vanishing mass. Equations (327) and (328) have the same qualitative solution and both of them indicate the critical frequency, $\beta_c^2 = 8\pi$ which is identical to that of obtained from the linearised RG flow. Thus, I can confirm that the linearised RG is sufficient to determine the exact value for the critical frequency of SG type models. In Fig. 20 I show the phase structure obtained by solving (328). The RG trajectories of the exact RG flow are straight lines

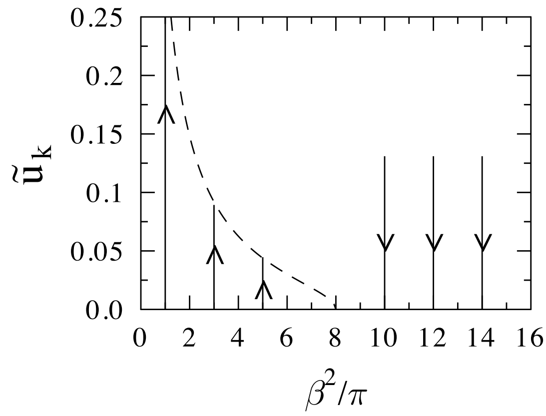

Figure 20: The figure shows the phase structure of the SG model obtained by the FRG equation using Litim's regulator in the scale-independent frequency case [10]. The two phases are separated by $\beta_c^2 = 8\pi$. The dashed line shows the line of IR fixed points of the broken phase.

(similarly to the linearised RG flow) because in LPA the frequency parameter is scale independent. However, there is an important difference between the linearised and the exact RG flows. The exact RG equations, (327) and (328) predict a line of IR fixed points below the critical frequency. Indeed, above (below) the critical frequency $\beta_c^2 = 8\pi$, the line of IR fixed points is at $\tilde{u}_{\rm IR} = 0$ ($\tilde{u}_{\rm IR} \neq 0$). For $\beta^2 < 8\pi$ the IR value for the Fourier amplitude depends on the particular value of β^2. The line of IR fixed points can be obtained by setting the derivative of the Fourier amplitude to zero in the differential equations (327) and (328) which then reduce to simple algebraic equations which can be solved analytically.

9.3 The MSG model in $d = 2$

As I argued, the MSG model has a Z_2 symmetry (just like the Ising model), therefore one expects two phases. Indeed, it was shown in [54] that the Ising-type phase transition is controlled by the dimensionless quantity u/M^2 related to the critical ratio $(m/e)_c$ of QED_2 which separates the confining and the half-asymptotic phases of the fermionic model. The critical ratio $(m/e)_c = 0.31 - 0.33$ has been calculated by the density matrix RG method for the fermionic model which implies [54]

$$\left(\frac{u}{M^2}\right)_c = \left(\frac{m}{e}\right)_c \frac{\exp{(\gamma)}}{2\sqrt{\pi}} = 0.156 - 0.168. \tag{329}$$

One of my goals is to reproduce this critical ratio in the framework of the FRG method [22]. Let me first consider the RG flow obtained by the optimized (Litim) regulator function (248) with $a = 1$ in the LPA (i.e. $z = 1/\beta^2$ =constant) which reads as [22]

$$(2 + k\partial_k)\tilde{u}_k = -\frac{1}{2\pi\beta^2\tilde{u}_k}\left[1 - \sqrt{\frac{(1+\tilde{M}_k^2)^2}{(1+\tilde{M}_k^2)^2 + \beta^4\tilde{u}_k^2}}\right], \qquad (2 + k\partial_k)\tilde{M}_k^2 = 0. \tag{330}$$

In the broken symmetric phase, the RG trajectories merge into a single trajectory in the deep IR region which is characterized by the critical ratio $[\tilde{u}/\tilde{M}^2]_c \approx 0.07957$ (for $\beta^2 = 4\pi$) and serves

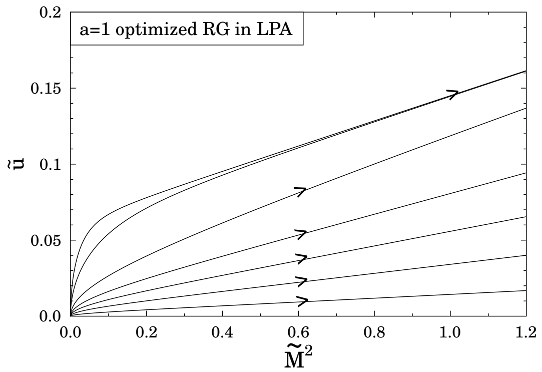

Figure 21: Phase diagram of the MSG model for $\beta^2 = 4\pi$ [22]. RG trajectories are obtained by the integration of Eq. (330). Since the so called, spinodal instability (SI) does not occur in the RG flow, the critical ratio of the MSG model can be determined, $[\tilde{u}/\tilde{M}^2]_c \approx 0.07957$. The arrows indicate the direction of the flow.

as an upper bound (see Fig. 21). The critical value obtained by Eq. (330) is less than the exact result (329), therefore it requires further improvement. Let me try to improve it by the optimized regulator (248) with $a \neq 1$ which has the following form in LPA [22]

$$
\begin{aligned}
(2 + k\partial_k)\tilde{u}_k =& \frac{a}{(a-1)2\pi\tilde{u}_k\beta^2}\Bigg[(1+\tilde{M}_k^2) - (a+\tilde{M}_k^2) \\
& + \sqrt{(a+\tilde{M}_k^2)^2 - \tilde{u}_k^2\beta^4} - \sqrt{(1+\tilde{M}_k^2)^2 - \tilde{u}_k^2\beta^4}\Bigg], \\
(2 + k\partial_k)\tilde{M}_k^2 =& 0.
\end{aligned} \tag{331}
$$

However, for $a \neq 1$, the so called, spinodal instability (SI) appears in the RG flow in the broken symmetric phase, i.e. RG equations become singular in the IR limit and the RG flow stops at some finite scale (see the dashed lines in Fig. 22). Although RG trajectories start to converge into a single one in the broken phase the critical value of the single-frequency MSG model cannot be determined unambiguously. In other words, the convergence properties of the optimized RG is weakened for $a \neq 1$. Let me try to use other types of RG equations, for example the power-law type RG with

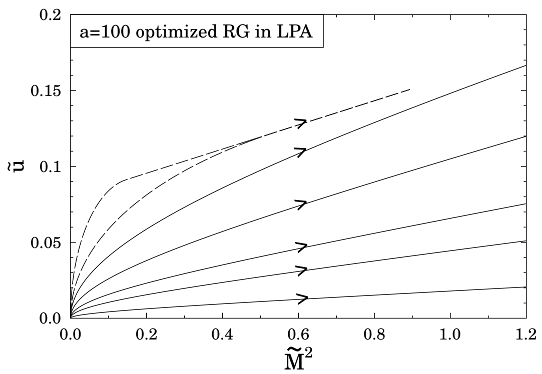

Figure 22: Phase diagram of the MSG model for $\beta^2 = 4\pi$ [22]. RG trajectories are obtained by the integration of Eq. (331) with $a = 100$. The dashed lines correspond to RG trajectories where SI occurs in the RG flow, thus the critical ratio of the MSG model cannot be obtained.

$b = 1$ in LPA which reads as [22]

$$
(2 + k\partial_k)\tilde{u}_k = \frac{(1+\tilde{M}_k^2) - \sqrt{(1+\tilde{M}_k^2)^2 - \tilde{u}_k^2\beta^4}}{2\pi\tilde{u}_k\beta^2}, \qquad (2 + k\partial_k)\tilde{M}_k^2 = 0. \tag{332}
$$

It is known that in the sharp limit, the optimized RG becomes identical to the power-law type RG with $b = 1$, i.e. Eq. (331) reduces to Eq. (332) for $a \to \infty$. Thus, SI is expected in case of Eq. (332). Indeed, the numerical solution of (332) indicates the appearance of SI in the broken symmetric phase (see the dashed lines in Fig. 23).

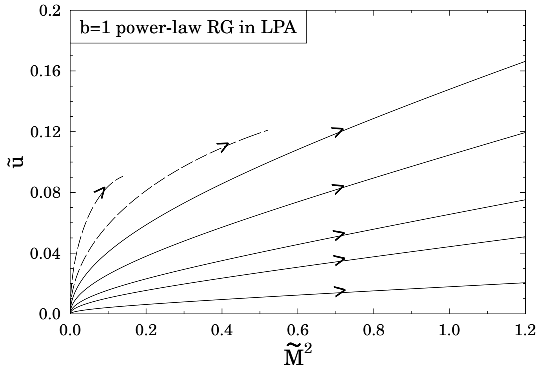

Figure 23: Phase diagram of the MSG model for $\beta^2 = 4\pi$ [22]. RG trajectories are obtained by the integration of Eq. (332). The dashed lines correspond to RG trajectories where SI occurs in the RG flow, thus the critical ratio of the single-frequency MSG model cannot be obtained.

It is also illustrative to compare the IR values of the ratio $\tilde{u}/\tilde{M}^2$ at the scale of SI given by the integration of optimized RG with various values for the parameter a using the same UV initial condition (see Fig. 24). This demonstrates that the best estimate for the critical ratio of the single-frequency MSG model, in the framework of the optimized RG can be achieved for $a = 1$. My findings [22] are consistent to the feature of the optimized RG namely that it increases the convergence properties of the truncated flow. For example, similar result is shown in Fig. 12 of [185] in the framework of the O(N) symmetric scalar theory in $d = 3$ dimensions.

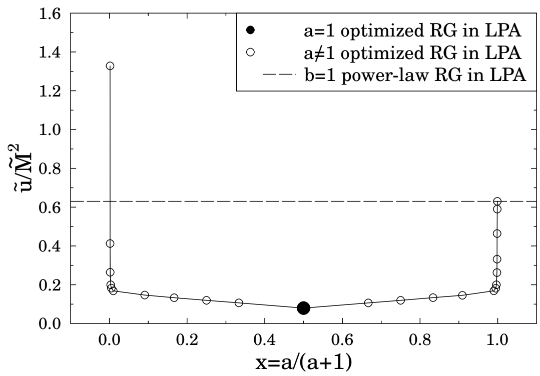

Figure 24: This figure shows how the IR value of the ratio $\tilde{u}/\tilde{M}^2$ obtained by the integration of RG Eqs. (330) and (331) depends on the parameter a of the regulator function [22]. The same initial condition has been used for the numerical integration, ($\tilde{u}(\Lambda) = 10^{-5}, \tilde{M}^2(\Lambda) = 10^{-9}$). SI has occurred for $a \neq 1$, thus the RG flow stops at some finite scale where the ratio has been read off and plotted. It is possible to avoid SI but only for $a = 1$. For $a \to \infty$ (i.e. $x \to 1$) the RG Eq. (331) becomes identical to that was obtained by the power-law type regulator with $b = 1$ (332); consequently in this case the IR values of the ratio coincide.

Since Eq. (330) has no singular behavior, the appearance of SI is expected to be the consequence of an inappropriate approximation, e.g. too drastic simplification of the functional subspace. Consequently, in order to obtain reliable results and to avoid SI in case of the MSG model one has to incorporate higher harmonics generated by RG equations which is discussed later.

9.4 Optimisation of the regulator-dependence based on the MSG model

The fact that the critical value of the MSG model which separates the two phases cannot be obtained for all type of regulators can be used to optimise the regulator-dependence of the FRG equations [23, 24]. For example, in Ref. [24] the optimization of the regulator-dependence of functional RG equations in LPA has been done in the framework of the PMS method by using the CSS regulator (249). The CSS regulator (249), has a very general functional form and it reduces to all major type of regulator functions in appropriate limits [23]. Due to its versatile functional form, the CSS regulator represents an excellent playground for the PMS method and allows to systematically investigate the optimization of the regulator within a rather wide class of regulators.

The RG trajectories corresponding to the broken phase of the single-frequency MSG model where the reflection symmetry (Z_2) is broken spontaneously, merge into a single one in the IR limit, see Fig. 21 and its slope defines the critical ratio. Thus, in the broken phase $\tilde{u}_k$ is a linear function of $\tilde{M}_k^2$ in the IR limit [24],

$$\tilde{u}_k = a \; \tilde{M}_k^2 + b. \tag{333}$$

Both the ratio of the dimensionful coupling to the mass term, as well as the dimensionless equivalent tend to the constant $[\tilde{u}_{k\to 0}/\tilde{M}^2_{k\to 0}] = [u_{k\to 0}/M^2_{k\to 0}] = a$ (since $\tilde{u}_k$ and $\tilde{M}_k^2$ are increasing in the IR limit, their ratio tends to a and the constant b term can be neglected), and the slope is independent of the initial conditions (in the symmetric phase, the linear functional form (333) holds but the slope depends on the initial values).

There is strong numerical evidence [24] for a scheme-independent slope $a = 1/(4\pi)$, within the single Fourier-mode approximation. This means, the critical value in this approximation is

$$\chi_c = \left[\frac{u}{M^2}\right]_c = \frac{1}{(4\pi)} \approx 0.07957. \tag{334}$$

In this case,the optimized regulator (248) with $b = 1$ and $c = 0.01$ leads to a ratio $[u/M^2]_c = 0.07964$ closer to the analytic one but other regulators such as the power-law type regulator with $b = 1$ run into a singularity and stop at some finite momentum scale, rendering the determination of the critical ratio impossible. Therefore, the use of the single Fourier mode approximation provides us with a tool to consider the convergence properties of the RG equations and to optimise the regulator functions [24].

Indeed, the convergence properties of the RG equation depend on the regulator chosen. The RG evolution stops at some finite scale $k_\mathrm{f} \neq 0$ and the ratio $[u_{k\to k_\mathrm{f}}/M^2_{k\to k_\mathrm{f}}]$ becomes scheme-dependent, where the optimization can be performed. This strategy is used to select the optimized regulator according to its convergence properties. The goal is to find the optimal set of parameters b, h, c of the CSS regulator (249). In Fig. 25, the critical ratio obtained by the CSS with exponential norm (249) is shown as a function of b, h and c. The one closest to the analytic formula (334), is obtained for $b = 1$, $c = 0.001$ and $h = 1$. This is the Litim limit of the CSS regulator. In Fig. 26, I plot the minimum values for the critical ratio obtained at every subgraph of Fig. 25 i.e., keeping b fixed, and thus the dependence of each minimum on the parameter b can be read off, confirming once more the nature of the optimized regulator. Again, this demonstrates that Litim's limit ($b \to 1$, $c \to 0$ and $h = 1$) leads to the optimum result [24].

9.5 The MSG model in $d > 2$

Finally, let me discuss the FRG study of the MSG model in $d = 4$ dimensions (or in general in higher dimensions) [19, 20]. Similarly to the $d = 2$ case, one finds two phases in higher dimensions [19, 20], controlled by the dimensionless quantity $\tilde{u}_k\tilde{\beta}_k^2/\tilde{M}_k^2$, which tends to a constant in the IR limit. In the (Z_2) symmetric phase the magnitude of this constant is arbitrary (and depends on the initial conditions), but always smaller than one, i.e., $\lim_{k\to 0} |\tilde{u}_k\tilde{\beta}_k^2/\tilde{M}_k^2| < 1$ (see the blue lines of Fig. 27).

In the spontaneously broken (SSB) phase, the IR value of the magnitude of the ratio is exactly one [19] (independently of the initial values) which serves as an upper bound, see green lines of

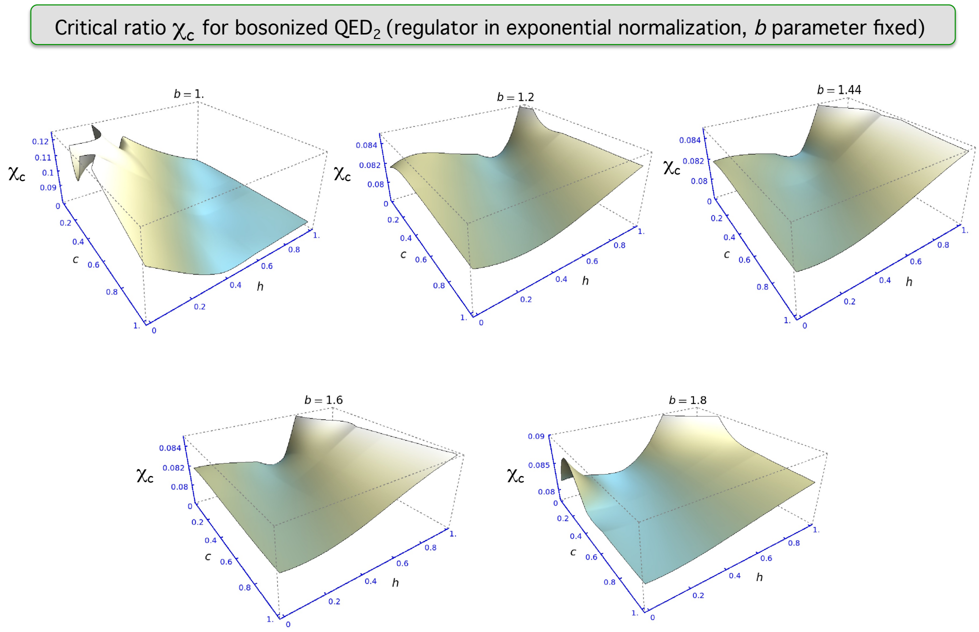

Figure 25: I plot the critical ratio χ_c of bosonized QED$_2$, obtained by various parameters of the CSS regulator with exponential norm [24]. The critical ratio χ_c is on the ordinate axis. Lower critical ratios indicate better regulators, with the optimum results being obtained for $b = 1$, $c = 0.001$ and $h = 1$ (Litim limit of the CSS).

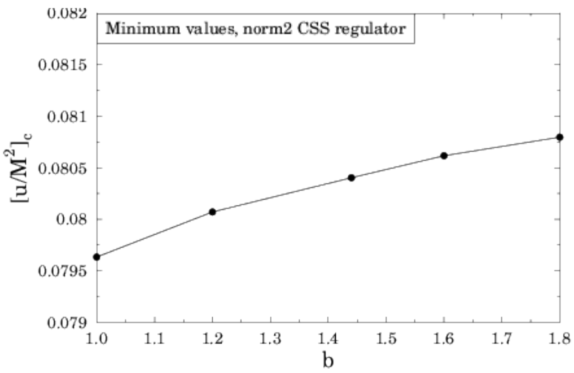

Figure 26: Minimum values (with respect to c and h for fixed b) of the critical ratio obtained by the CSS regulator with exponential norm are plotted for various values of the parameter b [24]. The most favorable results are obtained in the limit $b \to 1$, $c \to 0$ and $h \to 1$. Notably, the optimum critical ratio $[u/M^2]_c$ obtained for $b \to 1$ is lower than the "optimum" value (with respect to a variation of c and h) obtained for $b = 1.44$.

Fig. 27. The black line separates the two phases. In other words, trajectories in the SSB phase (green lines) merge into a master trajectory (green line parallel to the black one) of Fig. 27 which implies [19],

$$\bar{u}_k \tilde{\beta}_k^2 = 1 + \tilde{M}_k^2 \,, \qquad \to \qquad \frac{\bar{u}_k \tilde{\beta}_k^2}{\tilde{M}_k^2} - 1 = \frac{1}{\tilde{M}_k^2} = \frac{k^2}{M_{\mathrm{UV}}^2}. \tag{335}$$

This scaling relation is valid when the running is determined by the master trajectory which, apart from the very beginning of the running, is always the case in the SSB phase. Under these assumptions, Eq. (335) can be used to determine the IR value of the Higgs mass and VeV from the UV initial conditions (190a) and compared to the known results of (187). Indeed by substituting

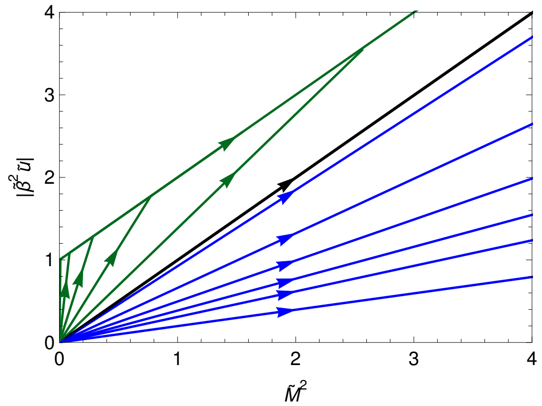

Figure 27: Flow of the MNI (i.e. MSG) model showing two phases separated by a black line with a unit slope [19]. The blue lines corresponds to the symmetric phase, while the green lines correspond to the SSB phase.

(335) into (186) and assuming running parameters $\tilde{u}_k\tilde{\beta}_k^2/\tilde{M}_k^2 = u_k\beta^2/M^2$, one gets [19]

$$M_h(k) = M_{\mathrm{UV}}\sqrt{2\left(\frac{\tilde{u}_k\tilde{\beta}_k^2}{\tilde{M}_k^2} - 1\right)}$$

$$\text{and from Eq.(335)}: \quad M_h(k) = M_{\mathrm{UV}}\,\sqrt{2}\sqrt{\frac{k^2}{M_{\mathrm{UV}}^2}} = \sqrt{2}\,\cancel{M_{\mathrm{UV}}}\,\frac{k}{\cancel{M_{\mathrm{UV}}}} = \sqrt{2}k, \tag{336}$$

where the cancellation of the UV mass has been made evident. Since $k_{\mathrm{IR}} = 250$ GeV, it provides

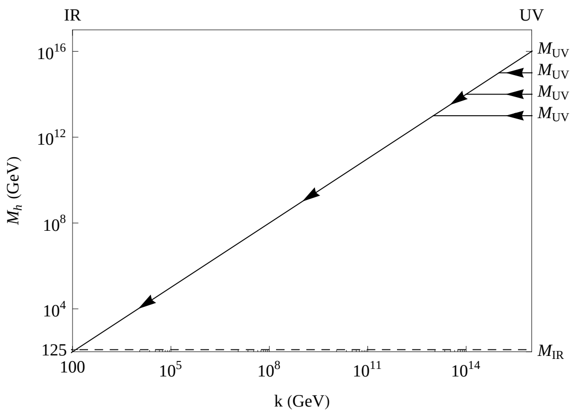

Figure 28: Flow of the Higgs mass from the cosmological (UV) scale to the electroweak (IR) scale obtained from the study of the MNI model [19]. The trajectories merge into a single line showing UV insensitivity.

the required IR value for the Higgs mass (187), at least the same order, in accordance with measurements. Furthermore, the IR values for the Higgs mass becomes independent of the UV initial parameter (see Fig. 28).

I showed that the Massive Natural Inflation (MNI) model which is an MSG type scalar theory serves as a viable model for cosmic inflation [19, 20]. By using PLANCK data, one can fix the parameters of the model at the scale of inflation which is around the GUT scale. It was shown that the value for the parameters chosen at the cosmological scale does not influence the results at the electroweak scale. By using the MNI model one can complete the theory towards low energies producing the correct order-of-magnitude for the Higgs mass [19].

10 Exact FRG equation with higher harmonics in LPA

In this section I discuss the FRG equations obtained in LPA for a general periodic (SG type) model where the running dimensionful (and dimensionless) potential contains higher harmonics [1, 4, 5]

$$\text{SG}: \qquad V_k(\varphi) = \sum_{n=1}^{\infty} u_n(k)\cos(n\beta\varphi), \qquad \tilde{V}_k(\tilde{\varphi}) = \sum_{n=1}^{\infty} \tilde{u}_n(k)\cos(n\tilde{\beta}_k\tilde{\varphi}), \tag{337}$$

and the dimensionful frequency β is scale-independent. Similarly, I consider a general periodic model with a mass term, i.e., the general MSG type model where the running dimensionful (and dimensionless) potential contains higher harmonics [4, 5]

$$\begin{aligned}
\text{MSG}: \qquad V_k(\varphi) &= \frac{1}{2}M^2\varphi^2 + U_k(\varphi), \qquad U_k(\varphi) = \sum_{n=1}^{\infty} u_n(k)\cos(n\beta\varphi), \\
\tilde{V}_k(\tilde{\varphi}) &= \frac{1}{2}\tilde{M}_k^2\tilde{\varphi}^2 + \tilde{U}_k(\tilde{\varphi}), \qquad \tilde{U}_k(\tilde{\varphi}) = \sum_{n=1}^{\infty} \tilde{u}_n(k)\cos(n\tilde{\beta}_k\tilde{\varphi}),
\end{aligned} \tag{338}$$

and again the dimensionful frequency β and the dimensionful mass M^2 are scale-independent.

10.1 Dimensionful potential and the convexity

Let me first show why the (dimensionful) effective potential should be convex [196]. The effective action is the Legendre transformation of $W[J]$, which is the generating functional for the connected Green-functions

$$\Gamma_{\text{eff}}[\phi] = -W[J] + \int J\phi\,. \tag{339}$$

If one fixes the field and the source to constant values ($\phi(x) = \phi_0,\ J(x) = J_0$) in the space-time volume Ω, then the effective action reduces to the effective potential as follows

$$\Gamma_{\text{eff}}[\phi] = \Omega\, V_{\text{eff}}(\phi_0)\,, \quad W[J] = \Omega w(J_0)\,. \tag{340}$$

Using the relations of the Legendre transformation for the reduced functions

$$\phi_0 = \frac{\delta w[J_0]}{\delta J_0}\,, \quad J_0 = \frac{\delta V_{\text{eff}}[\phi_0]}{\delta\phi_0}\,, \tag{341}$$

one obtains the following equation by differentiating the effective potential with respect to the field and the source by using the chain rule,

$$\left(\frac{\delta^2 V_{\text{eff}}}{\delta\phi_0\delta\phi_0}\right)\left(\frac{\delta^2 w}{\delta J_0\delta J_0}\right) = 1. \tag{342}$$

The second derivative of the generating functional of the connected Green functions with respect to the source term is the connected correlation function which should be positive. Thus, the convexity of the effective action comes from Eq. (342)

$$\left(\frac{\delta^2 V_{\text{eff}}}{\delta\phi_0\delta\phi_0}\right) \geq 0\,. \tag{343}$$

On one hand, the non-perturbative RG equation (240) should recover the full quantum effective action in the IR ($k \to 0$) limit. On the other hand, I have just shown that the effective action or more precisely the (dimensionful) effective potential should be convex. In the following I discuss the consequences for the SG and MSG models [1, 4, 5].

10.1.1 Convexity of the SG model – flattening of the dimensionful periodic potential

As it was shown previously, an important property of the effective potential is the convexity. The effective potential $V_{\text{eff}}(\varphi)$ is convex even if the classical potential is non-convex. It was also argued that the effective potential should be periodic as well, if the bare theory is periodic. This conflict between convexity and periodicity can be resolved only in a trivial manner, namely if V_{eff} is a constant function, which is periodic and convex at the same time. Thus, I conclude, that for the periodic field theory, the convexity and periodicity are so strong constraints on the effective potential that it should be constant [1]. Indeed, the FRG study of the SG model confirms the flattening of the dimensionful potential, see Fig. 29 in $d = 2$ [1] and in $d > 2$ [17], too. Let me

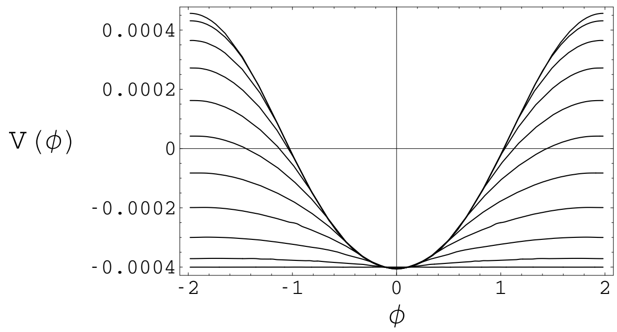

Figure 29: Flattening of the blocked dimensionful periodic potential under RG transformations in $d = 2$ dimensions [1].

note that this general statement (flattening of the SG potential in both phases) holds for the dimensionful potential, however, the dimensionless one can be used to distinguished between the phases of the periodic scalar field theory (which is known to have a non-trivial phase structure in $d = 2$).

10.1.2 Convexity of the MSG model – RG running induced cosmic inflation

Let me now study the convexity of the (dimensionful) MSG model in $d = 4$ dimensions which has relevance for cosmic inflation [20]. As it was discussed previously, the simple mass term has no RG evolution at all, i.e., the dimensionful mass remains unchanged, so that the dimensionless mass has a trivial RG scaling. The MSG model has a non-trivial RG scaling because of the periodic term which evolves under RG transformation and in the low-energy, i.e., IR limit the MSG potential should tend to a convex one even if the potential in the high-energy, i.e., UV limit is non-convex. This result can be used to a possible mechanism for inflation which combines the "old" (inflation form false vacuum) and the "new" (slow-roll) scenarios for inflation [20].

The main idea behind the proposed method for RG running induced inflation [20] is the fact that any scalar potential should tend to a convex one during the RG flow. The RG evolution of the potential starts from a concave potential at high energies, (i.e., at the Planck scale) which ensures that the VeV can be trapped; it tends to a less concave one at lower energies (i.e. at the GUT scale) and releases the VeV to initiate inflation. RG running is expected to provide a sufficient change in the shape of the potential between the two scales. Indeed, in the RG-inspired model, the VeV is assumed to be trapped in a false vacuum in the pre-inflationary period at very high energies, and then, due to quantum fluctuations, as described by the RG, the effective potential is modified releasing the VeV and leading to the classical inflationary evolution at the scale of inflation, see Fig. 30.

RG evolution forces the potential to tend a convex one which is valid for any potential, and so the MSG potential becomes shallow at the scale of inflation. At this stage, the VeV starts to roll down towards the real minimum, inducing inflation. A 3D visual representation of the change is given in Fig. 31, where the change in the shape of the potential against the running RG scale k is plotted for the MSG model [20].

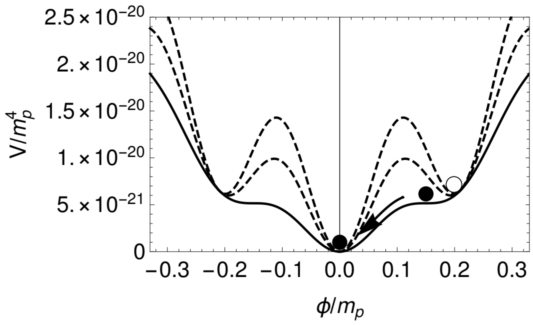

Figure 30: Potential of the MSG model (171) at various RG scales [20]. The solid line stands for the potential at the scale of inflation. The values of the potential parameters have been calculated by the slow-roll study of the MSG model which yield the solid line that represents the potential over the whole inflationary period, where the VeV (full black circle) rolls down inducing inflation. The dashed lines correspond to UV values (pre-inflation), obtained by RG considerations.

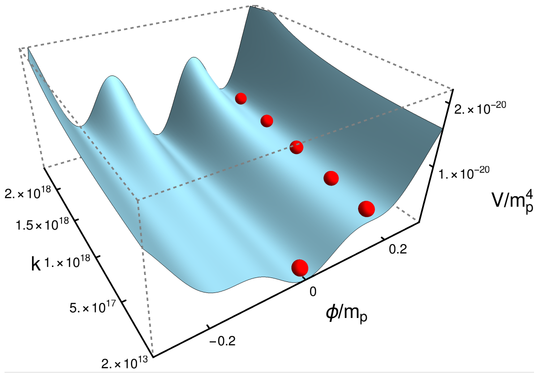

Figure 31: The RG scaling of the MSG potential is shown from the Planck scale towards the scale of inflation for the MSG potential [20]. The red ball denotes the VeV which is trapped in a false vacuum at the Planck scale and due to RG running it is released at the scale of inflation to roll down to the real ground state.

Here, I discussed the consequences of convexity on the dimensionful SG and MSG potentials. In the following I turn to the discussion of the dimensionless potentials and their RG running.

10.2 Dimensionless potential and the symmetry broken phase

As I argued, convexity puts strong constraints on the dimensionful effective potential. Let me consider the FRG equation (with various regulators) in LPA for $d = 2$ dimensions and draw some general conclusions on the shape of the dimensionless potential in the symmetry broken phase [1, 4, 5]. Let me first discuss the Wetterich RG equation with sharp cutoff (255) which is identical to the Wegner–Houghton RG equation in LPA. If one differentiates it with respect to the field variable and multiply it with $1 + \tilde{V}''_k$ one finds the following flow and fixed point equations [4, 5],

$$(2 + k\partial_k)\tilde{V}'_k = -\tilde{V}''_k(2 + k\partial_k)\tilde{V}'_k - \frac{1}{4\pi}\tilde{V}'''_k, \;\; \to \;\; 2\tilde{V}_\star + [\tilde{V}'_\star]^2 + \frac{1}{4\pi}\tilde{V}''_\star = c_1 \tag{344}$$

with the arbitrary constant c_1. The fixed point equation is exhibiting the trivial solution $\tilde{V}_* = c_1/2$ (Gaussian fixed point) and [4, 5],

$$\tilde{V}_\star = -\frac{1}{2}\phi^2 + c\phi + \text{const.} \tag{345}$$

which can also be obtained as the solution of $1 + \tilde{V}_k''$. The sharp cutoff scheme, i.e., the Wegner–Houghton RG equation can account for the spinodal instability (SI), which appears when the restoring force acting on the field fluctuations to be eliminated vanishes, $1 + \tilde{V}_k''(\phi) = 0$ at some finite scale $k_{\rm SI}$ and the resulting condensate generates tree-level contributions to the evolution equation. It was shown that the tree-level RG equation (228) leads to the local potential (345), too. The appearance of the spinodal instability or more general, the IR fixed point solution (345) are signatures of the symmetry breaking, thus, one refers to (345) as the IR convexity fixed point [1, 4, 5].

As a next step, let me study the Wetterich RG equation (254) with the power-law cutoff which reads as

$$(2 + k\partial_k)\tilde{V}_k = -\frac{1}{4\pi}\int_0^\infty \mathrm{d}y \frac{(-b)y^{-b}\, y}{y(1+y^{-b}) + \tilde{V}_k''} \tag{346}$$

with $y = p^2/k^2$. For arbitrary parameter value b, the propagator on the right hand side of Eq. (346) may develop a pole at some scale $k_{\rm SI}$ and at some value of the field ϕ for which $\tilde{V}_k''(\phi) = -C(b) = -b/(b-1)^{(b-1)/b}$ holds, which signals the occurring of SI. The infrared singularity of the functional RG equation is supposed to be related to the convexity of the effective action for theories within a phase of spontaneous symmetry breaking. It was shown that in such a case one has to seek the local potential for $k < k_{\rm SI}$ by minimizing Γ_k in the subspace of inhomogeneous (soliton like) field configurations and ends up with the result [4, 5],

$$\tilde{V}_\star = -\frac{1}{2}C(b)\phi^2 + c\phi + \text{const.} \tag{347}$$

which is again a parabolic shape fixed point solution.

As a final step, let me study the Wetterich RG equation (254) with the Litim (optimised) cutoff (256) which leads to the following fixed point equation and solution (after derivation with respect to the field and multiplication by $(1 + \tilde{V}_k'')^2$), see [4, 5],

$$(1 + \tilde{V}_\star'')^2 2\tilde{V}_\star' + \frac{1}{4\pi}\tilde{V}_\star''' = 0, \quad \rightarrow \quad \tilde{V}_\star = -\frac{1}{2}\phi^2 + c\phi + \text{const.} \tag{348}$$

where the fixed point solution is identical to that of obtained for the sharp cutoff case (345). Let me also note, that the IR convexity fixed point can be found for the symmetry broken phase of the ϕ^4 model, see Fig. 19, where $\tilde{g}_4 = 0$ and $\tilde{g}_2 = -1$.

10.2.1 Dimensionless SG potential with higher harminics

For the sake of completeness let me derive the RG flow equations for the Fourier amplitudes of the SG model with higher harmonics (337). The insertion of the ansatz (337) into the mass cutoff RG Eq. (344) yields the RG flow equations [1, 4, 5]

$$(2 + k\partial_k)n\tilde{u}_n = \frac{\beta^2}{4\pi}n^3\tilde{u}_n + \frac{\beta^2}{2}\sum_{s=1}^{\infty} sA_{n,s}(2 + k\partial_k)\tilde{u}_s, \tag{349}$$

for the couplings $\tilde{u}_n$ where $A_{n,s}(k) = (n-s)^2\tilde{u}_{|n-s|} - (n+s)^2\tilde{u}_{n+s}$. Eq. (344) is valid unless SI arises. In the strong-coupling phase $\beta^2 > 8\pi$ no SI occurs, Eq. (349) holds at any scale and every Fourier amplitude is irrelevant. The dimensionless blocked potential becomes flat, i.e. all couplings $\tilde{u}_n$ vanish in the IR limit $k \to 0$. For $\beta^2 < 8\pi$ the SI occurs in the RG flow when the propagator diverges, $k_{\rm SI}^2 + V_{k_{\rm SI}}''(\phi) = 0$ and one has to use the tree-level RG to obtain the IR effective potential [1, 4, 5].

As a final step, let me turn now to the discussion of the Wetterich RG flow using the optimized (Litim) regulator. The insertion of the ansatz (337) into the mass cutoff RG Eq. (348) yields the RG flow equations [4, 5]

$$(2 + k\partial_k)n\tilde{u}_n = \frac{\beta^2}{4\pi}n^3\tilde{u}_n + \beta^2\sum_{p=1}^{\infty} pA_{n,p}(2 + k\partial_k)\tilde{u}_p - \frac{1}{4}\beta^4\sum_{q=1}^{\infty}\sum_{p=1}^{\infty} pA_{n,q}A_{q,p}(2 + k\partial_k)\tilde{u}_p \tag{350}$$

for the Fourier amplitudes $\tilde{u}_n$ where $A_{i,j} = (i-j)^2\tilde{u}_{|i-j|} + (i+j)^2\tilde{u}_{i+j}$. For $\beta^2 < 8\pi$ the numerical solution of the system (350) of the coupled flow equations exhibits the following features. Values for β^2 taken in the vicinity of the critical value 8π and restricting oneself to the first few Fourier amplitudes, one finds no SI in the IR region and the dimensionless Fourier amplitudes tend to a constant value.

Thus, it is important to consider the connection between the non-trivial dimensionless effective potential obtained by the direct integration of various RG flow equations and the general properties of the dimensionless effective potential found in the broken phase. In case of a Z_2 symmetry, the linear term vanishes in (345), in (347) and in (348). I drop the field-independent terms, too. For the sake of simplicity let me consider the case of the Litim cutoff (348). In this case, in the so called weak coupling phase of the periodic (SG type) model ($\beta^2 < 8\pi$) the dimensionless effective potential remains a non-vanishing periodic one (the line of non-trivial IR fixed points in Fig. 32), graphically obtained by setting forth along the ϕ axis the section of the parabola [1, 4]

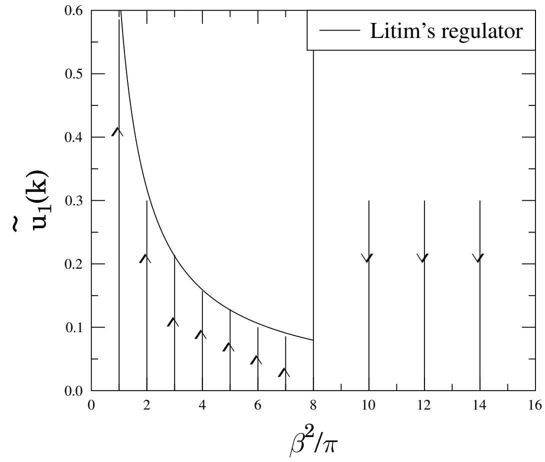

Figure 32: The phase structure of the periodic model obtained in LPA (β^2 is scale-independent) [4]. The non-trivial IR fixed points for $\beta^2 < 8\pi$ are obtained by the Litim cutoff. The line of IR fixed points is given by the analytic expression $\tilde{u}_1(0) = 2/\beta^2$.

$$\tilde{V}_{k\to 0}(\phi) = \frac{2}{\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\beta\phi) = -\frac{1}{2}\phi^2, \quad \to \quad \tilde{u}_1(0) = \frac{2}{\beta^2} \tag{351}$$

with $\phi \in [-\pi/\beta, \pi/\beta]$ periodically [1, 4]. Each parabola section is the same parabola that one would find as the non-trivial fixed point of the polynomial theory. The periodic dimensionless effective potential is continuous, sectionally differentiable.

10.2.2 Dimensionless MSG potential with higher harmonics

It was shown that the FRG equation results in an inverse parabolic dimensionless potential in the symmetry broken phase. Let us use this to calculate the critical value of the MSG model. In the broken phase, in the IR limit, the dimensionless potential of the MSG model (338) can be written as [4],

$$\tilde{V}_{k\to 0} = \frac{1}{2}\tilde{M}^2_{k\to 0}\phi^2 + \tilde{U}_{k\to 0}(\phi) \equiv -\frac{1}{2}\phi^2 \quad \to \quad \tilde{U}_{k\to 0}(\phi) = -\frac{1}{2}(1+\tilde{M}^2_{k\to 0})\phi^2 \tag{352}$$

where $\tilde{U}_{k\to 0}$ is periodic and contains higher harmonics, thus, one finds [4]

$$\tilde{U}_{k\to 0}(\phi) = (1+\tilde{M}^2_{k\to 0})\frac{2}{\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\beta\phi) \quad \to \quad \tilde{u}_1(k\to 0) = (1+\tilde{M}^2_{k\to 0})\frac{2}{\beta^2} \tag{353}$$

which results in the following critical value [4]

$$\left[\frac{\tilde{u}_1(k\to 0)}{\tilde{M}^2_{k\to 0}}\right]_c = \frac{1+\tilde{M}^2_{k\to 0}}{\tilde{M}^2_{k\to 0}}\frac{2}{\beta^2} = \frac{2}{\beta^2} \quad \to \quad \left[\frac{u_1}{M^2}\right]_c = \frac{2}{4\pi} \approx 0.159155 \tag{354}$$

where we used the scaling relation of the explicit mass $\tilde{M}^2_k \sim k^{-2}$. This critical value is twice as large as the one obtained for the single-frequency approximation and closer to the known value of the critical ratio of the corresponding fermionic model. Important to note that the Wetterich RG equation with the Litim cutoff and with the mass cutoff (the power-law regulator with $b = 1$ which is identical to the Wegner-Houghton RG in $d = 2$) gives the same critical value (354). The Litim regulator has the best convergence properties, so it can be used to avoid the appearance of SI in the RG flow which happens when the propagator diverges, $k^2_{\rm SI} + V''_{k_{\rm SI}}(\phi) = 0$ [4].

10.3 Periodicity and the reflection symmetry

Up to now I considered the SG model (either with or without higher harmonics) which has two discrete symmetries, the periodicity ($\phi \to \phi + 2\pi/\beta$) and the reflection ($\phi \to -\phi$) symmetry. In order to study the interplay of these two symmetries in the framework the FRG method let me start to consider the following very general periodic theory, i.e., the multi-frequency sine-Gordon (MFSG) model [197, 198, 199, 200, 201, 202, 203], see [5]

$$\mathcal{S}_{\rm MFSG} = \int \mathrm{d}^2x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \sum_i^n \mu_i \cos(\beta_i\phi + \delta_i)\right] \tag{355}$$

which contains n cosine terms where ϕ is a real scalar field, $\beta_i \in \mathbb{R}$ are the frequencies, $\beta_i \neq \beta_j$ if $i \neq j$, μ_i are the coupling constants (of dimension mass^2 at the classical level) and $\delta_i \in \mathbb{R}$ are the phases in the terms of the potential. Two cases can be distinguished according to the periodicity properties of the model. The first one is the rational case, when the potential is a trigonometric function: the ratios of the frequencies β_i are rational and consequently, the potential is periodic. Let the period of the potential be $2\pi/\beta$ in this case. The other case is the irrational one, when the potential is not periodic [5].

If the target space of the field ϕ is compactified: $\phi \equiv \phi + 2k\beta\pi$, where $k \in \mathbb{N}$ is arbitrary, the MFSG model is called the k-folded multi-frequency SG model [199]. It was shown by semi-classical (mean-field/Landau-Ginzburg) analysis [197] and by means of form factor perturbation and truncated conformal space approaches [197, 200, 202, 203] that (first and second order) phase transitions occur in the compact MFSG model as the coupling constants are tuned appropriately (assuming that $n > 1$). Consequently, it represents an excellent toy model to study the influence of the compactness on the phase structure and the low-energy behavior of the model. Here, I restrict the attention to the rational, non-compact case [5].

As a rule, the solution of the RG equations is sought for in a restricted functional subspace. Since the RG equations retain the symmetries of the bare action, the functional subspace should be chosen keeping the symmetries of the bare action unbroken. Furthermore, even this – generally infinite dimensional – subspace is reduced to a finite dimensional one by the truncation of the appropriate series expansion of the blocked potential. The truncated Fourier expanded form can be a straightforward approximation for scalar models with periodicity in internal space [5].

Let me now turn to the symmetries of MFSG models if the ratios of the frequencies are rational. Then the bare potential is periodic in the internal space, let its period be $2\pi/\beta$, and one has to look for the solution of the RG equations among the periodic functions with such a period. The bare potential may have however further symmetries as well. For example, the MFSG models can exhibit a reflection symmetry besides periodicity. Three cases can be distinguished [5].

- Let us suppose that the bare potential of the MFSG model contains a single cosine mode with $\delta_1 = 0$

$$\tilde{V}_\Lambda(\phi) = \tilde{\mu}_1 \cos(\beta\,\phi)\,. \tag{356}$$

In this case the model has a discrete reflection symmetry ($\phi \to -\phi$), which is preserved by the FRG equations. Since the RG transformations generate higher harmonics, one is inclined to look for the solution in its Fourier decomposed form

$$\tilde{V}_k(\phi) = \sum_{n=0}^{N} \tilde{u}_n(k) \cos(n\,\beta\,\phi)\,, \tag{357}$$

exhibiting periodicity in the internal space. The dimensionless couplings are represented by the Fourier amplitudes $\tilde{u}_n(k)$ (with $\tilde{u}_1(k=\Lambda) = \tilde{\mu}_1$) and the 'frequency' β is a scale-independent, dimensionless parameter in the LPA.

- If the bare potential of the MFSG model contains a single sine mode (i.e. $\delta_1 = 3\pi/2$)

$$\tilde{V}_\Lambda(\phi) = \tilde{\mu}_1 \sin(\beta\,\phi)\,, \tag{358}$$

the model has another discrete Z_2 symmetry ($\phi \to -\pi/\beta - \phi$) which is preserved by the RG equations. The potential is antisymmetric but the RG equations are not, consequently, one has to look for the solution of the RG equations as

$$\tilde{V}_k(\phi) = \sum_{n=0}^{N} \left[\tilde{u}_{2n}(k) \cos(2n\beta\phi) + \tilde{v}_{2n+1}(k) \sin((2n+1)\beta\phi)\right] \tag{359}$$

with the dimensionless Fourier amplitudes $\tilde{u}_{2n}(k)$ and $\tilde{v}_{2n+1}(k)$ (and $\tilde{v}_1(k=\Lambda) = \tilde{\mu}_1$).

- Finally, if the bare potential of the MFSG model contains both cosine and sine modes (i.e. $\delta_1 = 0$ and $\delta_2 = 3\pi/2$)

$$\tilde{V}_\Lambda(\phi) = \tilde{\mu}_1 \cos(\beta\,\phi) + \tilde{\mu}_2 \sin(\beta\,\phi)\,, \tag{360}$$

the model has no Z_2 symmetry, consequently, all the Fourier modes are generated during the RG flow and the solution has the general form

$$\tilde{V}_k(\phi) = \sum_{n=0}^{N} \left[\tilde{u}_n(k) \cos(n\beta\phi) + \tilde{v}_n(k) \sin(n\beta\phi)\right], \tag{361}$$

with the dimensionless Fourier amplitudes $\tilde{u}_n(k)$ and $\tilde{v}_n(k)$ (and $\tilde{u}_1(k=\Lambda) = \tilde{\mu}_1$, $\tilde{v}_1(k=\Lambda) = \tilde{\mu}_2$).

Since Eq.(361) represents the blocked potential for the most general MFSG model with rational frequency ratios, let me further discuss that case. Inserting the ansatz (361) into the derivative form of the Wetterich RG equation with mass cutoff (344) (which is identical to the sharp cutoff, i.e, the Wegner–Houghton RG equation in $d = 2$ dimensions) one can read off flow equations for the Fourier amplitudes, i.e. for the scale-dependent dimensionless couplings $\tilde{u}_n(k)$, $\tilde{v}_n(k)$ which read as [5]

$$(2 + k\partial_k) n\tilde{u}_n = \tfrac{\beta^2}{4\pi} n^3 \tilde{u}_n + \tfrac{\beta^2}{2} \textstyle\sum_{s=1}^{N} \left(s A^{(1)}_{n,s} (2 + k\partial_k)\tilde{u}_s + s A^{(4)}_{n,s} (2 + k\partial_k)\tilde{v}_s \right), \tag{362}$$

$$(2 + k\partial_k) n\tilde{v}_n = \tfrac{\beta^2}{4\pi} n^3 \tilde{v}_n - \tfrac{\beta^2}{2} \textstyle\sum_{s=1}^{N} \left(s A^{(2)}_{n,s} (2 + k\partial_k)\tilde{u}_s + s A^{(3)}_{n,s} (2 + k\partial_k)\tilde{v}_s \right), \tag{363}$$

where

$$\begin{aligned}
A^{(1)}_{n,s}(k) &= (n-s)^2 \tilde{u}_{|n-s|} - (n+s)^2 \tilde{u}_{n+s} \Theta(n+s \le N),\\
A^{(2)}_{n,s}(k) &= \mathrm{sgn}(s-n)\ \ (n-s)^2 \tilde{v}_{|n-s|} + (n+s)^2 \tilde{v}_{n+s} \Theta(n+s \le N),\\
A^{(3)}_{n,s}(k) &= -(n-s)^2 \tilde{v}_{|n-s|} - (n+s)^2 \tilde{v}_{n+s} \Theta(n+s \le N),\\
A^{(4)}_{n,s}(k) &= \mathrm{sgn}(s-n)\ \ (n-s)^2 \tilde{v}_{|n-s|} - (n+s)^2 \tilde{v}_{n+s} \Theta(n+s \le N),
\end{aligned}$$

with $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$, and $\Theta(n \le N) = 1$ if $n \le N$ and $\Theta(n \le N) = 0$ if $n > N$. Let me note that Eq. (344) and, consequently, Eq. (362), Eq. (363) are valid unless SI arises [5].

Let me consider the IR effective theory of the MFSG model in the framework of the Wetterich RG equation with mass cutoff, i.e., by solving Eqs. (362), (363) numerically. For $\beta^2 > 8\pi$, the Fourier amplitudes are irrelevant in the limit $k \to 0$, independently of the initial conditions, see Fig. 33 [5]. The numerical solution provides the IR scaling of the model which is determined by

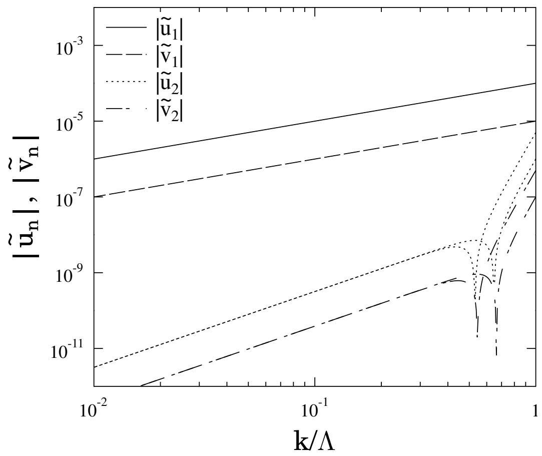

Figure 33: The scaling of the first few Fourier amplitudes of the MFSG model is obtained by the Wetterich RG equation with mass cutoff, i.e., by solving Eqs. (362), (363) numerically for $\beta^2 = 12\pi$ with various initial conditions for the higher harmonics [5]. The peaks in the scaling of $\tilde{u}_2(k)$ and $\tilde{v}_2(k)$ indicate the change of their sign during the RG flow.

two independent parameters ($\tilde{u}_1(\Lambda)$, $\tilde{v}_1(\Lambda)$), if the bare action has no Z_2 symmetry and depends on a single parameter (either $\tilde{u}_1(\Lambda)$ or $\tilde{v}_1(\Lambda)$) in case of a Z_2 symmetric bare action, and it is independent of the initial conditions of the higher harmonics, see Fig. 33.

For $\beta^2 < 8\pi$, the IR scaling behavior turns all the Fourier amplitudes into relevant coupling constants, consequently, an SI could appear in the RG flow. Indeed, in Fig. 34 the scaling of the coupling constants of the MFSG model is presented for $\beta^2 = 4\pi$ and the vertical line shows the appearance of SI.

Beyond the momentum scale k_{SI}, the RG equation loses its validity and one has to use the tree-level RG equation (228) which leads to the IR effective potential (345) in the deep IR limit ($k \to 0$). In order to preserve periodicity, the IR effective potential of the MFSG model has a parabola-shape for $\phi \in [-\pi/\beta, \pi/\beta]$ and such parabola sections are repeated along the ϕ axis [5]. Let me analyze the sensitivity of the IR effective theory on the UV initial conditions. In case of a reflection symmetry $\phi \to -\phi$, the linear term vanishes in (345), i.e. $c = 0$, and the potential is superuniversal, i.e. independent of any initial conditions. If the bare action has another type of reflection symmetry $\phi \to -\phi - \pi/\beta$, then the constant in (345) is non-zero but fixed, i.e. $c = -\pi/(2\beta)$, consequently, the IR potential is again superuniversal. If the bare action of the MFSG model has no Z_2 symmetry then the deep IR behavior depends on a single parameter c [5].

Let me consider the IR scaling of the MFSG model by solving the Wetterich RG equation with mass cutoff (344) (which is identical to the sharp cutoff, i.e., the Wegner–Houghton RG equation in $d = 2$ dimensions) by a computer algebraic code. The solution found is expanded in Fourier series in order to compare the results to those obtained by Eqs. (362), (363). For $\beta^2 = 4\pi$ the scalings of the first few Fourier amplitudes are plotted in Fig. 34. There is a quantitative agreement between the results obtained by Eq.(344) and Eqs. (362), (363) in the UV and IR scaling regimes. However, the important difference is that no SI is found in the RG flow when Eq.(344) is solved directly. This indicates that SI occurs in the RG approach as an artifact due to the truncated Fourier-expansion applied to the almost degenerate blocked action of the MFSG model, at least for $\beta^2 = 4\pi$ [5].

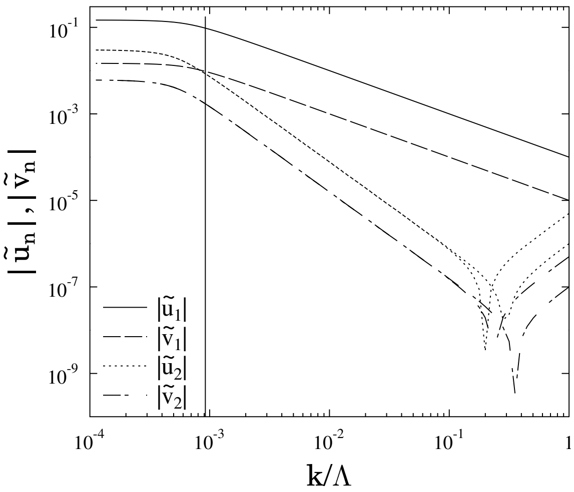

Figure 34: The scaling of the first few Fourier amplitudes of the MFSG model is obtained for $\beta^2 = 4\pi$ [5] by solving numerically either Eqs. (362), (363) or Eq. (344). In the latter case, the partial differential equation (344) is solved by a computer algebraic code and then the solution is expanded in Fourier series. The vertical line indicates the momentum scale of SI (k_{SI}) where Eqs. (362), (363) lose their validity. Above this scale, $k_{\mathrm{SI}} < k$, the results obtained by Eqs. (362), (363) and by Eq. (344) coincide. Below the scale of SI, $k < k_{\mathrm{SI}}$, the scaling of the Fourier amplitudes is determined by the direct integration of Eq. (344).

The IR effective potential of the MFSG model was found to be different above and below $\beta_c^2 = 8\pi$. For $\beta^2 > 8\pi$, the deep IR behavior of the MFSG model with Z_2 symmetry (i.e. $\phi \to -\phi$ or $\phi \to -\pi/\beta - \phi$) depends on the UV initial condition for either the fundamental cosine or the fundamental sine mode, respectively, and for $\beta^2 < 8\pi$ it is superuniversal, i.e. independent of any initial conditions. If the MFSG model has no Z_2 symmetry, for $\beta^2 > 8\pi$ the IR effective potential depends on the UV initial conditions both for the fundamental cosine and sine modes (i.e. it depends on two independent parameters) and for $\beta^2 < 8\pi$ it is universal [5], i.e. depends on only a single parameter, namely the ratio $\tilde{u}_1(\Lambda)/\tilde{v}_1(\Lambda)$ [5].

11 Exact FRG equation in the single Fourier mode approximation beyond LPA

In this section I apply the FRG method for SG type models beyond LPA [18, 2, 3, 22, 10, 11, 25, 194]. It was shown in [3] that the Polchinski RG equation in LPA' is not suitable to recover the known dilute gas RG equations obtained for the 2D Coulomb-gas (CG) which is equivalent to the 2D-SG model. Similarly, the Wegner-Houghton RG equation is also problematic beyond LPA: although flow equations can be obtained at LPA' [2] but it confronts with the gradient expansion, so, it is not a good choice beyond LPA. Thus, I use the Wetterich RG equation in this section.

11.1 The SG model in arbitrary dimension

Let me first discuss the SG model. In order to solve the FRG equation in LPA', I consider the following ansatz for the SG model [18]

$$\Gamma_k[\varphi] = \int d^dx \left[\frac{1}{2}z_k(\partial_\mu\varphi)^2 + V_k(\varphi)\right], \quad V_k(\varphi) = u_k\cos(\beta\varphi) \tag{364}$$

where the local potential contains a single Fourier mode. The FRG equations at LPA' level have been discussed previously and they are given by Eq. (263). Inserting the ansatz (364) into Eqs. (264), (265), flow equations for the dimensionful couplings can be derived [18]

$$k\partial_k u_k = \int_p \frac{k\partial_k R_k}{\beta^2 u_k}\left(\frac{P-\sqrt{P^2-(\beta^2u_k)^2}}{\sqrt{P^2-(\beta^2u_k)^2}}\right), \tag{365}$$

$$k\partial_k z_k = \int_p \frac{\beta^2\, k\partial_k R_k}{2}\left[\frac{-(\beta^2u_k)^2P(\partial_{p^2}P+\frac{2}{d}p^2\partial^2_{p^2}P)}{[P^2-(\beta^2u_k)^2]^{5/2}} + \frac{(\beta^2u_k)^2p^2(\partial_{p^2}P)^2(4P^2+(\beta^2u_k)^2)}{d\,[P^2-(\beta^2u_k)^2]^{7/2}}\right] \tag{366}$$

where $P = z_kp^2+R_k$ and $\int_p = \int dp\, p^{d-1}\Omega_d/(2\pi)^d$ with the d-dimensional solid angle Ω_d. Since the dimensionful frequency is scale-independent, it is convenient to merge it with the scale-dependent wave function renormalization z_k. Thus, I introduce $\hat{z}_k = z_k/\beta^2$, $\hat{R}_k = R_k/\beta^2$ and $\hat{P} = P/\beta^2 = \hat{z}_kp^2+\hat{R}_k$ and the RG flow equations (365) and (366) can be written as [18],

$$k\partial_k u_k = \int_p \frac{k\partial_k\hat{R}_k}{u_k}\left(\frac{\hat{P}-\sqrt{\hat{P}^2-u_k^2}}{\sqrt{\hat{P}^2-u_k^2}}\right), \tag{367}$$

$$k\partial_k \hat{z}_k = \int_p \frac{k\partial_k\hat{R}_k}{2}\left[\frac{-u_k^2\hat{P}(\partial_{p^2}\hat{P}+\frac{2}{d}p^2\partial^2_{p^2}\hat{P})}{[\hat{P}^2-u_k^2]^{5/2}} + \frac{u_k^2p^2(\partial_{p^2}\hat{P})^2(4\hat{P}^2+u_k^2)}{d\,[\hat{P}^2-u_k^2]^{7/2}}\right], \tag{368}$$

Before I further study the RG flow equations, I show that they can be derived by using the rescaled version of the original action for the SG model [18],

$$\Gamma_k[\theta] = \int d^dx \left[\frac{1}{2}\hat{z}_k(\partial_\mu\theta_x)^2 + u_k\cos(\theta_x)\right], \tag{369}$$

where the rescaled (dimensionless) field $\theta = \beta\varphi$ is introduced. Let me note, that the field carries a dimension for $d \neq 2$ thus the frequency of the SG model (39) becomes a dimensionful parameter for $d \neq 2$, i.e., $\beta^2 = k^{2-d}\tilde{\beta}_k^2$ where $\tilde{\beta}_k$ is dimensionless, so the rescaled wavefunction renormalization has a dimension of k^{d-2}, i.e., $\hat{z}_k = \tilde{z}_kk^{d-2}$. The corresponding FRG equation reads as [18]

$$k\partial_k\Gamma_k[\theta] = \frac{1}{2}\mathrm{Tr}\left[\frac{k\partial_kR_k}{R_k+\beta^2\delta^2_\theta\Gamma_k[\theta]}\right] = \frac{1}{2}\mathrm{Tr}\left[\frac{k\partial_k\hat{R}_k}{\hat{R}_k+\delta^2_\theta\Gamma_k[\theta]}\right] \tag{370}$$

where the rescaled regulator function $\hat{R}_k = R_k/\beta^2$ has been used. Inserting the ansatz (369) into Eqs. (370), the RG flow equations (367) and (368) can be obtained Thus, equations (367) and (368) are derived in two different ways.

Momentum integrals have to be performed numerically, except the linearized form of Eqs. (367), (368) around the Gaussian fixed point where analytical results are available. This requires a special choice for the regulator function R_k such as the power-law [179] or the Litim-type [184] ones. In general, the regulator function beyond LPA should be given by the inclusion (multiplicative approach) or the exclusion (additive approach) of the field independent wavefunction renormalization z_k,

$$\text{multiplicative}: \quad R_k(p) = z_k\, p^2\, r(p^2/k^2), \qquad \text{additive}: \quad R_k(p) = p^2\, r(p^2/k^2), \tag{371}$$

In the multiplicative approach, the rescaled regulator $\hat{R}_k$ contains the rescaled wavefunction renormalization $\hat{z}_k$. In the additive approach, the frequency can be absorbed by the overall multiplicative constant of the rescaled regulator or can be chosen arbitrarily since it is a scale-independent free parameter of the model. Important to note, that the additive approach requires the use of the power-law regulator function.

The phase structure should be independent whether one uses the multiplicative or additive approaches for the definition of the regulator and of its parameters such as b. Regarding the regulator-dependence I note that the linearised RG flow equations can always be obtained analytically for the power-law type regulator. The full RG flow (with higher harmonics) requires a numerical treatment anyway (even for the Litim-type regulator) and produces us a complete picture of the phase diagram.

Analytic solutions are always available for the power-law type regulator if one considers the approximated flow equation where the exact RG equation (367) and (368) are expanded in Taylor series with respect to u_k around zero where the dimensionless couplings $\tilde{u}_k = k^{-d} u_k$, and $\tilde{z}_k = k^{2-d}\hat{z}_k$ are introduced. In the additive approach the leading order flow equations have the following forms for $d = 1$ dimension [18]

$$(1 + k\partial_k)\tilde{u}_k = \frac{1}{4b\sin(\frac{\pi}{2b})}\tilde{z}_k^{\frac{1}{2b}-1}\tilde{u}_k + \mathcal{O}(\tilde{u}_k^2), \tag{372}$$

$$(-1 + k\partial_k)\tilde{z}_k = -\frac{c_1(b)}{4\pi}\tilde{z}_k^{\frac{5}{2b}-2}\tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3), \tag{373}$$

with $c_1(b) = \frac{5\pi(55+4b(5b-18))}{48b^2\sin[5\pi/(2b)]}$. Eqs. (372), (373) have a non-trivial fixed point at $\tilde{z}_\star = [4b\sin(\pi/(2b)]^{2b/(1-2b)}$, $\tilde{u}_\star^2 = (4\pi/c_1)\tilde{z}_\star^{3-5/(2b)}$. For $d = 2$, leading order RG flow equations are [18]

$$(2 + k\partial_k)\tilde{u}_k = \frac{1}{4\pi}\tilde{z}_k^{-1}\tilde{u}_k + \mathcal{O}(\tilde{u}_k^2) \tag{374}$$

$$k\partial_k\tilde{z}_k = -\frac{c_2(b)}{8\pi}\tilde{z}_k^{\frac{2}{b}-2}\tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3) \tag{375}$$

with $c_2(b) = \frac{2\pi(b-2)(b-1)}{3b^2\sin[2\pi/b]}$ and result in a KTB type (i.e. infinite order) phase transition with $\tilde{z}_\star = 1/(8\pi)$. Important to note that $c_2(b) > 0$ for $b > 1$. Finally, for $d = 3$, the leading order RG flow equations are [18]

$$(3 + k\partial_k)\tilde{u}_k = \frac{1}{8\pi b\sin(\frac{\pi}{2b})}\tilde{z}_k^{-\frac{1}{2b}-1}\tilde{u}_k + \mathcal{O}(\tilde{u}_k^2) \tag{376}$$

$$(1 + k\partial_k)\tilde{z}_k = -\frac{c_3(b)}{8\pi^2}\tilde{z}_k^{\frac{3}{2b}-2}\tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3) \tag{377}$$

with $c_3(b) = \frac{\pi(3+4b(b-2))}{16b^2\sin[3\pi/(2b)]}$. Due to the tree-level scaling of $\tilde{z}_k$, the non-trivial fixed point appears for $d < 2$ and disappears for $d > 2$ in the RG flow. However, a "turning point" can be identified for $d > 2$ where the irrelevant coupling $\tilde{u}_k$ turns to a relevant one. For $d = 3$ the turning point is at $\tilde{z}_\star = [24\pi b\sin(\pi/(2b)]^{-2b/(1+2b)}$.

In the multiplicative approach where the definition of the regulator R_k includes z_k beyond LPA, the linearized RG flow equations obtained for the dimensionless couplings are almost identical to those obtained by the additive case. The prefactors of the r.h.s of the flow equations and the power

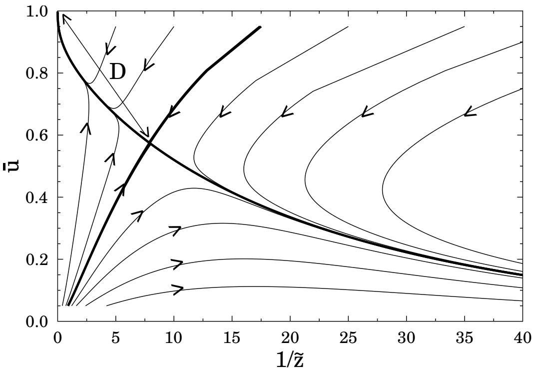

Figure 35: The exact phase diagram of the SG and the equivalent CG models for $d = 1$ dimensions (similar RG flow can be drawn for $d < 2$). Arrows indicate the direction of the flow [26].

of the Fourier amplitude $\tilde{u}$ are identical in the multiplicative and additive cases. The difference is due to the power of the dimensionless wavefunction renormalization $\tilde{z}_k$. In the multiplicative case, the r.h.s of the flow equations of the Fourier amplitude contain $\tilde{z}_k^{-1}$ and the flow equations of the wavefunction renormalization have $\tilde{z}_k^{-2}$ independently of choice of the dimension d and the parameter b. Since the flow equations of the multiplicative and additive approaches should give the same phase structure, here I focus on the more complex additive case [18].

In general, RG equations (365) and (366) (exact for a single Fourier mode) have to be solved numerically. An important feature of the exact RG flow is the emergence of a new low-energy/infrared (IR) fixed point related to the degeneracy of the blocked action. Namely, Eqs. (365), (366) become singular at the momentum scale where $\bar{k} - k^{2-d}u_k = 0$ with $\bar{k} = \min_{p^2} P = bk^2[\tilde{z}_k/(b-1)]^{1-1/b}$. Therefore, it is convenient to redefine the dimensionless coupling constant as $\bar{u}_k \equiv k^{2-d}u_k/\bar{k} = k^2\tilde{u}/\bar{k}$ which tends to one in case of degeneracy. Exact RG equations (365), (366) were solved for $b = 2$ and flow diagrams are plotted in Fig. 35 for $d = 1$, in Fig. 36 for $d = 2$, and in Fig. 37 for $d = 3$ dimensions. The IR (symmetry breaking) fixed point ($\bar{u}_{\mathrm{IR}} = 1$, $1/\tilde{z}_{\mathrm{IR}} = 0$) which corresponds to

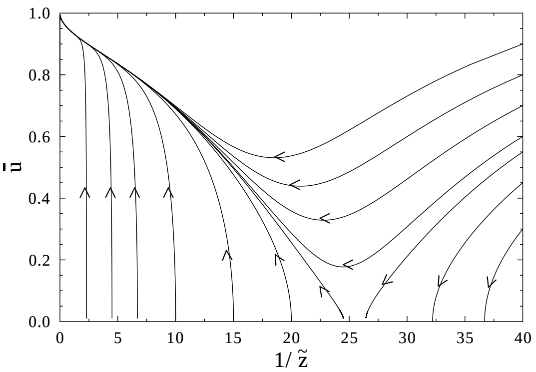

Figure 36: The exact phase diagram of the SG and the equivalent CG models for $d = 2$ dimensions which shows the Kosterlitz-Thouless-Berezinski type phase transition [18].

degeneracy was found in any dimensions. Let me emphasize that only the exact RG flow with the inclusion of the wave function renormalization is suitable for the determination of the IR (symmetry breaking) fixed point since the perturbative (truncated) RG equations are non-singular. For $d = 2$ dimensions the critical value $1/\tilde{z}_c = \beta_c^2 = 8\pi$ which separates the phases of the model (see Fig. 36) is found to be scheme-independent [18].

The $d = 1$ case needs further improvement because in quantum mechanics spontaneous symmetry breaking is not allowed [26]. Indeed, for the $d = 1$ dimension, based on the RG flow equations

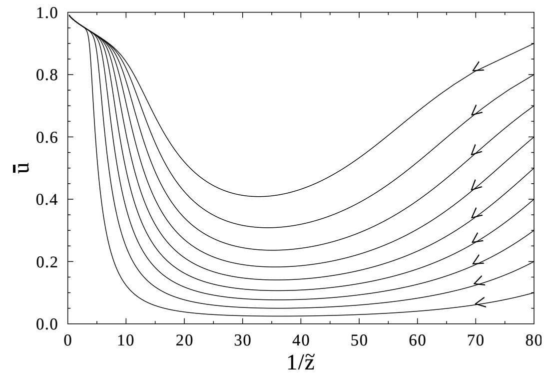

Figure 37: The exact phase diagram of the SG and the equivalent CG models for $d = 3$ dimensions which indicates a single phase for $d > 2$ [18].

(367) and (368), a saddle point $\bar{u}_\star$, $1/\bar{z}_\star$ appears in the RG flow, see Fig. 35 and thus the SG model seems to have two phases [26]. In fractal dimensions, $1 < d < 2$ the nontrivial saddle point appears in the RG flow, too [26]. However, there is an important difference between the cases of fractal dimensions and of the $d = 1$ dimension; namely, the spontaneously broken phase should vanish for $d = 1$ which indicates that the saddle point and the nontrivial IR fixed point ($1/\bar{z}_{\rm IR} \equiv 0$, $\bar{u}_{\rm IR} \equiv 1$) should coincide. Thus, the distance between the nontrivial IR fixed point and the saddle point (see Fig. 35) [26],

$$D \equiv \sqrt{(\bar{u}_{\rm IR} - \bar{u}_\star)^2 + (1/\bar{z}_{\rm IR} - 1/\bar{z}_\star)^2} = \sqrt{(1 - \bar{u}_\star)^2 + 1/\bar{z}_\star^2} \tag{378}$$

can be used to optimize the scheme dependence of RG equations; i.e., the better the RG scheme the smaller the distance D is. This strategy is used in Ref. [26] for the very general form of the CSS regulator and its Litim limit was found to be the optimal choice in LPA'. The other attractive IR fixed point ($\bar{u}_{k\to 0} = 0$, $1/\bar{z}_{k\to 0} = \infty$) corresponds to the symmetric phase. Let me note, connection between spontaneous symmetry breaking and truncation of the FRG equation has been studied for $O(N)$ models in Ref.[26].

11.1.1 Phase structure of the neutral CG in arbitrary dimension

The CG model has been the subject of an intense study in last decades [204, 67, 205, 206, 207] and there is a continuous interest in the use of the SG representation of CG systems [205, 206, 207]. Indeed, since the mapping between the CG and SG models holds in arbitrary dimension [207] (and it is exact in case of point-like charges) the RG study of the d-dimensional SG model can be directly used to map out the phase structure of the CG model [18]. In the framework of the real (or coordinate) space RG approach one can use the dilute gas approximation for the CG model which is equivalent to the low fugacity, i.e., small Fourier amplitude limit of the SG model. In this RG approach the charges (vortices) are considered as rigid discs with finite diameter, so, this can be seen as sharp cutoff version performed in the coordinate space which corresponds to a smooth cutoff version in the momentum space. The approximated RG equations of the CG model in arbitrary dimension is given in [67] and reads as

$$\frac{dx}{dl} = -x(xy^2 + d - 2), \quad \frac{dy}{dl} = -y(x - d) \tag{379}$$

with $\partial_l = -k\partial_k$, $y \sim \tilde{u}_k$ and $x \sim 1/\tilde{T} \sim 1/\tilde{z}_k$ where $\tilde{T}$ is the temperature and y is the fugacity. Using these identities Eq. (379) can be rewritten as

$$((d - 2) + k\partial_k)\tilde{z}_k = -c_z \tilde{u}_k^2, \quad (d + k\partial_k)\tilde{u}_k = c_u \frac{1}{\tilde{z}_k} \tilde{u}_k, \tag{380}$$

with constants c_u, c_z and they are found to be similar to the leading order RG flow equations obtained for the SG model, see Eqs. (372),(373), Eqs. (374),(375) and Eqs. (376),(375). For example, the non-trivial fixed point of the 1-dimensional SG model can be identified in the flow generated by (380), too.

It is clear that in $d = 2$ dimensions, the SG flow equations of the additive case (374) and (375) in the limit $b \to 1$ have identical functional form to (380). Since the critical value $1/\hat{z}_c = 8\pi$ is independent of the actual choice of the regulator, one has to find $c_u = 1/(4\pi)$. By the rescaling of the Fourier amplitude, the value of the other constant c_z can be chosen to be identical to $c(b = 1)/(8\pi) = 1/(24\pi)$ which results in exactly identical flow equations to the additive power-law case with $b = 1$. Indeed, the power-law regulator is a smooth cutoff in the momentum scape RG and this is found to be identical to the sharp-cutoff of the real space RG approach as expected [18, 194].

In arbitrary dimension one has to take multiplicative case or the sharp cutoff limit ($b \to \infty$) of the additive case in order to find agreement with the second equation of (380) where the leading order flow equation for the Fourier amplitude is $(d + k\partial_k)\tilde{u}_k = \Omega_d \tilde{u}_k/(\tilde{z}_k 2(2\pi)^d)$. However, to recover the first equation of (380), one has to choose a finite value for the parameter b and this is possible only for the additive case [18, 194].

The exact RG flow signals the existence of the high temperature ($\bar{u}_\star = 1$, $1/\tilde{z}_\star = 0$) and the show the absence of new further non-trivial fixed points. Since the mapping between the SG and CG models is exact for point-like charges the exact RG flow indicates a single phase for the CG for $d > 2$.

11.1.2 Checking the results

RG equations can be obtained either directly to the SG model or indirectly to its equivalent CG and XY spin model representations. As I argued, the mapping between the CG and SG models holds in arbitrary dimension [207] (and it is exact in case of point-like charges). Thus, the RG study of the SG model can be directly used to map out the phase structure of the CG model and vice versa.

Let me first rewrite the leading order RG flow equations (372), (373), (374), (375) and (376), (377) in a single form valid in arbitrary dimensions (for the additive definition of the power-law regulator) [18]

$$(d + k\partial_k)\tilde{u}_k = A_{b,d}\, \tilde{z}_k^{\frac{(2-d)-2b}{2b}}\, \tilde{u}_k + \mathcal{O}(\tilde{u}_k^2) \tag{381}$$

$$((d-2) + k\partial_k)\tilde{z}_k = -B_{b,d}\, \tilde{z}_k^{\frac{(6-d)-4b}{2b}}\, \tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3) \tag{382}$$

where $A_{b,d}$ and $B_{b,d}$ are constants depending on the dimension d and the regulator parameter b. Let me use the frequency instead of the wave function renormalization ($\tilde{\beta}^2 = 1/\tilde{z}$), where the additive case reads as [18]

$$k\partial_k \tilde{u}_k = \left[A_{b,d}\,(\tilde{\beta}_k^2)^{\frac{2b-(2-d)}{2b}} - d\right]\tilde{u} \tag{383}$$

$$k\partial_k \tilde{\beta}_k^2 = \tilde{\beta}_k^2\left[(d-2) + B_{b,d}\,\tilde{u}_k^2(\tilde{\beta}_k^2)^{\frac{6b-(6-d)}{2b}}\right]. \tag{384}$$

From Eqs. (383) and (384) one can take the limit sharp cutoff limit $b \to \infty$ and the additive and multiplicative cases (371) can be compared to each other

$$\text{additive, b} \to \infty: \quad k\partial_k \tilde{u}_k = \left[A_{\infty,d}\,\tilde{\beta}_k^2 - d\right]\tilde{u}, \quad k\partial_k \tilde{\beta}_k^2 = \tilde{\beta}_k^2\left[(d-2) + B_{\infty,d}\,\tilde{u}_k^2\tilde{\beta}_k^6\right] \tag{385}$$

$$\text{multiplicative}: \quad k\partial_k \tilde{u}_k = \left[A_{b,d}\,\tilde{\beta}_k^2 - d\right]\tilde{u}, \quad k\partial_k \tilde{\beta}_k^2 = \tilde{\beta}_k^2\left[(d-2) + B_{b,d}\,\tilde{u}_k^2\tilde{\beta}_k^6\right]. \tag{386}$$

which have the same functional but the coefficients are different. For the multiplicative case these coefficients are well-defined (and depend on the parameter b and d) but for the additive case the coefficient $B_{\infty,d}$ cannot be defined unambigously, since the sharp cutoff confronts to the derivative expansion.

Let me first compare Eqs. (385) and (386) to the RG equations (3.2.8) and (3.2.9) of [208] which is obtained in the dilute gas approximation and reads

$$k\partial_k \tilde{u}_k = \left[\frac{K_d}{2}\,\tilde{\beta}_k^2 - d\right]\tilde{u}, \quad k\partial_k \tilde{\beta}_k^2 = \tilde{\beta}_k^2\left[(d-2) + \left(\frac{I_1 K_d}{2} + B\right)\frac{\tilde{u}_k^2}{4}\tilde{\beta}_k^6\right]. \tag{387}$$

by using the following identifications $2z \equiv \tilde{u}$, $\alpha \equiv \beta$ and $\partial_l \equiv -k\partial_k$ where K_d, I_1 and B are constants. It is clearly demonstrated that Eqs. (385) and (386) are identical to (387) up to some constant but the coefficient $B_{\infty,d}$ of the additive case (385) cannot be defined unambigously. So, the multiplicative power-law gives better result then the additive one and again one can conclude that the sharp-cutoff real space RG corresponds to a smooth cutoff regulator of the momentum RG [18, 194].

As a next step, let me consider the flow equations above (45) in [209] which are obtained directly for the SG model in $d = 2$ dimensions using the Wilson-Kadanoff blocking relation up to leading order terms and reads as

$$k\partial_k \tilde{u}_k = \left(\frac{\tilde{\beta}_k^2}{4\pi} - 2\right)\tilde{u} \tag{388}$$

$$k\partial_k \tilde{\beta}_k^2 = \frac{3\tilde{\beta}_k^6 \tilde{u}_k^2}{4\pi\Lambda^3} \tag{389}$$

where the identifications $g \equiv \tilde{u}$ and $\partial_l \equiv -k\partial_k$ are used. One finds agreement between Eqs. (383) (384) and Eqs. (388) (389) since $A_{b,2} = 1/(4\pi)$ and $b = 2$ is chosen [18, 194]..

However, in [209] the same non-perturbative (Wilson-Kadanoff) RG method was used, so, all what one can conclude is that the choice for the regulator functions is identified. Thus, if one wants to find connections between renormalization schemes and regulators a better choice is Ref. [33] because there one finds a standard perturbative RG approach for the SG model with the following perturbative RG flow equations, see Eqs. (159) and (160) in [33]

$$k\partial_k \tilde{u}_k = \tilde{u}_k\left(\frac{\beta_k^2}{4\pi} - 2\right) - \tilde{u}_k^3\frac{\beta_k^4}{32\pi}, \quad k\partial_k \beta_k^2 = \tilde{u}_k^2\frac{\beta_k^6}{32\pi}, \tag{390}$$

where $\tilde{\alpha}_k = \tilde{u}_k\beta_k^2$ and $t_k = \beta_k^2$. These are identical (keeping only the leading order terms) to the additive power-law flow equations Eqs. (383), (384) with $b = 2$. Of course, one can use again a scheme transformation to relate the FRG flow equations with arbitrary choice of the regulator to the perturbative RG flow equations. However, in this case the transformation is not trivial since one has to rescale the Fourier amplitude by the frequency, so, practically one should consider different models to be able to relate the FRG and perturbative RG β functions. It is more reasonable to fix the model and look for a particular choice of the regulator by which the FRG flow equations reproduce the functional form of the perturbative RG β functions. Based on this logic, the additive power-law flow equations Eqs. (383), (384) with $b = 2$ are the best choice [194].

11.1.3 Isotropic classical XY spin model in $d = 3$

The partition function of the d-dimensional isotropic classical XY spin model can be expressed in terms of topological excitations of the original degrees of freedom. For $d = 2$ dimensions, the dual theory is the vortex gas which is known to belong to the class of universality of the neutral CG [210, 211, 214]. Two-dimensional generalized models are well known where both the CG and the vortex gas are included as particular limiting cases [210, 211, 214] and are self-dual under the duality transformation. For $d = 3$, the dual theory is the gas of interacting vortex loops [212, 213] (i.e. the lattice CG [214]). Corresponding flow equations have been derived for the parameters K (i.e. the coupling between the spins) and y (i.e. the fugacity of the vortex loops) by real-space RG method ($a \sim 1/k$ where a is the running cutoff in the coordinate space while k is the running momentum cutoff) in the limit of low fugacity [212, 213],

$$a\partial_a K = K - \frac{4\pi^3}{3}K^2 y^2, \quad a\partial_a y = \left(6 - \pi^2 KL\right) y, \tag{391}$$

where L approaches a constant in the IR limit ($L \to 1$) [212], or it is weakly divergent ($L \to \ln(a/a_c) + 1$) [213]. Since $a\partial_a = -k\partial_k$ and by using the identities $y \equiv \tilde{u}$, $K \equiv 1/\tilde{z}$, RG equations of (391) are rewritten as [18].

$$k\partial_k \tilde{z}_k = \tilde{z}_k - \frac{4\pi^3}{3}\tilde{u}_k^2, \quad k\partial_k \tilde{u}_k = 2\left(\frac{1}{\tilde{z}_k}\frac{\pi^2 L}{2} - 3\right)\tilde{u}_k, \tag{392}$$

which can be compared to the linearized RG obtained for the SG (376), (377) and the CG (380) models in $d=3$ dimensions. The only qualitative disagreement is the sign of the tree-level scaling term of $\tilde{z}_k$ but it has important consequences since the RG flow of the vortex-loop gas has a non-trivial fixed point which is absent in case of the 3-dimensional SG and CG models. Therefore, it demonstrates that the vortex-loop gas has a different scaling behaviour, thus it belongs to a class of universality different from that of the SG and CG models for $d \neq 2$. For $d=2$, the couplings $K, \tilde{z}$ have no tree-level scaling thus SG, CG, XY models are in the same universality class. Let me emphasize that RG equations are compared at the linearized level, however the exact RG study of the SG and CG models is required since it shows the absence of new non-trivial fixed points, thus there is no room to find a mapping between the parameters of the vortex-loop gas and the 3-dimensional CG which could produce the same phase structure [18]..

11.1.4 Central charges of the SG model in $d=2$

Let me finish the RG study of the SG model at LPA' by the discussion of its conformal properties and their connection to RG flow equations [10]. In $d=2$ dimensions, by using the mass cutoff, i.e. power-law type regulator with $b=1$, the momentum integrals of (367) and (368) can be performed and the RG equations reads as [10],

$$(2+k\partial_k)\tilde{u}_k = \frac{1}{2\pi z_k \tilde{u}_k}\left[1-\sqrt{1-\tilde{u}_k^2}\right], \qquad k\partial_k z_k = -\frac{1}{24\pi}\frac{\tilde{u}_k^2}{[1-\tilde{u}_k^2]^{\frac{3}{2}}} \tag{393}$$

with the dimensionless coupling $\tilde{u} = k^{-2}u$. The phase diagram obtained at this approximation level is sketched in Fig. 38 and obtained by power-law type regulator with $b=2$, where one evidences three different regions, [10].

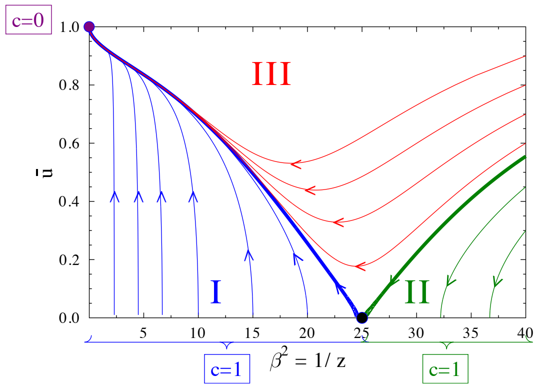

Figure 38: The flow diagram of the SG model in the scale-dependent frequency approximation. The phase space is divided into three regions [10]. In region I we have a line of UV repulsive Gaussian fixed points ($\bar{u}=0, \beta^2 < 8\pi$). Every trajectory starting in the vicinity of this line ends in an IR attractive fixed point (purple full circle, $\bar{u}=1, \beta^2=0$). The Δc observed along the trajectories of this region should be equal to 1. Region II contains a line of IR attractive Gaussian fixed points ($\bar{u}=0, \beta^2 > 8\pi$,) which are the end points of trajectories starting at $\beta^2 \approx \infty$ below the thick green line, i.e. the separatrix. Region III contains those trajectories starting at $\beta^2 \approx \infty$ which end in the IR attractive fixed point (purple full circle).

In Ref. [195] an explicit expression for the flow equation of the c-function in the LPA scheme has been derived with the mass cutoff

$$k\partial_k c_k = \frac{[k\partial_k \tilde{V}_k''(\varphi_{0,k})]^2}{[1+\tilde{V}_k''(\varphi_{0,k})]^3}, \tag{394}$$

with the dimensionless blocked potential $\tilde{V}_k(\varphi)$ which is evaluated at its running minimum $\varphi = \varphi_{0,k}$ (i.e. the solution of $\tilde{V}_k'(\varphi) = 0$). Based on (394) a detailed study of the c-function of the SG model in the framework of FRG method is discussed in [10]. The Δc is strictly well defined only in region I, where I start from a Gaussian fixed point $c_{UV} = 1$ and I end up on a massive IR fixed point $c_{IR} = 0$. The massive IR fixed point related to the degeneracy of the blocked action is an important feature of the exact RG flow. In region II the trajectories end in the Gaussian fixed points $c = 1$ but they are coming from infinity where actually no fixed point is present. Thus, Δc is not defined in this region. Region III contains those trajectories which start at $\beta = \infty$ but end in the IR massive fixed point at $c = 0$. Even in this case the Δc is not well defined.

11.2 The ShG model in $d = 2$

In this subsection I consider the phase structure and conformal properties of the ShG model in $d = 2$ dimensions by using the FRG method in LPA' [11]. It is important to note that (56) has a Z_2 symmetry, and that the ShG model is not periodic. Therefore, in order to study the RG flow of the ShG model and to map out its phase structure one can use the Taylor-expanded form of Eq. (56)

$$\tilde{V}_k(\varphi) = \tilde{u}_k \left[1 + \frac{1}{2}\beta^2\varphi^2 + \frac{1}{4!}\beta^4\varphi^4 + ...\right] = \sum_{n=0}^{\infty} \frac{1}{(2n)!} g_{2n}\varphi^{2n}, \qquad g_{2n} = \tilde{u}_k\beta^{2n}. \tag{395}$$

Thus, the ShG model can be considered as an Ising-type model but with restricted initial values for the couplings. The key point is that with ShG-type initial values the RG flow *always* starts from the symmetric phase, see Fig. 19, so, the ShG model has a single phase with $\Delta c = 1$.

The phase structure of the ShG model can also be mapped out by using analytic continuation. The simplest way of doing that if one replaces the frequency of the SG model by an imaginary one directly which leads to the following RG flow equations [11]

$$(2 + k\partial_k)\tilde{u}_k = -\frac{\beta^2}{2\pi\tilde{u}_k}\left[1 - \sqrt{1 - \tilde{u}_k^2}\right] \tag{396}$$

$$k\partial_k\beta_k^2 = -\frac{1}{24\pi}\frac{\beta_k^4\tilde{u}_k^2}{[1 - \tilde{u}_k^2]^{\frac{3}{2}}}. \tag{397}$$

Important to note that the RG flow equations of the SG model contains β^2 and β^4. This means that one can apply the following replacement $\beta^2 \to -\beta^2$ in the flow equations of the SG theory in order to obtain the flow equations for the ShG model. Indeed, the flow diagram of the ShG can be represented by extending the SG flow diagram for negative values of the frequency β^2, see Fig. 39, which can be compared to figure 1 of Ref. [209]. There is a disagreement between the two figures,

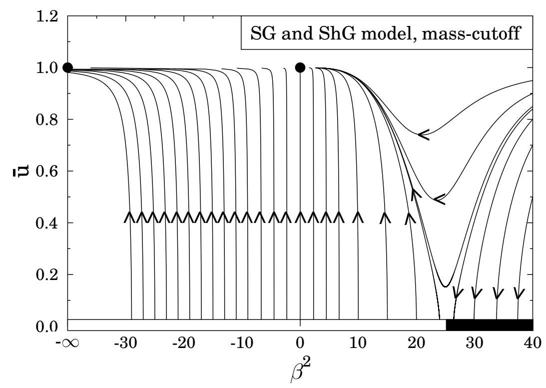

Figure 39: Phase structure of the SG model for regions of positive and negative β^2, i.e., the RG flow diagram of the SG and ShG models are merged into a single one [11]. Black circles denote the IR attractive fixed points.

namely in [209] the RG trajectories of the negative β^2 regime run into the IR (symmetry breaking) fixed point of the SG model which signals the presence of spontaneous symmetry breaking (SSB). At variance, we argued here that the ShG model has no SSB, since it has a single phase which is the symmetric one. Moreover, the flow diagram plotted in figure 1 of [209] suggests that the negative and positive β^2 regions are basically reflected to each other, implying in turn the reflection of the critical value of the frequency ($\beta_c^2 = 8\pi$) too. However, it was also shown that no such critical frequency exists for the ShG model i.e., the negative β^2 case of the SG theory [11]. Therefore, I conclude that figure 1 of [209] is misleading.

11.3 The SnG model in $d = 2$

I am now in position to perform the functional RG study of the SnG model [11]. According to the previous discussion, it is based on the Fourier decomposition where the frequency b^2 of the fundamental mode plays a crucial role in the determination of the phase structure. Thus, beyond LPA, the SnG model can be treated the way as the SG model, so the RG equation has to be solved over the functional subspace spanned by the following ansatz [11]

$$\Gamma_k = \int d^2x \left[\frac{1}{2} z_k (\partial_\mu \varphi_x)^2 + V_k(\varphi_x)\right], \qquad V_k(\varphi) = -\sum_{n=1}^{\infty} u_n(k) \cos(n\,\varphi), \tag{398}$$

where the local potential contains infinitely many Fourier modes and I use the notation [11]

$$z \equiv \frac{1}{b^2} = \frac{\left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, m\right)\right]^2}{\beta^2} \tag{399}$$

via the rescaling of the field $\varphi \to \varphi/b$ in (61) and z_k again stands for the field-independent wave-function renormalization. It is important to note that m remains a non-scaling parameter even beyond LPA. By using the mass cutoff, i.e., the power-law type regulator with $b = 1$, the RG equations for the couplings of the SnG reads as [11],

$$(2 + k\partial_k)\tilde{u}_k = \frac{1}{2\pi z_k \tilde{u}_k}\left[1 - \sqrt{1 - \tilde{u}_k^2}\right], \qquad k\partial_k z_k = -\frac{1}{24\pi}\frac{\tilde{u}_k^2}{[1 - \tilde{u}_k^2]^{\frac{3}{2}}} \tag{400}$$

with the dimensionless coupling $\tilde{u} = k^{-2}u$ which is identical to the flow equations of the SG model but with the different definition for z. In order to compare the flow diagrams of the SnG and SG models it is convenient to use the squared frequency β^2 instead of the wave function renormalization z. Then, the flow diagram of the SnG model obtained in the single-Fourier approximation beyond LPA is shown in Fig. 40.

I finally comment on the limit $m \to 1$ of the SnG model [11]. I showed that the SnG model, being periodic, has a BKT transition in all points but for $m = 1$ where it reduces to the ShG model. Therefore, let me discuss whether the limit $m \to 1$ is analytic or not. Two facts that would support the analytic behaviour are following [11]: (i) the ShG as well as the SnG model with $m = 1$ show a single phase; (ii) this phase is the high-temperature one, where the Fourier amplitude is relevant. However, in favour of the fact that the limit $m \to 1$ is not analytic one can argue that (i) the frequency is relevant in the ShG model, but irrelevant in the $m \to 1$ limit; and (ii) the single phase of the ShG model is the symmetric one, but the $m \to 1$ limit suggests SSB. In order to clearly make a conclusion on the subtleties of the $m \to 1$ limit, one has to show a physical quantity which has different value at the two cases. To this purpose I propose the susceptibility of the topological charge

$$\chi = \langle Q^2 \rangle - \langle Q \rangle^2 \tag{401}$$

where Q is the winding number, see [1]. This serves as a disorder parameter, since the topological susceptibility is vanishing whenever the Fourier amplitude is zero. This quantity can be shown to be non-zero in the limit $m \to 1$ of the SnG model, but vanishing for the ShG theory. Therefore, I conclude that the limit $m \to 1$ is non analytic.

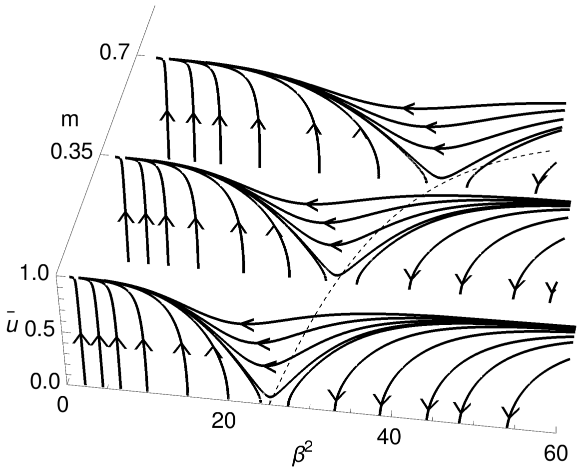

Figure 40: Phase structure of the SnG model for $m = 0, 0.35, 0.7$ [11]. The dashed line indicates the critical frequency $\beta_c^2(m)$ of the BKT phase transition.

11.4 The MSG model in $d = 2$

Let me study the MSG model beyond LPA where I use the following ansatz [22]

$$\Gamma_k = \int_x \left[\frac{1}{2} z(k)(\partial_\mu \theta)^2 + V_k(\theta)\right], \qquad V_k(\theta) = \frac{1}{2}\mathbf{M}^2(k)\ \theta^2 + u(k)\cos(\theta), \qquad \mathbf{M}^2 \equiv \frac{M^2}{\beta^2}, \quad (402)$$

where the field $\theta = \beta\varphi$ is rescaled in (41) and $z(k)$ stands for the field-independent wave-function renormalization. The flow equations for the MSG model are identical to those obtained for the rescaled SG model (for $d = 2$), i.e., (367) and (368) with the only modification, $\hat{P} = \hat{z}(k)p^2 + \mathbf{M}^2 + \hat{R}_k$ and with $\partial_k \mathbf{M} = 0$. By using the power-law type regulator function with $b = 1$, the RG flow equations for MSG model reads as [22],

$$(2 + k\partial_k)\tilde{u} = \frac{1}{2\pi z \tilde{u}}\left[1 + \tilde{\mathbf{M}}^2 - \sqrt{(1+\tilde{\mathbf{M}}^2)^2 - \tilde{u}^2}\right],$$
$$k\partial_k z = -\frac{1}{24\pi}\frac{\tilde{u}^2}{[(1+\tilde{\mathbf{M}}^2)^2 - \tilde{u}^2]^{\frac{3}{2}}}, \qquad (2 + k\partial_k)\tilde{\mathbf{M}}^2 = 0, \quad (403)$$

with dimensionless couplings $\tilde{u} = k^{-2}u$, $\tilde{\mathbf{M}}^2 = k^{-2}\mathbf{M}^2$. In the massless limit ($\tilde{\mathbf{M}} \to 0$), Eqs. (11.4) reduce to those derived for the SG model (393).

For nonvanishing mass ($\tilde{\mathbf{M}} \neq 0$), the linearized form of Eq. (11.4) is equivalent to the perturbative RG equations of [52, 87]. However, the perturbative RG flow produces arbitrary large ratio $u/M^2 = \tilde{u}/\tilde{M}^2$ depending on the initial conditions (there is no upper bound), hence no critical value can be determined. Consequently, the perturbative RG flow is not suitable for the prediction of the Ising-type phase transition of the MSG model [22].

So, let me first consider the exact RG flow equations for the MSG model in the mass cutoff scheme, (i.e., for the power-law regulator with $b = 1$) where RG flow equations are given by Eq. (11.4). SI appears in the RG flow in the symmetry broken phase, i.e. the RG equation becomes singular in the IR limit and the flow stops at some finite momentum scale. In order to eliminate (or suppress) the appearance of SI, one has to increase the convergence properties of the RG equations which can be done by choosing $b > 1$ for the power-law regulator. In this case the (exact) RG flow equations have no analytic form and they are solved numerically (see Fig. 41). Independently of the actual value of b, the potential was found to become degenerate in the broken symmetric phase and the RG flow is determined by the degeneracy condition [22]

$$c(b)z^{1-1/b} - \tilde{u} + \tilde{\mathbf{M}}^2 = 0 \quad (404)$$

where $c(b) = b/(b-1)^{1-1/b}$. Therefore, in the IR limit the ratio [22]

$$r_b(k) = \frac{\tilde{u}}{c(b)z^{1-1/b} + \tilde{\mathbf{M}}^2} \tag{405}$$

tends to one (i.e. $r_b(k \to 0) = 1$) in the broken phase (see the dashed lines in Fig. 41 for $b = 2$). Therefore, the ratio becomes universal in the broken phase. In the symmetric phase it tends to a constant IR value depending on the initial conditions. The critical ratio $r_b^c(k)$ which separates the phases of the single-frequency MSG model is represented by the thick solid line in Fig. 41 [22].

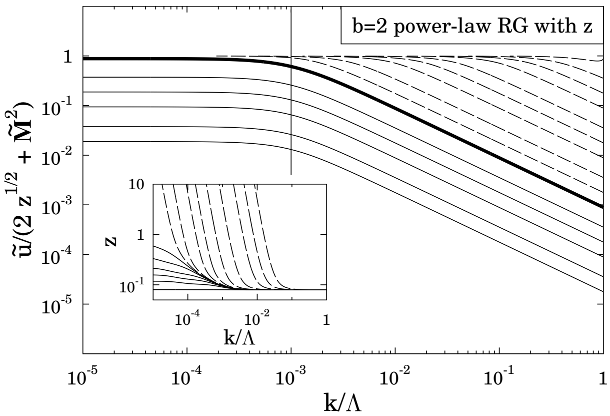

Figure 41: RG trajectories are obtained for the power-law regulator with $b = 2$ for the MSG model at LPA' [22]. Dashed lines correspond to RG trajectories of the broken symmetric phase. Vertical line shows the dimensionful mass scale which remains unchanged under RG transformations. The inset shows the scaling of the wave-function renormalization in the two phases.

Let me note that the RG flow always stops at a finite momentum scale in the broken phase independently of b but a better convergence is obtained for $b > 1$ [22]. In general, the single-mode approximation at LPA is "improved" by the inclusion of the wave-function renormalization. For example, the critical exponent ν of the MSG model can be obtained in the framework of the power-law RG with $b > 1$ if $z(k)$ is kept scale-dependent [22]. It is known [54] that the MSG model belongs to the two-dimensional Ising universality class, thus the correlation length is a power-law function of the reduced temperature $\xi \sim t^{-\nu}$ with $\nu = 1$. Indeed, if one defines [22] the correlation length in the symmetric (disordered) phase by the constant IR values of the ratio, $\xi \sim [1 - r_b(k \to 0)]^{-1}$ and the reduced temperature is given by the initial UV ($k = \Lambda$) values, $t = [r_b(\Lambda)^{-1} - r_b^c(\Lambda)^{-1}]/r_b^c(\Lambda)^{-1}$ then $\nu = 1$ is obtained (see solid lines in Fig. 41 for $b = 2$). Let me note that the correlation length can be defined as $\xi \sim (\mathbf{M} - k_c)^{-1}$ in the broken phase where k_c represents the momentum scale at which the ratio (405) becomes constant during the RG flow. If the reduced temperature is $t = [r_b(\Lambda) - r_b^c(\Lambda)]/r_b^c(\Lambda)$ then one obtains again the power-law behavior with $\nu = 1$. Consequently, the RG equations derived for the single-frequency MSG model beyond LPA are sufficient to indicate that the model undergoes a second order Ising-type phase transition [22].

Let me consider the RG evolution of the wave-function renormalization which is equivalent to the inverse frequency, i.e. $z(k) \equiv 1/\beta^2(k)$. In the symmetric phase $z(k)$ becomes a constant in the IR limit depending on the initial conditions (see the solid lines in the inset of Fig. 41). In the broken phase, however $z(k)$ runs into infinity for $k \to 0$ (see dashed lines in the inset of Fig. 41), i.e. it has a universal behavior in the broken phase thus $\beta(k)$ tends to zero. Therefore, if one assumes that bosonization identifications between the parameters of the fermionic and the corresponding bosonic theory hold also for the blocked action then my result has a drawback on bosonization, namely it indicates the necessity to construct the fermionic counterpart of the MSG model for $\beta^2 \neq 4\pi$ [22].

12 Summary

1. **The application of the FRG method for sine-Gordon models**

   The renormalization of sine–Gordon (SG) type models represents a challenge in quantum field theory since one has to use a method which retains the symmetry (periodicity) of the system. Consequently, the usual perturbative treatment which is based on the Taylor expansion of the potential does not work for SG type models, at least in $d > 2$ dimensions. The Functional Renormalization Group (FRG) approach is one of the examples which is suitable to perform the renormalization of such models non-perturbatively.

   1.1 The non-perturbative RG analysis of SG type models has been started with the work [1] where I developed a method based on the Fourier expansion of the periodic potential to perform the FRG study of SG type models and I have succeed in elucidating the phase structure of the 2-dimensional SG model within the framework of the FRG approach at the first time in the literature. I have argued that the periodicity and the convexity are so strong constraints on the dimensionful effective potential that it always becomes flat which was reproduced by integrating out the RG equation.

   1.2 I have shown [3] that the Polchinski RG equation obtained in the next-to-leading order of the gradient expansion (LPA') is not suitable to recover the known dilute gas RG equations obtained for the 2-dimensional Coulomb-gas (CG) which is equivalent to the 2-dimensional SG model. Similarly, the use of the Wegner-Houghton RG equation is also problematic: although flow equations can be obtained at LPA and LPA' [2] but it confronts with the gradient expansion, so, it is not a good choice beyond LPA.

   1.3 I have shown that linearised RG flow equations obtained in LPA are suitable to determine the critical value of the frequency parameter of SG type models and it is independent of the choice of the regulator function [4]. I have determined the low energy behaviour of the dimensionless effective potential of the SG model which can be used to distinguish between the phases of the model.

   1.4 I have shown that the critical value of the frequency parameter of SG type models is not influenced by the compactness of the field variable [5] and not affected by the reflection symmetry. In particular, I have studied the interplay between periodicity and the reflection symmetry with regard to the phase structure of an SG model where apart from the usual cosine one finds sine modes which break the reflection symmetry.

   1.5 I have generalised the FRG approach for layered sine-Gordon (LSG) type models [6, 7] and I have shown that the massive sine-Gordon (MSG) model which has an explicit mass term in addition to the periodic self-interaction, has no topological phase transition.

   1.6 I have shown that the breakdown of periodicity of the LSG model is only partial and it depends on the number of non-zero eigenvalues of the mass matrix which couples the SG fields [8] and showed that the critical frequency of the topological phase transition of the LSG model depends on the number of layers N.

2. **Sine-Gordon models in low-dimensions**

   The goal was to determine the low-energy behaviour and conformal properties of various 2-dimensional SG type field theoretical models including the massive sine-Gordon (MSG) and layered sine-Gordon (LSG) models which are bosonised version of the multi-flavor QED_2 and the multi-color QCD_2 and to construct an LSG type model which can be used to describe the vortex behavior of layered superconductors and study the vortex dynamics by the FRG study of this LSG theory.

   2.1 I have constructed an FRG approach combined with a suitable rotation of the fields which makes the mass matrix diagonal of the LSG model with N-layers and provides us the tool to investigate the phase structure of the bosonised QED_2 with N_f flavors and QCD_2 with N_c colors [9] and shown that the magnetically coupled LSG model with

N-layer is identical to the bosonised version of the QED_2 with N_f flavors where the number of flavors and layers are the same $N = N_f$

2.2 I studied the c-function and the central charge of the SG model from the FRG flow [10]. The integration of the c-function along trajectories of the FRG flow gives access to the central charges of the model in the fixed points. I have shown that the central charge obtained by integrating the trajectories starting from the repulsive low-frequencies fixed points (below the critical frequency) to the infrared limit is in good quantitative agreement with the expected $\Delta c = 1$ result.

2.3 I have introduced the so called sn-Gordon periodic model which interpolates between the sine– and the sinh–Gordon theories in $1+1$ dimensions. I have performed the FRG study of the model which was written in terms of Jacobi functions [11] and derived the critical frequency as a function of the elliptic modulus and discussed its conformal properties.

2.4 I have constructed the effective action [12] of the LSG model proposed previously in the corresponding literature with a Josephson type mass matrix and showed that it is not suitable to describe the vortex dynamics in Josephson-type superconductors [13].

2.5 I have constructed a 2-dimensional LSG model and mapped onto the corresponding gas of topological excitations and shown that with a special choice of its mass matrix, this magnetically coupled LSG model can be used to recover the well known interaction potentials of fractional flux vortices of magnetically coupled layered superconductors in an extremely simple manner [14]. This provided us a tool to study the vortex dynamics in magnetically coupled layered superconducting systems by means of the FRG method. I determined the dependence of the transition temperature on the number of layers for the magnetically coupled superconducting system by the FRG method from first principles at the first time in the literature which was found to be in agreement with known results based on other methods.

2.6 I studied the role of amplitude fluctuations in the lattice XY spin model and the equivalent ϕ^4 theory [15]. In the so called amplitude-phase representation, I have shown that amplitude fluctuations can be integrated out which yields an effective SG model which proves that amplitude fluctuations cannot affect the topological phase transition. In addition, I suggested a coupled XY model [16] where the Kosterlitz-Thouless-Berezinskii paired phase can be observed.

3. **Sine-Gordon models in higher dimensions**

It is a natural question to ask whether one can find any role of SG type models in higher dimensions. The answer can be found if one looks for special cases where scalar fields have applications such as Branon, Inflaton, Higgs and Axion physics.

3.1 I have shown that the SG model has a single phase for $d > 2$ dimensions [17] based on its FRG study at LPA. Since the neutral Coulomb-gas and the SG model can be mapped onto each other, they belong to the same universality class for arbitrary dimension which is not true for the XY spin model. By using the FRG method and the Villain-transformation, I argued that the SG scalar theory and the XY model have different phase structures for $d > 2$ dimensions.

3.2 By the FRG study of the SG model at LPA' for $d > 2$ dimensions [18], I have shown that the dimensionful periodic potential flattens out in the IR limit which confirmed previous results on the flattening of the axion potential. I determined the FRG running of the frequency parameter and used it to show that starting from a small value, it does not evolve along the corresponding RG trajectory, thus, it supports the viability of a periodic inflationary potential. I have shown that the dimensionless SG potential remains bounded from below which opened a new platform to periodic type Higgs potentials.

3.3 I have proposed the so-called Massive Natural Inflation (MNI) as a viable model for a pseudo-periodic Higgs-inflation [19]. This model is identical to the MSG scalar theory and I have shown by its slow-roll analysis that its theoretical predictions are in excellent agreement with the PLANCK data and by its FRG study, I argued that it can play a role in Higgs physics, too.

3.4 I have shown that pre-inflationary quantum fluctuations can provide a scenario for the long-sought initial conditions for the inflaton field [20]. I have proposed an RG running induced cosmic inflation where at very high energies the vacuum-expectation value of the field was trapped in a false vacuum and then, due to RG running, at low-energies the flattening of the potential allows the field to roll down to its true vacuum.

3.5 I have studied the renormalisation of non-differentiable potentials and shown that quantum fluctuations smoothen the bare singularity of the potential [21]. An example for such, non-differentiable potentials is the effective action for branons in the framework of the brane world scenario which contains a Liouville-type V-shaped interaction and, in principle, can be replaced by SG or MSG type models.

4. **Methodical issues of the FRG approach**

The FRG method is designed to perform renormalization non-perturbatively, however, it has its limitation, too. Approximations are required to solve this partial integro-differential equation and the solution of the approximated FRG equation depends on the choice of the regulator function which requires optimisation.

4.1 I have shown that known results on critical value of the $N_f = 1$ QED_2 can be used to study the dependence of FRG equations on the choice of the regulator function and approximations applied for its bosonized version [22] which is the MSG model. I have shown that the MSG model has an Ising-type phase transition with two phases where the reflection symmetry can be broken spontaneously.

4.2 I have introduced a new type of regulator function, i.e, the Compactly Supported Smooth (CSS) regulator [23]. It reduces to all major type of regulator functions (exponential, power-law, optimised, sharp) in appropriate limits of its parameters thus it provides us the tool to compare various regulators in the framework of the Principle of Minimum Sensitivity (PMS) optimisation method. In addition, it has derivatives of all orders, so, it solves the problem of differentiability in case of the Litim–Pawlowski optimization method, thus, one can consider the Litim limit at any order of the gradient expansion.

4.3 I have determined the optimised parameters of the CSS regulator by the PMS method for the Ising and MSG models [24] and I have found its Litim limit as the best choice.

4.4 I have suggested a new optimisation method based on the requirement for the absence of Spontaneous Symmetry Breaking (SSB) in the SG model for $d = 1$ dimensions [25]. It is based on the idea that for special cases where SSB is not allowed, like in the 1-dimensional SG model, the truncated FRG equation still signals the appearance of SSB, so, the best parameters of the CSS can be chosen to minimise this fake SSB phase. Again, the Litim limit of the CSS was found to be the optimal choice. In addition, I have studied various aspects of truncation effects in the FRG approach for SSB [26].

4.5 I have suggested a special subtraction method to handle the problem of UV divergent nature of the RG evolution of the field independent term which due to its construction, requires special care in the FRG approach [27]. In several instances, the constant term of the potential has no physical meaning, however, in low dimensions, this is associated with the ground-state energy and in higher dimensions it is identical to the vacuum energy term and it plays a role in cosmic inflation, too. The subtraction was needed if the Gaussian fixed point is missing in the FRG flow once the constant term is included and the suggested subtraction method was suitable to restore the Gaussian fixed point in these cases.

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