Lecture Note

Classical Mechanics I.

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Contents

1 Linear Momentum

Newton's laws of motion laid the foundation for classical mechanics. They describe the relationship between a body and the forces acting upon it, and its motion in response to those forces.

1.1 Newton's first law - The law of Inertia

- Let us consider the change in the state of motion of various objects.
- wrong question: Why do they move?
- good question: Why does the state of motion change?

Newton's 1st law: In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

- In non-inertial frames of reference, Newton's 1st law does not necessary hold.
- Inertial frame can be constructed by fixing the frame of reference with respect to distant stars.
- If an inertial frame exists then infinitely many exist.

Extended objects require special care.

- It is not true that in inertial frames, all points of an extended object either remain at rest or continue to move at a constant velocity, unless acted upon by a force.
- However, one can always associate at least one point (the so called inertial point) to an extended objects which either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
- velocity of an extended object is the velocity of its inertial point.

Postulate:

- inertial frame exists
- inertial point exists (in case of extended objects)

1.2 Experimental laws of two body interactions; inertial mass

Let us consider the planar motion, more precisely the collisions of two discs.

1. Our first observation is that the change in the velocities of the two object are anti-parallel.

$$\Delta \mathbf{v}_\mathrm{i} \upharpoonleft\downharpoonright \Delta \mathbf{v}_\mathrm{j}\,. \tag{1}$$

   where $\Delta \mathbf{v}_\mathrm{i} = \mathbf{v}''_\mathrm{i} - \mathbf{v}'_\mathrm{i}$ and $\Delta \mathbf{v}_\mathrm{j} = \mathbf{v}''_\mathrm{j} - \mathbf{v}'_\mathrm{j}$.

2. Colliding the same two objects one finds that the following quantity

$$\frac{|\Delta \mathbf{v}_\mathrm{j}|}{|\Delta \mathbf{v}_\mathrm{i}|} = C_\mathrm{ij}\,. \tag{2}$$

   remains constant, i.e., the value of C_ij is unchanged over the collisions.

3. In case of three different objects, their relative inertia C_ij, C_ik and C_jk are not independent.

$$C_\mathrm{ij} C_\mathrm{jk} = C_\mathrm{ik}\,, \quad \text{or} \qquad C_\mathrm{ij} = \frac{C_\mathrm{ik}}{C_\mathrm{jk}} \tag{3}$$

   Thus, the relation C_ij is transitive.

4. In case of two identical objects one finds $C_{\mathrm{ii}} = 1$. If one observes $C_{\mathrm{ij}}= 1$ than $C_{\mathrm{ji}} = 1$. Thus the relation C_{ij} is reflexive and symmetric.

Therefore, the relation C_{ji} is an equivalence relation. As a consequence of the reflexive, symmetric, and transitive properties, any equivalence relation provides a partition of the underlying set into disjoint equivalence classes. Two elements of the given set are equivalent to each other, if and only if they belong to the same equivalence class. This classification can be used to define the inertial mass,

$$\textbf{Definition}: \qquad m_{\mathrm{body}} = C_{\mathrm{body,ref}}\, m_{\mathrm{ref}}\,. \tag{4}$$

where we introduced a reference mass $m_{\mathrm{ref}} = 1\,\mathrm{kg}$.
By using the transitivity of the relative inertia C_{ji} one can derive the following expression for the inertial mass

$$C_{\mathrm{ij}} = \frac{C_{\mathrm{i,ref}}}{C_{\mathrm{j,ref}}} = \frac{m_{\mathrm{i}}}{m_{\mathrm{j}}}\,, \qquad \to \qquad \frac{|\Delta \mathbf{v}_{\mathrm{j}}|}{|\Delta \mathbf{v}_{\mathrm{i}}|} = \frac{m_{\mathrm{i}}}{m_{\mathrm{j}}} \qquad \to \qquad m_{\mathrm{i}}\Delta\mathbf{v}_{\mathrm{i}} = -m_{\mathrm{j}}\Delta\mathbf{v}_{\mathrm{j}}\,. \tag{5}$$

Let us note that the inertial mass is found to be an additive quantity in classical mechanics,

$$m_{\mathrm{A+B}} = m_{\mathrm{A}} + m_{\mathrm{B}}\,, \tag{6}$$

but this is not true in quantum mechanics.

1.3 Linear momentum; conservation of linear momentum

It is useful to introduce a new vector quantity $m\mathbf{v}$ which is the so called linear momentum

$$\textbf{Definition}: \qquad \mathbf{p} = m\mathbf{v}\,. \tag{7}$$

where $\dim p =$ M L T^{-1}, SI unit $[p]$ = kg·m/s. Thus, if one considers the case where the inertial masses of the objects do not change over the interactions, one finds

$$\Delta(m_{\mathrm{i}}\mathbf{v}_{\mathrm{i}}) = -\Delta(m_{\mathrm{j}}\mathbf{v}_{\mathrm{j}})\,. \qquad \to \qquad \Delta\mathbf{p}_{\mathrm{i}} + \Delta\mathbf{p}_{\mathrm{j}} = 0 \tag{8}$$

which indicates that **the vector sum of the linear momenta remains constant over the two-body interaction, i.e., the linear momentum is conserved**,

$$\mathbf{p}_{\mathrm{i}} + \mathbf{p}_{\mathrm{j}} = \text{constant.} \tag{9}$$

This can be generalised for many body interactions: the linear momentum is conserved for isolated (closed) systems

$$\sum_{\mathrm{i}} \mathbf{p}_{\mathrm{i}} \equiv \sum_{\mathrm{i}} m_{\mathrm{i}}\mathbf{v}_{\mathrm{i}} = \text{constant.} \tag{10}$$

1.4 Centre of mass - inertial point

Let us come back to the concept of the inertial point. One can show that the centre of mass is identical to the inertial point. In case of two bodies (A and B), the position vector of the centre of mass is defined as

$$\mathbf{r}_{\mathrm{c}} = \mathbf{r}_{\mathrm{A}} + (\mathbf{r}_{\mathrm{B}} - \mathbf{r}_{\mathrm{A}})\frac{m_A}{m_A + m_B} \qquad \to \qquad \mathbf{r}_{\mathrm{c}} = \frac{m_A\mathbf{r}_{\mathrm{A}} + m_B\mathbf{r}_{\mathrm{B}}}{m_A + m_B} \tag{11}$$

which can be generalised for N bodies

$$\mathbf{r}_{\mathrm{c}} = \frac{\sum_{i=1}^{N} m_i\mathbf{r}_{\mathrm{i}}}{\sum_{i=1}^{N} m_i}. \tag{12}$$

Its time derivative gives the velocity vector of the centre of mass which is constant for isolated (closed) systems,

$$\frac{d}{dt}\mathbf{r}_{\mathrm{c}} = \mathbf{v}_{\mathrm{c}} = \frac{\sum_{i=1}^{N} m_i \frac{d}{dt}\mathbf{r}_{\mathrm{i}}}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\mathbf{v}_{\mathrm{i}}}{\sum_{i=1}^{N} m_i} = \text{constant,} \tag{13}$$

which shows the equivalence of the inertial point and the centre of mass.

1.5 Definition of Force; Newton's 3rd Law

How can one quantify the change of the state of motion?

- Changing the velocity $\Delta\mathbf{v}$? Insufficient since the mass matters!
- Changing the momentum $\Delta\mathbf{p}$? Insufficient since it matters how fast!
- proposal: $\Delta\mathbf{p}/\Delta t$

Based on the above proposal one can introduce the concept of the force: a body A exerts a force on body B,

$$\textbf{Definition}: \qquad \mathbf{F}_{A\to B} = \mathbf{F}_{BA} \equiv \lim_{\Delta t\to 0}\frac{\Delta\mathbf{p}_\mathrm{B}}{\Delta t} = \frac{d\mathbf{p}_\mathrm{B}}{dt} \tag{14}$$

where $\dim F =$ M L T^{-2}, SI unit $[F] =$ kg$\cdot$ m/s^2. If one assumes a constant mass then it can be written

$$\mathbf{F}_{BA} = \frac{d(m_B\mathbf{v}_\mathrm{B})}{dt} = m_B\frac{d\mathbf{v}_\mathrm{B}}{dt} = m_B\ \mathbf{a}_\mathrm{B}. \tag{15}$$

In case of an isolated (closed) system $\Delta\mathbf{p}_\mathrm{A} = -\Delta\mathbf{p}_\mathrm{B}$, thus we can write

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \tag{16}$$

which the law of action-reaction, i.e., **Newton's 3rd law of motion: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.**

Knowing the force acting on a body in a given invironment is made possible by the existence of force laws. On the fundamental level, we know of four basic forces: (i) *the gravitational force*, which originates in mass of matter and is responsible for binding the solar system and the galaxies; (ii) *the electromagnetic force*, which originates in electric charges and is responsible for binding the constituents of atoms; (iii) *the weak force*, responsible for the radioactive processes; and (iv) *the strong force*, which is responsible for binding the constituents of nuclei together. On the microscopic scale, these forces have very different relative strength. Compared to the strong force between two touching protons, the electromagnetic one has relative strength of 10^{-2}, the weak force has that of 10^{-7} and the gravitational has that of 10^{-38}. The large difference in strength between the electromagnetic and gravitational forces ensures that our Universe is neutral: if there were just one surplus of positive, or negative charge for 10^{36} protons and electrons, it would not be possible to describe the motion of planets and stars by assuming only gravitational interaction between them.

Most of the forces we meet in daily life involves only two forces: gravity and electromagnetism. The gravitational force is apparent in the Earth's attraction of objects. The relative weakness of the gravitational attraction between laboratory objects is negligible in comparison to the other forces, such as elastic force of a spring, tensile force of a stretched rope, frictional force, viscous forces (for instance air drag). Most of these forces are effective forces originating from the electromagnetic forces between atoms. In analysing the motions of bodies, we can ignore the microscopic nature of these forces and replace the complicated structure with a single effective force of a specific force law.

1.6 Force laws

In order to have more quantitaive understanding of force laws, let us consider several specific cases, often encountered in ordinary circumstances.

(i) **Force law of the elastic interaction**

Let us consider the motion of an extended body exerted by a stretched spring placed on a frictionless table. Our goal is to measure the acceleration of the body at various stretched (current) length of the spring. The acceleration is measured (calculated) by the following formula

$$a \approx \frac{s_2/\Delta t_2 - s_1/\Delta t_1}{\frac{\Delta t_1 + \Delta t_2}{2}}$$

where s_1 and s_2 are length intervals associated to the external object and Δt_1 and Δt_2 are the corresponding time intervals measured over the tabletop experiment. As a result of the measurement one finds that the acceleration is always a linear function of the current (stretched) length of the spring with different slope with different masses. However, if one plots $m|\mathbf{a}|$ (instead of $|\mathbf{a}|$) as a function of the current (stretched) length of the spring, the slope becomes identical $ma = c|\Delta\mathbf{l}|$ which means

$$m\mathbf{a} = -c\Delta\mathbf{l}\,. \tag{17}$$

and $|\Delta\mathbf{l}| = l - l_0$ where l is the current (stretched) length and l_0 is the rest length of the spring. Therefore, the force law of the elastic interaction can be written as

$$\mathbf{F} = -c\Delta\mathbf{l} \tag{18}$$

and with a special chose of a cartesian coordinate system, the x-component of the vector is given by

$$F_x = -c(x - x_0) \tag{19}$$

(i) **Force law of the gravitational interaction**

Let us perform free fall experiments with a sheet of paper and a metal ball.

- one finds that the metal ball falls faster (it has a larger acceleration). Air resistance matters!
- after removing the air, the sheet of paper and the metal ball have exactly the same acceleration!
- the acceleration caused by the gravitational field of the Earth is independent of the inertial mass of the object.
- gravitational acceleration (g) is independent of the vertical and horizontal velocity of the object.

Let us note that one can observe a very small variation of the gravitational acceleration (g) on the position. For example, in Hungary on finds $g_{\text{Hungary}} = 9.81\ \text{m/s}^2$ and on the North and South poles $g_{\text{poles}} = 9.85\ \text{m/s}^2$. Therefore, the force law of the gravitational interaction (on the surface of the Earth) can be written as

$$\mathbf{F}_{\text{g}} = \mathbf{g}m\,, \tag{20}$$

where the direction of $\mathbf{g}$ is towards the centre of the Earth.

Let us consider whether the above force law depends on the distance from the centre of the Earth! For example, let us study the orbiting motion of the Moon around the Earth which is caused by the gravitational interaction between the Earth and the Moon. This is exactly the same kind of gravitational interaction as we studied in case of free fall experiments on the surface of the Earth. The difference between the surface free fall experiments and the study of the orbiting Moon is just the distance from the centre of the Earth. The ratio of the radius of the Moon's orbit (384.000 km) and the radius of the Earth (6400 km) is 60. Let us compare the gravitational accelerations at the two distances. The centripetal acceleration of the Moon can be calculated

$$a_{\text{H}} = \omega^2 r_{\text{H}} = \left(\frac{2\pi}{27,3\cdot 24\cdot 3600}\right)^2 \cdot 3,84\cdot 10^8\,\frac{\text{m}}{\text{s}^2} \approx 0,27\frac{\text{cm}}{\text{s}^2}\,, \tag{21}$$

so, it gives $g/a_{\text{H}} \approx 3\,600$. Therefore, one can conclude that

$$g = \frac{C}{r^2}\,, \tag{22}$$

Due to Newton's 3rd law $\mathbf{F}_{\text{Earth,Moon}} = -\mathbf{F}_{\text{Moon,Earth}}$ and their magnitudes are the same, thus one should write

$$F_{\text{G}} = G\,\frac{Mm}{r^2}\,, \tag{23}$$

or in a vector form

$$\mathbf{F}_{\text{grav}} = -G\,\frac{Mm}{r^2}\mathbf{r}^0\,, \tag{24}$$

where $\mathbf{r}^0 \equiv \mathbf{r}/r$ is a unit vector towards $\mathbf{r}$. G is the Newton constant which has the following value,

$$G = 6,67 \cdot 10^{-11}\,\text{Nm}^2/\text{kg}^2\,. \tag{25}$$

which is first measured by Henry Cavendish in 1798.

The simplified version of the Cavendish experiment can be performed in the lecture room where two identical small balls with a mass $m = 15$g are placed on the edges of a rod which is attached to a torsion spring. The length of the rod is 30cm, so, the distance between each ball and the torsion spring is $l = 15$cm. The system is at rest and then we place two larger balls with a mass $M = 1.5$kg close to the small ones with a separation $r = 5$cm. Due to the gravitational attraction between the small and large balls the torsion sprint starts to rotate. A light beam is reflected from a mirror attached to the spring to the wall of the lecture hall on the other side with a distance $L = 15.5$m. There is a length scale placed on the wall, so, the displacement can be read off directly. This experimental setup is suitable to measure the displacement with respect to time, so, one can measure the gravitational acceleration. As a result, one finds that the displacement is 5cm in 66s and 10cm in 95s, so, the acceleration is found to be a constant since $66^2/95^2 \approx 5/10$. Important to note that the angular displacement φ of the torsion spring is small, so, one can neglect its torque and indeed the acceleration is entirely determined by the gravitation attraction between the balls (this is why we observe a constant acceleration). Since the displacement of the ray of light on the scale is $s = 2\varphi L$, the displacement of the small balls can be written as

$$s' = \ell\varphi = \frac{s\ell}{2L} = \frac{0,10\,\text{m}\cdot 0,050\,\text{m}}{2\cdot 15,5\,\text{m}} = 1,6\cdot 10^{-4}\,\text{m}\,.$$

which again confirms that the displacement of the small balls compared to their original distance from the large ones is negligible, so, the distance between the balls can be considered as a constant over the measurement. Thus, the acceleration is

$$a = \frac{2s'}{t^2} = \frac{2\cdot 1,6\cdot 10^{-4}\,\text{m}}{(95\,\text{s})^2} = 3,6\cdot 10^{-8}\,\frac{\text{m}}{\text{s}^2}\,.$$

and from (23) one finds

$$G = \frac{ar^2}{M} = \frac{3,6\cdot 10^{-8}\cdot 0,050^2}{1,5}\,\frac{\text{Nm}^2}{\text{kg}^2} \approx 6\cdot 10^{-11}\,\frac{\text{Nm}^2}{\text{kg}^2}\,,$$

which is in a very good agreement with the known value of G.

Based on the result of the Cavendish experiment one can estimate the mass of the Earth since one can combine the acceleration of the free fall on the surface of the Earth (20) and the general force law of gravitation (23)

$$g = G\,\frac{M}{R^2}\,, \tag{26}$$

where R is the radius of the Earth. Inserting the measured values of g, G and R one finds $M \approx 6\cdot 10^{24}\,\text{kg}$.

(i) **Force law of friction and drag forces**

On the fundamental level, the surface forces acting between two touching surfaces are due to the electromagnetic interactions among the microscopic parts of the two bodies. On the phenomenological level it is possible to describe these interactions approximately assuming a single effective force acting between the two surfaces. If the relative velocity of the surfaces is non-zero, this effective force is called the force of *kinetic friction*, which has a direction such that it is parallel to the surface and reduces the relative velocity $\mathbf{v}$ of the two bodies,

$$\mathbf{F} = -f_k \frac{\mathbf{v}}{v}. \tag{27}$$

The magnitude of the force of kinetic friction f_k will be discussed later. Friction is very important in our daily lives. It dissipates a lot of ordered kinetic energy (to be defined precisely later). It causes wear and seizing of moving parts, and much enginieering effort is devoted to reducing it. On the other hand, without friction we would not be able to write, walk, as well many other means of transport would not be possible.

When an object moves through fluid (or gas) medium, frictional force emerges between the object and fluid, which is called drag. The direction of drag is always opposite to the relative velocity $\mathbf{v}$ of the body and the fluid. Its magnitude depends on the relative velocity such that it cannot be expressed by a simple function. Nevertheless, for sufficiently small velocities the dependence can be approximated by a linear function,

$$\mathbf{F_I} = -C\mathbf{v}, \tag{28}$$

while for sufficiently large velocities the functional form is approximately quadratic,

$$\mathbf{F_{II}} = -Cv\mathbf{v}. \tag{29}$$

(i) **Force law of electrostatic interaction between two point charges (Coulomb's law)**

The magnitude of the force acting between two pointlike charges of charge Q_A and Q_B is inversely proportional to the square of the distance r between them. The direction of the force exerted by charge A on charge B depends on the relative sign of the two charges. If the relative signs are the same the force is repulsive, if those are different, the force is attractive. This can be summarized in a single formula where the charges are considered positive or negative:

$$\mathbf{F}_{BA} = K\frac{Q_A Q_B}{r^2}\mathbf{r_0}\,, \tag{30}$$

where the unit vector is defined as $\mathbf{r_0} = \mathbf{r}_{A\to B}/r$ and the constant K has the approximate numerical value of $9 \cdot 10^9\,\mathrm{N\,m^2/C^2}$.

(i) **Force law of a charged particle moving in magnetic field (Lorentz's law)**

The force acting on a pointlike charge of charge Q that moves in a magnetic field $\mathbf{B}$ with velocity $\mathbf{v}$ is

$$\mathbf{F} = Q\,\mathbf{v} \times \mathbf{B}. \tag{31}$$

(i) **Force law of Van der Waals interaction**

The Van der Waals force acts between two dipoles and it has relevance in solid state physics. We do not discuss the explicit form of this force law but mention a few properties. This is a short range force, it drops to zero at a distance $r \sim 10^{-6}$ cm. For very short distances $r < 10^{-8}$ cm it is repulsive and for larger distances $r > 10^{-8}$ it is attractive.

1.7 Superposition principle – independence of forces

The definition of force is often supplemented with another statement, which is consistent with its vector nature. It is not a necessity that net force can be calculated as a linear sum of all forces. This is true only if there are no polarisation effects.

One can perform the following experiment. Let us attach a spring to an object with a rest mass (m) and measure its acceleration $\mathbf{a_1}$. Attach a different spring to the same object and again measure its acceleration $\mathbf{a_2}$. Not attach both springs to the object and measure its net acceleration

$\mathbf{a_{sum}}$. The measurement confirms that $\mathbf{a_{sum}} = \mathbf{a_1} + \mathbf{a_2}$ which results in the linear superposition of the two individual forces $\mathbf{F_{sum}} = \mathbf{F_1} + \mathbf{F_2}$. This can be generalised for many body interactions and summarised in a new law of motion.

Newton's fourth law: when several forces act on a body, then the net force, that appears in Newton?s second law, is a linear sum of the individual forces.

1.8 Newton's second law – equation of motion

According to the Galilean transformation of velocities, if two frames of reference have a relative motion of constant velocity, the accelaration of any object in the two frames are the same. Therefore, the forces cannot depend on absolute coordinates and velocities (which are different in the two frames), but on relative ones. For instance, the force exerted by an alongated spring depends only on the deformation of the spring. Thus force laws express the forces as functions of relative coordinate differences, velocities, physical properties of bodies in the environment of the object (such as the spring constant), labelled i whose motion is being analysed,

$$\mathbf{F} = \mathbf{F}(\mathbf{r}_{ij}, \mathbf{v}_{ij}, \dots)\,. \tag{32}$$

It is now possible to formulate the most important law of motion, the law of dynamics. **Newton's second law: in an inertial frame of reference, the vector sum of the forces on an object is equal to the time derivative of the momentum vector of the object.** In other words, the momentum of a body changes only if it comes into interaction with another body. The rate of change of momentum is equal to the force acting on the body,

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{r}_{ij}, \mathbf{v}_{ij}, \dots)\,. \tag{33}$$

If we assume that the inertial mass (m) of the object is constant,

$$m\mathbf{a} = \mathbf{F}(\mathbf{r}_{ij}, \mathbf{v}_{ij}, \dots)\,. \tag{34}$$

According to Newtons second law, the sum of these forces acting on the body is equal to $m\,\mathbf{a}_i = m\,d^2\mathbf{r}_i/dt^2$. Therefore, using the force laws, the law of dynamics is a differential equation of second order, this is the equation of motion,

$$\frac{d^2\mathbf{r}_i}{dt^2} = \frac{1}{m}\mathbf{F}(\mathbf{r}_{ij}, \mathbf{v}_{ij}, \dots)\,. \tag{35}$$

The solution of this differential equation is the position vector of the body at any time, $\mathbf{r}_i(t)$. Let us summarise Newton's laws of motion:

(N1) **In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.**

(N2) **In an inertial frame of reference, the vector sum of the forces on an object is equal to the time derivative of the momentum vector of the object.**

(N3) **When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.**

(N4) **Forces add up like vectors, that is, that forces obey the principle of superposition.**

Finally, let us discuss the principle of relativity in the Newtonian physics. The special principle of relativity was first explicitly enunciated by Galileo Galilei in 1632 in his Dialogue Concerning the Two Chief World Systems, using the metaphor of Galileo's ship. When formulated in the context of these laws, the special principle of relativity states that the laws of mechanics are invariant under a Galilean transformation.

1.9 Solving the equation of motion

1) Analytic solution for projectile motion

Near the surface of Earth the gravitational force acting on the stone can be considered constant, therefore in the force law $\mathbf{F} = \mathbf{g}m$, $\mathbf{g}$ does not depend on the position of the stone. Let us assume that we can neglect the drag force, so this is the only force acting on the stone during its motion. Furthermore, we assume that the starting position is $\mathbf{r}_0$, and the starting velocity is $\mathbf{v}_0$. These are the *initial conditions*. Using $\mathbf{a} = \ddot{\mathbf{r}}$ and Newton's 2nd law, $\mathbf{F} = m\mathbf{a}$, the equation of motion can be written as follows:

$$\ddot{\mathbf{r}} = \mathbf{g}\,, \tag{36}$$

which is a vectorial equation, containing the second derivative of the unkown function $\mathbf{r}$, hence called second order differential equation. We want to answer the following question: Which function $\mathbf{r} = \mathbf{r}(t)$ satisfies this equation? As the acceleration does not depend on the position, it is convenient to compute the velocity as a function of time first. The velocity satisfies the $\dot{\mathbf{v}} = \mathbf{g}$ first order differential equation, with the solution

$$\dot{\mathbf{r}} = \mathbf{v} = \mathbf{v}_0 + \mathbf{g}t\,, \tag{37}$$

because the first derivative of this function with respect to time is indeed $\mathbf{g}$, and its value at time $t = 0$ is $\mathbf{v}_0$. Then the solution of the first order differential equation (37), that also satisfies the initial conditions, is

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 \tag{38}$$

because the derivative of this equation with respect to time is indeed $\mathbf{v}_0 + \mathbf{g}t$, and its value at $t = 0$ is $\mathbf{r}_0$. We see that we must fix as many initial conditions as the order of the differential equation. In order to solve a second order differential equation (the typical equation of motion) we must give two *initial conditions*.

Let us choose the y axis pointing vertically upward and the x axis pointing horizontally such that the initial velocity and position lies in the $x - y$ plane. The gravitational acceleration has direction $-y$, therefore the vector equation (38) has no z component. The motion occurs in the $x - y$ plane, we do not even use the z coordinates. We choose the origin of the coordinates such that the initial x coordinate of $\mathbf{r}_0$ is zero, $x_0 = 0$ and its y coordinate is $ys_0 = h$, denoting the initial height. Let the initial speed be v_0, and the angle between the initial velocity and the x axis beα. Then the inital values of the coordinates and velocity components are

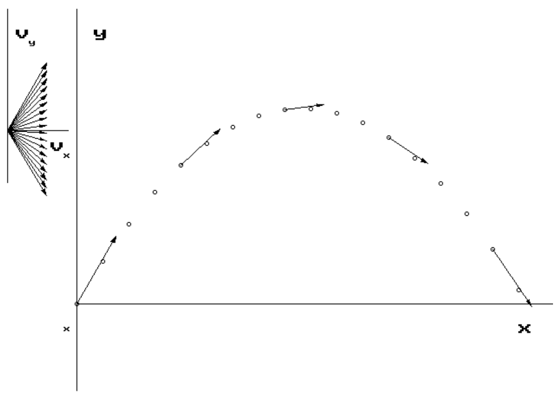


Figure 1:

$$x_0 = 0\,, \quad y_0 = h\,, \tag{39}$$

and

$$v_{0x} = v_0 \cos\alpha\,, \quad v_{0y} = v_0 \sin\alpha\,. \tag{40}$$

Together with g these must be considered constants, we just have not yet specified their magnitudes. The coordinates depend on time as

$$x(t) = v_{0x}t\,, \quad y(t) = h + v_{0y}t - \frac{1}{2}gt^2\,, \tag{41}$$

while the components of velocity depend on time as

$$v_x(t) = v_{0x}\,, \quad v_y(t) = v_{0y} - gt\,. \tag{42}$$

We used equations (41) to draw the plots in figure 1., and the equations (42) to draw the velocity vectors at several different times.

2) Numerical solution for ballistic motion

The drag force increases rapidly with increasing speed, therefore, it is certainly not negligible in analysing the motion of bullets. The real motion of bullets cannot be described with equation (38). For high speeds drag is proportional to the square of speed, $\mathbf{F}_{\text{drag}} = -Kv^2\mathbf{v}^0 \equiv -Kv\mathbf{v}$. The equation of motion is

$$m\mathbf{a} = m\mathbf{g} - Kv\mathbf{v} \tag{43}$$

or dividing by m we obtain

$$\mathbf{a} = \mathbf{g} - Cv\mathbf{v}\,. \tag{44}$$

We choose the same coordinate system as in the previous case, therefore, as neither the acceleration nor the initial position and velocity vectors have z component, the motion occurs in the $x-y$ plane. Writing the equation of motion in components,

$$\dot{v}_x = -C\,v_x\sqrt{v_x^2+v_y^2}\,, \qquad \dot{v}_y = -g - C\,v_y\sqrt{v_x^2+v_y^2} \tag{45}$$

we obtain a system of two *coupled differential equations*. The acceleration is still independent of position, so it is convenient to compute the velocity first and then the coordinates of position. The initial condition are the same as in the previous case with the choice $h = 0$. We can compute the position using the following algorithm (:= means *let it equal*, i.e in the last line the new and the old values appear on the left and right hand sides, respectively):

$$\begin{array}{c}
\boxed{x := 0\,;\ y := 0\,; \quad v_x = v_0\cos\alpha\,;\ v_y = v_0\sin\alpha} \\
\downarrow \\
\boxed{b := C\sqrt{v_x^2+v_y^2}} \\
\downarrow \\
\boxed{a_x := -bv_x\,;\ a_y := -g - bv_y} \\
\downarrow \\
\boxed{v_x := v_x + a_x\Delta t\,;\ v_y := v_y + a_y\Delta t}
\end{array} \tag{46}$$

Using the result $\mathbf{v} = \mathbf{v}(t)$ we obtain the dependence of the position coordiantes on time using the approximation of integration:

$$x(t_n) = \sum_{i=1}^{n}\langle v_x\rangle_i\Delta t\,, \qquad y(t_n) = \sum_{i=1}^{n}\langle v_y\rangle_i\Delta t\,, \tag{47}$$

where $\langle v_x\rangle_i$ and $\langle v_y\rangle_i$ denote the mean of the velocity components during the ith time interval.

In figure 2. we show the path and speed vectors of a bullet compared with and without drag. The path with drag is called ballistic path.

3) Harmonic oscillator

Let us consider a one-dimensional motion on a frictionless table where an object is attached to a spring. It is convenient to chose the x-coordinate along the spring. The spring force law is $F_x = -c(x - x_0)$, and it is free to choose the zero position at the rest length of the spring, so, the force law is reduced to $F_x = -cx$. In this case the equation of motion reads as

$$m\ddot{x} = -cx\,, \qquad \to \qquad \ddot{x} = -\frac{c}{m} \tag{48}$$

Let us look for the solution $x(t)$ among the periodic functions,

$$x(t) = A\sin(\omega t + \phi) \tag{49}$$

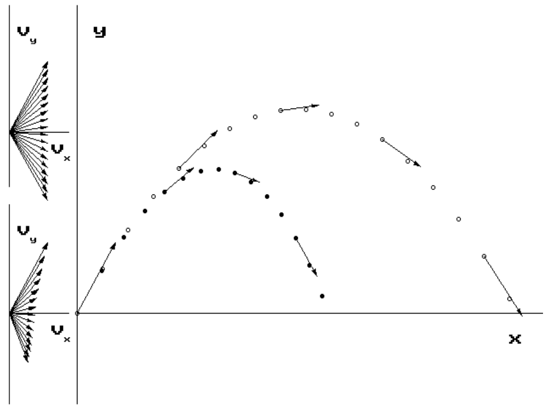


Figure 2:

where A, ω and ϕ are constants which need to be determined. The dimension of A and $x(t)$ should be the same and angular frequency $\dim \omega = \ \mathrm{T}^{-1}$ where T is the period time. In order to prove that (49) is indeed a solution of (48) one has to derive $x(t)$ with respect to time

$$\dot{x} = A\omega\,\cos(\omega t + \phi)\,, \qquad \ddot{x} = -A\omega^2\,\sin(\omega t + \phi)\,, \tag{50}$$

and they should be substituted into the equation (48) which gives

$$-A\omega^2\,\sin(\omega t + \phi) = -A\frac{k}{m}\sin(\omega t + \phi) \tag{51}$$

which holds only if the angular frequency is chosen to be

$$\omega = \sqrt{\frac{c}{m}}\,. \tag{52}$$

The two remaining parameters, the amplitude A and the phase ϕ are fixed by the initial conditions. Assume that the position x_k and the velocity v_k are known at $t = 0$. Then one can write $x_\mathrm{k} = A\sin\phi$,, $v_\mathrm{k} = A\omega\cos\phi$ which can be used to determine the amplitude and the phase,

$$A = \sqrt{x_\mathrm{k}^2 + \frac{v_\mathrm{k}^2}{\omega^2}}\,, \qquad \phi = \mathrm{arctg}\,\frac{x_\mathrm{k}\omega}{v_\mathrm{k}}. \tag{53}$$

A remark is in order. What if we consider an object attached to a spring vertically. In this case apart from the spring force (which depends on the position) a gravitational force is acting on the object,

$$m\ddot{x} = -c(x - x_0) + mg = -c\left[x - \left(x_0 + \frac{mg}{c}\right)\right]\,.$$

which shows that even if gravitational interaction is present, the equation of motion (54) is identical to the equation of motion of the harmonic oscillator (48) if one redefines the rest length $x_0 \to x_0 + mg/c$.

4) Exponentially damped oscillator

Let us take into account the air resistance which can be done by the drag force depending on the velocity,

$$m\ddot{x} = -cx - c_v v_x = -cx - c_v\dot{x} \tag{54}$$

where the equation of motion can be rewritten by introducing $\delta = c_v/(2m)$ and $\omega_0 = \sqrt{c/m}$,

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0. \tag{55}$$

If the damping term is chosen appropriately, then one finds a decreasing amplitudes where its odd and even maxima have the following property,

$$\frac{x_1}{x_3} \approx \frac{x_3}{x_5} \approx \frac{x_5}{x_7}\ldots \tag{56}$$

which results in a geometric series and an exponential damping. Let us look for the solution in the following form

$$x(t) = A\mathrm{e}^{-bt}\sin(\omega t + \phi) \tag{57}$$

where its derivatives with respect to time read

$$\dot{x}(t) = A\mathrm{e}^{-bt}[-b\sin(\omega t+\phi)+\omega\cos(\omega t+\phi)]\,, \qquad \ddot{x}(t) = A\mathrm{e}^{-bt}[(b^2-\omega^2)\sin(\omega t+\phi)-2b\omega\cos(\omega t+\phi)]\,. \tag{58}$$

Inserting the solution and its derivatives into the equation of motion one finds an equation with the following form $X\sin(\omega t + \phi) + Y\cos(\omega t + \phi) = 0$ which can be satisfied if and only if $X = 0$ and $Y = 0$,

$$X = b^2 - \omega^2 - 2b\delta + \omega_0^2 = 0 \qquad Y = -2b\omega + 2\delta\omega = 0 \tag{59}$$

which results in the following solution for b and ω,

$$b = \delta\,, \quad \text{és} \quad \omega^2 = \omega_0^2 - \delta^2\,. \tag{60}$$

The amplitude and the phase are determined by initial conditions. Let us note, that the solution is periodic if $\omega_0 > \delta$,

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}}\,. \tag{61}$$

which is larger then the period of time $T > T_0 \equiv 2\pi/\omega_0$ of the corresponding harmonic oscillator without any damping term. If $\omega_0 < \delta$ it is overdamped.

5) Forced oscillator

So far we have discussed natural oscillations of a body when it is displaced from its equilibrium. A different situation is when the body is subject to a sinusoidal external force, which is characterised by the following time-dependent displacement $\xi(t) = b\sin\omega t$. Thus the external force is

$$F_x = -c[x + x_0 - \xi(t) - x_0] = -cx + cb\sin\omega t = -cx + F_0\sin\omega t\,. \tag{62}$$

The resulting oscillation is called forced oscillation. Incorporating the damping term, too the equation of motion reads

$$m\ddot{x} = -cx + F_0\sin\omega t - c_v\dot{x} \tag{63}$$

which can be rewritten as

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = f_0\sin\omega t\,, \tag{64}$$

where δ are ω_0 are defined previously and $f_0 = F_0/m$.

The forced oscillation is also a harmonic oscillation with the frequency of the external force and not of the natural frequency of the body. However, the response of the body, that is the amplitude and phase of the oscillation, depends on the relation between the forcing and natural frequencies. A particular solution to the equation of motion is

$$x(t) = A(\omega)\sin(\omega t - \phi(\omega))\,, \tag{65}$$

where the dependence of the amplitude and the phase constant on the frequency of the external force can be obtained by substituting the solution into the equation of motion and requiring that the equality holds for any time t,

$$A(\omega) = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\delta^2\omega^2}}\,, \tag{66}$$

and a phase shift is

$$\phi(\omega) = \operatorname{arctg}\frac{2\delta\omega}{\omega_0^2 - \omega^2}\,. \tag{67}$$

If the damping is absent, then the amplitude of the forced oscillator aproaches infinity as the forcing frequency aproaches the natural frequency. In reality, the damping is always present. Nevertheless, the amplitude of the forced oscillation has a clear maximum when the denominator in (66) attains its minimum, which is close to, but always somewhat smaller than the natural frequency (an effect of the damping). The position of the maximum is called resonant frequency and the oscillation with maximal amplitude is a resonance. At resonance the phase shift is $\pi/2$.

6) Equation of motion in case of the Lorentz force

Let us solve the equation motion in case of the Lorentz force, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. The force, and consequently the acceleration are perpendicular to the velocity, thus $a_{\mathrm{t}} = \dot{v} = 0$. So, the velocity is constant over the motion. If the initial velocity is perpendicular to the magnetic field then the trajectory of the motion is a circle with the following centripetal acceleration,

$$a_{\mathrm{n}} = \frac{q}{m} vB \,, \tag{68}$$

where the corresponding angular frequency (cyclotron frequency) is given by

$$\omega = \frac{a_{\mathrm{n}}}{v} = \frac{qvB}{mv} = \frac{q}{m} B \,, \tag{69}$$

with the following radius

$$r = \frac{v}{\omega} = \frac{mv}{qB} \,. \tag{70}$$

If the initial velocity is not perpendicular to the magnetic field then the trajectory of the motion is a spiral.

7) Equation of motion in case of central forces

In case of central forces, the force $\mathbf{F}$ is parallel (or antiparallel) to the position vector $\mathbf{r}$, thus with appropriate initial conditions the trajectory of the motion is a circle. However, this requires a particular initial velocity, which has to be perpendicular to the position vector and its magnitude cannot be chosen arbitrarily. The magnitude of this critical velocity is determined by the kinematic expression, $a_{\mathrm{cp}} = v^2/r$ which relates the centripetal acceleration and the (tangent) velocity.

If one considers the gravitational force (which is an example for central forces) the magnitude of the critical velocity is given by,

$$m\frac{v^2}{r} = F(r) \,. \tag{71}$$

where the force law has a well-known form close to the Earth,

$$m\frac{v^2}{r} = mg \,, \tag{72}$$

which results in the so-called first cosmic speed, $v_{\mathrm{I}} = \sqrt{gR} \approx \sqrt{10\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot 6,4 \cdot 10^6\,\mathrm{m}} = 8\,\frac{\mathrm{km}}{\mathrm{s}}$.

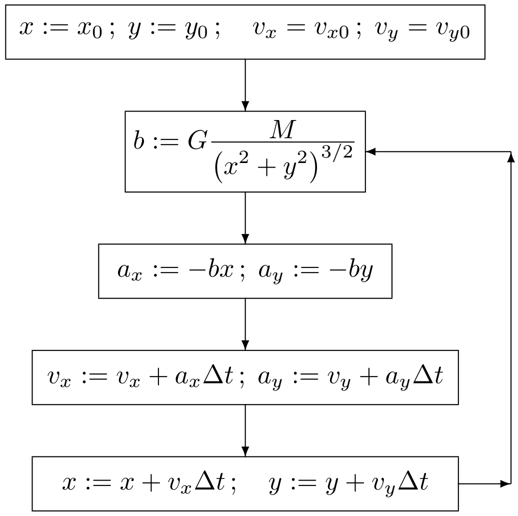
If the initial value of he velocity is not the critical one, then the trajectory of the motion is not a circle and the equation of motion can be solved only numerically. For example in case of the gravitational force one finds

$$m\mathbf{a} = -G\,\frac{Mm}{r^3}\,\mathbf{r} \,, \tag{73}$$

which results in the following components,

$$a_x = -G\,\frac{M}{(x^2+y^2)^{3/2}}\,x \,, \qquad a_y = -G\,\frac{M}{(x^2+y^2)^{3/2}}\,y \,. \tag{74}$$

and it can be solved by the algorithm,



(75)

Based on the above numerical solution, one can confirm Kepler's (Johannes Kepler 1571–1630) first law of planet motion. In astronomy, Kepler's laws of planet motion are three scientific laws describing the motion of planets around the Sun, published by Johannes Kepler between 1609 and 1619. These modified the heliocentric theory of Nicolaus Copernicus, replacing its circular orbits and epicycles with elliptical trajectories, and explaining how planetary velocities vary. The laws state that:

1. **The orbit of a planet is an ellipse with the Sun at one of the two foci. (Trajectory of the motion is a conic section.)**
2. **A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. (The area-velocity is constant.)**
3. **The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.** ($T^2/a^3 =$ **constant.)**

1.10 Inertial and gravitational mass

In case of an inertial motion one can always find an inertial frame of reference where the object (in case of extended object its centre of mass) is at rest. The required condition is

$$\sum_{\mathrm{i}} \mathbf{F}_{\mathrm{i}} = 0\,. \tag{76}$$

By using this condition, one can measure the mass of any object. Assume an object with a mass m attached to a spring vertically is at rest. This means, $\mathbf{F}_{\text{rugó}} + m_0\mathbf{g} = 0$, so thus $F_{\text{rugó}} = m_0 g$. Therefore, if one measures the current length of the spring and its rest length, the spring force can be calculated and the mass of the object is determined.

However, there is an important issue which needs to be mentioned here. The above argument is true only if we assume that the inertial and the gravitational masses are equal. This is, in principle, not a consequence of the gravitational force law or Newton's laws of motion. One has to clarify or falsify it by experiments. Indeed, it has been confirmed by various different type of measurements with good accuracy. As an example one can mention the Eötvös pendulum. The equivalence of the inertial and gravitational masses has very important consequences which are, however, not discussed here and we reserve it for further advanced studies on the topic.

1.11 Constrained motion

In many cases the elastic forces are observed without any apparent change in the geometry of the object that exerts the elastic force. For instance, the surface of the table appears flat after we put a block on it, or the length of a rope after hanging some object on it. In these cases the geometry of the motion is fixed, and the force exerted on the body is called constraint, the motion is called constrained motion.

a) Normal force

Thus, instead of applying the force law of some elastic force, we rather know the geometry of the motion. For instance, putting a block of mass m on a horizontal surface, we find that the block does not move, its acceleration is zero. Then it follows from Newton's second law that the force exerted by the surface of the table on the body $\mathbf{F_N}$ is equal in magnitude, but opposite in direction to the gravitational force $m\mathbf{g}$ exerted by the Earth,

$$m\mathbf{g} + \mathbf{F_N} = 0\,, \tag{77}$$

so, the force of constraint, in this case the so called normal force is $\mathbf{F_N} = -m\mathbf{g}$.

What if the block has a non-vanishing acceleration? Knowing the motion of the block, we can conclude about the force of the constraint. If the acceleration of the block is $\mathbf{a}$, then we obtain the force of the constraint from

$$m\mathbf{g} + \mathbf{F_N} = m\,\mathbf{a} \tag{78}$$

as $\mathbf{F_N} = -m(\mathbf{g} - \mathbf{a})$. Important to note, that the constrained force has no force law, it depends on the apparent geometry of the motion.

b) Mathematical pendulum

Let us consider the motion of a point-like object with a mass m attached to a rope with a length l which is the so called mathematical pendulum. The change in the length of a rope after hanging the object on it, is not observed, thus, the force exerted by the rope is considered as a force of the constraint. Let us denote the path of the hanged object with s

$$s = r\varphi \tag{79}$$

where the tangent of the velocity and acceleration of the object can be written as

$$v_\mathrm{t} = \dot{s} = l\dot{\varphi}\,, \qquad a_\mathrm{t} = \dot{v}_\mathrm{t} = l\ddot{\varphi} \tag{80}$$

and there is no tangent of the force exerted by the rope, so one has to consider the tangent of the gravitational force only

$$F_\mathrm{t} = -mg\sin\varphi\,, \tag{81}$$

which results in the following equation of motion

$$ml\ddot{\varphi} = -mg\sin\varphi \tag{82}$$

Let us divide both sides of the equation of motion with ml

$$\ddot{\varphi} = -\frac{g}{l}\sin\varphi\,, \tag{83}$$

and if one approximates the sine function with its arguments which is valid for small angles, $\sin\varphi \approx \varphi$

$$\ddot{\varphi} = -\frac{g}{l}\varphi\,, \tag{84}$$

then the resulting differential equation can be solved analytically

$$\varphi(t) = \varphi_0\sin(\omega t + \phi)\,, \tag{85}$$

where ω is defined as

$$\omega = \sqrt{\frac{g}{l}}\,, \tag{86}$$

and consequently, $T = 2\pi\sqrt{l/g}$. The constrained force can be calculated by solving the equation of motion for the radial component,

$$m\frac{v^2}{l} = K - mg\cos\varphi\,, \tag{87}$$

where $K = m(g\cos\varphi + v^2/l)$.

We obtained an explicit form for the constrained force of the mathematical pendulum. As discussed in the case of the normal force, the constrained force has no force law, so, it is useful to consider another example for constrained force exerted by the rope. Let us study a statical problem where a point like object is hanged on two different ropes where the ropes are not vertical. Let us denote the angles between the ropes and the vertical direction by α and β. In this case one finds

$$\sum_{\mathrm{i}} F_{\mathrm{i}x} = 0 \Rightarrow F_1 \sin\alpha - F_2 \sin\beta = 0\,, \tag{88}$$

$$\sum_{\mathrm{i}} F_{\mathrm{i}y} = 0 \Rightarrow F_1 \cos\alpha + F_2 \cos\beta - mg = 0\,, \tag{89}$$

with the solution

$$F_1 = \frac{mg}{\sin\alpha(\mathrm{ctg}\,\alpha + \mathrm{ctg}\,\beta)}\,, \tag{90}$$

$$F_2 = \frac{mg}{\sin\beta(\mathrm{ctg}\,\alpha + \mathrm{ctg}\,\beta)}\,. \tag{91}$$

If $\alpha = \beta$ the two forces are identical and equal to $F_1 = F_2 = mg/(2\cos\alpha)$. However, if $\alpha = \beta = 0°$ then the equation for the x-components carries no information since all terms are zero, thus, the problem is underdetermined, and $F_1 = xmg$, $F_2 = (1-x)mg$ is a solution for any real number $0 \leq x \leq 1$.

3) Frictional force, motion on a slope

Let us consider a block of mass m on a table which is not frictionless. In this case one can identify two constrained forces. If the block has no velocity one can identify the static frictional force which is parallel to the surface of the table. Since the block is in equilibrium, the vector sum of all forces acting on the block should be zero. The maximum of the static frictional force is given by

$$F_{\mathrm{s}} \leq F_{smax} = \mu_s F_{\mathrm{n}}\,, \tag{92}$$

where F_{n} is the normal force and μ_s is the static frictional constant.

If the block has non-vanishing velocity, then one has to take into account the kinetic frictional force,

$$F_{\mathrm{k}} = \mu F_{\mathrm{n}}\,, \tag{93}$$

where F_{n} is the normal force and and μ is the kinetic frictional constant.

Another example of constrained motion is putting a block of mass m on a slope of angle α with respect to the horizontal axis. The block may either stay on the slope in equilibrium (the motion is completely constrained), or may slide down with acceleration parallel to the plane (the motion is partially constrained). We find that there are two external agents that exert force on it: (i) the gravitational pull of the earth, represented by the force $m\mathbf{g}$, where $\mathbf{g}$ points vertically downwards, and (ii) the surface of the the slope, represented by a force $\mathbf{F}$, pointing to some direction upwards. If the block stays in equilibrium, then these two forces sum to zero,

$$\mathbf{F} + m\mathbf{g} = 0\,, \tag{94}$$

which specifies the force of constraint completely. Such an equilibrium is possible if the coefficient of static friction is sufficiently large,

$$F_{||} = F\sin\alpha \leq \mu_s F_{\perp} = \mu_s F\cos\alpha\,, \tag{95}$$

that is if $\mu_s \geq \tan\alpha$. If $\mu_s < \tan\alpha$, then equilibrium is not possible, the block slides down.

If the block slides down, then it does not leave the surface of the inclined plane, therefore, its acceleration is parallel to the surface. Thus, the vector components of $\mathbf{F}$ and $m\mathbf{g}$ that are perpendicular to the plane add to zero, so the scalar components are equal,

$$F_{\perp} = mg\cos\alpha\,. \tag{96}$$

The parallel component of $\mathbf{F}$ is the force of friction that is always oppposite to the relative velocity of the touching surfaces, therefore, we can write

$$mg\sin\alpha - F_{||} = ma\,. \tag{97}$$

These two equations are not sufficient to determine $F_\perp$, $F_{||}$ and a. However, we can use the expression for the kinetic friction,

$$F_{||} = \mu F_\perp\,. \tag{98}$$

We can solve the three equations (96), (97) and (98) to find the acceleration of the block, that is a constant,

$$a = g(\sin\alpha - \mu\cos\alpha)\,. \tag{99}$$

which gives the following critical value for the kinetic friction constant $\mu = \tan\alpha$.

Finally, let us pay the attention of the reader that strictly speaking the kinetic friction has no force law. Although, the expression $F_k = \mu F_N$ has been used in the above example but the normal force F_N is a constrained force and thus, it has no force law, so does the kinetic friction. Indeed, if one considers a constrained motion, the corresponding force of constraint has never had a force law as opposed to free forces like for example the gravitational force where one finds a well defined force law.

1.12 Bulk and surface forces

There are two different ways to build up a viable model description for objects (gases, liquids and solids): one can relies on the molecular model where the objects are made up of atoms, molecules, ions, or one can use the continuum model where objects are considered as a continuum media.

If we use the continuum model, an object can be divided into infinitely many small pieces having the same properties of the original object. Let consider two forces, the gravitational and the normal force. The gravitation force is expected to act on all pieces of the object, thus one can define the force density

$$\vec{\gamma} = \lim_{\Delta V\to 0}\frac{\Delta \mathbf{F}_{\rm g}}{\Delta V} \tag{100}$$

and similarly, the mass density

$$\rho = \lim_{\Delta V\to 0}\frac{\Delta m}{\Delta V} \tag{101}$$

where $\vec{\gamma} = \rho\mathbf{g}$ with $\dim\gamma = \text{ M (L T)}^{-2}$, $\dim\rho = \text{ M L}^{-3}$; and their SI units are $[\gamma] = \text{ N/m}^3$, $[\rho] = \text{ kg/m}^3$.

Concerning the normal force it is clear that it acts on the surface of the object, so it is relevant to introduce the area (or surface) density, $\lim_{\Delta A\to 0}\frac{\Delta \mathbf{F_N}}{\Delta A}$. The area-density of the normal force is called the pressure and it reads as

$$p = \lim_{\Delta A\to 0}\frac{\Delta F_{\rm n}}{\Delta A}\,. \tag{102}$$

1.13 Newton's laws for extended objects

We have already discussed how to generalise Newton's first law for extended objects where the centre of mass played an important rule. As a next step let us discuss how Newton's second law can be understood for extended objects, in particular whether the internal forces play any role or one just have to take into account the vector sum of external forces. Since the centre of mass is expected to have importance let us first give an example where it is calculated for an extended object.

1.13.1 Centre of mass of a half cylinder with homogeneous mass distribution

If one considers a system of particles, the centre of mass is given by

$$\mathbf{r}_\mathrm{c} = \frac{\sum_\mathrm{i} m_\mathrm{i}\mathbf{r}_\mathrm{i}}{\sum_\mathrm{i} m_\mathrm{i}} = \frac{\sum_\mathrm{i} m_\mathrm{i}\mathbf{r}_\mathrm{i}}{m} \tag{103}$$

which can be used to calculate the centre of mass of a half cylinder with homogeneous mass distribution where R is its radius and m is its total mass. On can choose a reference frame with cartesian coordinates where based on symmetry considerations, the y and the z components of the position vector of the centre of mass are $y_c = 0$ and $z_c = l/2$ where l is the hight of the cylinder. Thus, only x_c has to be determined.

Slice the half circle (the projection of the half cylinder into the $x - y$ plane) into very narrow stripes which are parallel to the y axis,

$$x_\mathrm{c} = \frac{\sum_\mathrm{i} \Delta m_\mathrm{i} x_\mathrm{i}}{m} \tag{104}$$

Since the cylinder has a homogeneous mass distribution one can write $\Delta m_\mathrm{i}/m = \Delta A_\mathrm{i}/A$, which results in

$$\Delta m_\mathrm{i} = \frac{2m}{R^2\pi}\Delta A_\mathrm{i} = \frac{2m}{R^2\pi} 2\sqrt{R^2 - x_\mathrm{i}^2}\Delta x_\mathrm{i}\,. \tag{105}$$

Then substituting it into Eq. (104) one finds

$$x_\mathrm{c} = \frac{2}{R^2\pi}\sum_\mathrm{i} 2x_\mathrm{i}\sqrt{R^2 - x_\mathrm{i}^2}\Delta x_\mathrm{i} \tag{106}$$

Taking the limit $\Delta x_\mathrm{i} \to 0$

$$x_\mathrm{c} = \frac{2}{R^2\pi}\int_0^R 2x\sqrt{R^2 - x^2}\,\mathrm{d}x = \frac{2}{R^2\pi}\left[\frac{\left(R^2 - x^2\right)^{3/2}}{3/2}\right]_0^R = \frac{4R}{3\pi} \tag{107}$$

The above strategy can be generalised for an arbitrary case, $\Delta m_\mathrm{i} = \rho\,\Delta V_i$,

$$\mathbf{r}_\mathrm{c} = \frac{\sum_\mathrm{i} \Delta m_\mathrm{i}\mathbf{r}_\mathrm{i}}{\sum_\mathrm{i} \Delta m_\mathrm{i}} = \frac{\sum_\mathrm{i} \Delta m_\mathrm{i}\mathbf{r}_\mathrm{i}}{m} = \frac{\sum_\mathrm{i} \rho\Delta V_i\mathbf{r}_\mathrm{i}}{m} = \int_V \frac{\rho\,\mathbf{r}_\mathrm{i}}{m} dV_i = \frac{1}{V}\int_V \mathbf{r}_\mathrm{i} dV_i \tag{108}$$

1.13.2 Newton's second law for extended bodies

In order to generalise Newton's second law for an extended objects let us divide it into N point like parts. If one applies Newton's second law on one of these parts both the vector sum of external ($\mathbf{F}_i$) and the vector sum of internal ($\sum_{j=1}^N \mathbf{F}_{ij}$) forces should be taken into account,

$$\Delta m_\mathrm{i}\mathbf{a}_\mathrm{i} = \mathbf{F}_\mathrm{i} + \sum_{j\neq i}^N \mathbf{F}_\mathrm{ij}\,, \tag{109}$$

and then one can sums up both sides of the equation of motion which results in

$$\sum_\mathrm{i}^N \Delta m_\mathrm{i}\mathbf{a}_\mathrm{i} = \sum_\mathrm{i}^N \mathbf{F}_\mathrm{i} + \sum_\mathrm{i}^N\sum_{j\neq i}^N \mathbf{F}_\mathrm{ij}\,. \tag{110}$$

According to Newton's third law, $\mathbf{F}_\mathrm{ij} = -\mathbf{F}_\mathrm{ji}$, the double sum on the internal forces is zero. Thus, the internal forces give no contribution to the equation of motion. In addition, if we use the definition for the acceleration of the centre of mass, $m\mathbf{a}_\mathrm{c} = \sum_\mathrm{i} \Delta m_\mathrm{i}\mathbf{a}_\mathrm{i}$, then our final result reads as

$$m\mathbf{a}_\mathrm{c} = \sum_\mathrm{i} \mathbf{F}_\mathrm{i}\,, \tag{111}$$

1.14 Newton's second law for variable-mass systems

In mechanics, a variable-mass system is a collection of matter whose mass varies with time. Let us discuss how Newton's second law can be generalised for such cases where the key issue is that one has to take into account the momentum carried by mass entering or leaving the system.

1.14.1 Equation of motion for variable-mass systems

One has to redefine the concept of an object, or body because its mass varies with time. So, let us consider a closed volume and define the body as the content of this finite volume which has the surface-area A. Then, one can identify the mass and momentum which entering and leaving the system. Thus, the mass current is defined as

$$J^{(m)} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} \tag{112}$$

and the momentum current is given by

$$\mathbf{J}^{(\mathbf{p})} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}}{\Delta t} \tag{113}$$

If we assume a finite, closed volume the net-current can be defined by the escaping current minus the incoming current, $J = J_{\text{esc}} - J_{\text{inc}}$. If the velocity of the escaping (or incoming) parts is constant, the mass and the momentum current can be related to each other,

$$\mathbf{J}^{(\mathbf{p})} = \mathbf{u} J^{(m)} \tag{114}$$

where $\mathbf{u}$ is the velocity of the escaping (or incoming) parts in the inertial frame.

After this introduction, let us consider s simple case where the system gains some mass (mass accretion) and simultaneously looses some mass (mass ejection). Assume a volume and three objects where two of these are in the volume at t_1 and their masses are given by m_{stay} and Δm_{esc} and the third object is outside of the volume at t_1 and its mass is given by Δm_{inc}. We are interested in the system at $t_2 = t_1 + \Delta t$ and we assume the following. The body with the mass m_{stay} stays in the volume, the body with the mass Δm_{esc} is now outside of the volume and the third object with the mass Δm_{inc} is in the volume. Let $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$ denote the momentum associated to the mass content of the closed volume,

$$\mathbf{p}(t_1) = \mathbf{p}_{\text{stay}}(t_1) + \Delta \mathbf{p}_{\text{esc}}(t_1) \tag{115}$$

$$\mathbf{p}(t_2) = \mathbf{p}_{\text{stay}}(t_2) + \Delta \mathbf{p}_{\text{inc}}(t_2). \tag{116}$$

Apply Newton's second law on the system of three object which is our starting point,

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\mathbf{p}_{\text{stay}}(t_2) + \Delta \mathbf{p}_{\text{inc}}(t_2) + \Delta \mathbf{p}_{\text{esc}}(t_2) - \big(\mathbf{p}_{\text{stay}}(t_1) + \Delta \mathbf{p}_{\text{inc}}(t_1) + \Delta \mathbf{p}_{\text{esc}}(t_1)\big)\right] = \mathbf{F}. \tag{117}$$

Rearranging the terms one finds

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{\left[\mathbf{p}_{\text{stay}}(t_2) + \Delta \mathbf{p}_{\text{inc}}(t_2)\right] - \left[\mathbf{p}_{\text{stay}}(t_1) + \Delta \mathbf{p}_{\text{esc}}(t_1)\right] + \left[\Delta \mathbf{p}_{\text{esc}}(t_2) - \Delta \mathbf{p}_{\text{inc}}(t_1)\right]\right\} = \mathbf{F}. \tag{118}$$

Since the momentum change associated to all objects in the closed volume is given by $\mathrm{d}\mathbf{p} = \mathbf{p}(t_2) - \mathbf{p}(t_1)$ one finds

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}_{\text{esc}}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}_{\text{inc}}}{\Delta t} = \mathbf{F} \tag{119}$$

which results in the following expression

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} - \left(\mathbf{J}^{(\mathbf{p})}_{\text{esc}} - \mathbf{J}^{(\mathbf{p})}_{\text{inc}}\right) = \mathbf{F} - \mathbf{J}^{(\mathbf{p})} \tag{120}$$

1.14.2 Equation of motion of rockets

If there is no mass entering the system, which is the case of rocket motion, the general expression (120) can be simplified

$$\frac{\mathrm{d}m(t)\mathbf{v}}{\mathrm{d}t} = \mathbf{F} - \mathbf{J}^{(\mathbf{p})}_{\mathrm{esc}} \tag{121}$$

and then

$$\mathbf{v}\frac{\mathrm{d}m}{\mathrm{d}t} + m\mathbf{a} = \mathbf{F} - \mathbf{J}^{(\mathbf{p})}_{\mathrm{esc}}\,. \tag{122}$$

For rockets the escaping current is $J^{(m)}_{\mathrm{esc}} = -\mathrm{d}m/\mathrm{d}t$, so thus, $\mathbf{J}^{(\mathbf{p})}_{\mathrm{esc}} = \mathbf{u}J^{(m)}_{\mathrm{esc}}$, where $\mathbf{u}$ is the velocity of the mass (fuel) leaving the rocket with respect to the inertial reference frame. Therefore, one finds

$$m\mathbf{a} = \mathbf{F} + (\mathbf{u}-\mathbf{v})\frac{\mathrm{d}m}{\mathrm{d}t} \equiv \mathbf{F} + \mathbf{u}_{\mathrm{rel}}\frac{\mathrm{d}m}{\mathrm{d}t} \tag{123}$$

where $\mathbf{u}_{\mathrm{rel}} = \mathbf{u} - \mathbf{v}$ is the relative velocity of the escaping or incoming mass (fuel) with respect to the center of mass of the rocket which is considered to be constant. If we study the motion of the rocket (ideal rocket equation) where external forces can be neglected ($\mathbf{F} = 0$), the equation of motion is written as

$$m\mathbf{a} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}t} = \mathbf{u}_{\mathrm{rel}}\frac{\mathrm{d}m}{\mathrm{d}t} \tag{124}$$

which results in the Tsiolkovsky-equation

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}m} = \mathbf{u}_{\mathrm{rel}}\frac{1}{m}\,, \tag{125}$$

where the independent variable is the mass. Its solution has the form $\mathbf{v} = \mathbf{u}_{\mathrm{rel}}\ln(Cm)$ where C is the constant determined by the initial conditions. For $m = m_0$ the initial velocity is $\mathbf{v} = \mathbf{v}_0$, which means $\mathbf{v}_0 = \mathbf{u}_{\mathrm{rel}}\ln(Cm_0)$. Then the final result reads

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{u}_{\mathrm{rel}}\ln\frac{m_0}{m} \tag{126}$$

1.15 Non-inertial reference frames; pseudo-forces

A non-inertial reference frame is a frame of reference that is undergoing acceleration with respect to an inertial frame. While the laws of motion are the same in all inertial frames, in non-inertial frames, they vary from frame to frame depending on the acceleration.

1.15.1 Pseudo-forces

In classical mechanics it is often possible to explain the motion of bodies in non-inertial reference frames by introducing additional fictitious forces also called inertial forces, pseudo-forces, to Newton's second law. To discuss these pseudo-forces one can start with the kinematical relation between absolute $\mathbf{a}$ and relative $\mathbf{a}'$ accelerations,

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}_{\mathrm{sz}} + 2\vec{\omega}_{\mathrm{sz}} \times \mathbf{v}'\,, \tag{127}$$

where $\mathbf{a}$ is the absolute acceleration of the object in a reference frame V, $\mathbf{a}'$ and $\mathbf{v}'$ are the relative acceleration and velocity of the same object in a non-inertial frame V$'$, finally $\vec{\omega}_{\mathrm{sz}}$ and $\mathbf{a}_{\mathrm{sz}}$ are the angular velocity and the acceleration of the system V$'$ with respect to the inertial frame V.

In case of pure translational acceleration where the non-inertial frame V$'$ does not rotate, one finds

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}_{\mathrm{sz}}. \tag{128}$$

and by inserting it into the equation of motion which stands for the absolute acceleration given by Newton's second law,

$$m\mathbf{a} = \sum_{\mathrm{i}}\mathbf{F}_{\mathrm{i}}\,, \tag{129}$$

one finds the following expression for the relative acceleration

$$m\mathbf{a}' = \sum_{\mathrm{i}} \mathbf{F}_{\mathrm{i}} + \mathbf{F}_{\mathrm{pseudo}}\,, \qquad (\mathbf{F}_{\mathrm{pseudo}} = -m\mathbf{a}_{\mathrm{sz}})\,. \tag{130}$$

where $\mathbf{F}_{\mathrm{pseudo}} = -m\mathbf{a}_{\mathrm{sz}}$ is the translational fictitious or pseudo force. One can say that Newton's second law holds in any coordinate system provided the term 'force' is redefined to include the pseudo-forces. Important to note that $\mathbf{F}_{\mathrm{pseudo}}$ is not a real force, it has no counter-force, so, one cannot apply Newton's third law in this case.

In case of pure rotational acceleration where the non-inertial frame V′ does not have a translational acceleration, one finds

$$\mathbf{a}' = \mathbf{a} - \mathbf{a}_{\mathrm{sz}} - 2\vec{\omega}_{\mathrm{sz}} \times \mathbf{v}' = \mathbf{a} - \mathbf{a}_{\mathrm{sz}} + 2\mathbf{v}' \times \vec{\omega}_{\mathrm{sz}}\,, \tag{131}$$

then by using Newton's second law one writes

$$m\mathbf{a}' = \sum_{\mathrm{i}} \mathbf{F}_{\mathrm{i}} - m\mathbf{a}_{\mathrm{sz}} + 2m\mathbf{v}' \times \vec{\omega}_{\mathrm{sz}}\,. \tag{132}$$

where a new type of pseudo forces can be identified,

$$\mathbf{F}_{\mathrm{Co}} = 2m\mathbf{v}' \times \vec{\omega}_{\mathrm{sz}}\,. \tag{133}$$

which is the Coriolis-force. If one considers pure rotation, the acceleration of the non-inertial frame reads as $\mathbf{a}_{\mathrm{sz}} = -\omega_{\mathrm{sz}}^2 \mathbf{r} = \vec{\omega}_{\mathrm{sz}} \times \mathbf{v}_{\mathrm{sz}}$, and so the corresponding pseudo force is given by

$$\mathbf{F}_{\mathrm{cf}} = m\omega_{\mathrm{sz}}^2 \mathbf{r} = m\mathbf{v}_{\mathrm{sz}} \times \vec{\omega}_{\mathrm{sz}} \tag{134}$$

which is the centrifugal force. Therefore, the final expression of Newton's second law in this case reads as

$$m\mathbf{a}' = \sum_{\mathrm{i}} \mathbf{F}_{\mathrm{i}} + \mathbf{F}_{\mathrm{cf}} + \mathbf{F}_{\mathrm{Co}}\,. \tag{135}$$

1.15.2 Pseudo-forces on the rotating Earth

In popular usage of the term "Coriolis effect", the rotating reference frame implied is almost always the Earth. Since the Earth spins with the angular velocity ω, Earth-bound observers need to account for the centrifugal and the Coriolis force to correctly analyze the motion of objects.

The centrifugal force acts on objects which either stay at rest or have non-vanishing velocity with respect to the Earth. Its direction is always perpendicular to the axis of rotation, in particular, it acts outwards in the radial direction and its magnitude is given by $m\omega^2 R\cos\psi$. Since the centrifugal force is a bulk force like the gravitational one, it contributes to the measured weight of an object. However, this contribution is, of course very small. Even if one takes its maximum, the ratio of the centrifugal and gravitational force is

$$\frac{F_{\mathrm{cf,max}}}{mg_0} = \frac{m\omega^2 R}{mg_0} \approx \left(\frac{2\pi}{24 \cdot 3\,600\,\mathrm{s}}\right)^2 \frac{6.4 \cdot 10^6\,\mathrm{m}}{10\,\mathrm{m/s^2}} \approx 3.4 \cdot 10^{-3}\,. \tag{136}$$

where the gravitational acceleration is denoted by g_0. Although this is indeed a small contribution but if one would like to incorporate this effect, the observed weight (mg) of an object is written as,

$$mg = m\sqrt{g_0^2 + (\omega^2 R\cos\psi)^2 - 2g_0\omega^2 R\cos^2\psi} \approx mg_0\sqrt{1 - 2\omega^2 R\cos^2\psi/g_0} \tag{137}$$

which can be well approximated

$$mg \approx mg_0\left(1 - \omega^2 R\cos^2\psi/g_0\right) = m\left(g_0 - \omega^2 R\cos^2\psi\right)\,, \tag{138}$$

so thus the measured value of g is smaller than g_0

$$g \approx g_0 - \omega^2 R\cos^2\psi\,. \tag{139}$$

The Coriolis force acts on objects which have non-vanishing velocity with respect to the Earth. The Earth completes one rotation for each day/night cycle, so for motions of everyday objects the Coriolis force is usually quite small compared with other forces; its effects generally become noticeable only for motions occurring over large distances and long periods of time, such as large-scale movement of air in the atmosphere or water in the ocean; or where high precision is important, such are long range artillery or missile trajectories. Such motions are constrained by the surface of the Earth, so only the horizontal component of the Coriolis force is generally important. This force causes moving objects on the surface of the Earth to be deflected to the right (with respect to the direction of travel) in the Northern Hemisphere and to the left in the Southern Hemisphere. The horizontal deflection effect is greater near the poles, since the effective rotation rate about a local vertical axis is largest there, and decreases to zero at the equator. Rather than flowing directly from areas of high pressure to low pressure, as they would in a non-rotating system, winds and currents tend to flow to the right of this direction north of the equator (anticlockwise) and to the left of this direction south of it (clockwise). This effect is responsible for the rotation and thus formation of cyclones.

2 Angular momentum

The generalisation of Newton's laws for extended objects is straightforward where the centre of mass plays a crucial role. If we are interested in the motion of the centre of mass only, the mechanics of any extended object can be considered as a point-like mechanics where the total mass of the object is placed in its centre of mass. However, if we would like to describe the motion of all points of the extended object one has to solve a set of coupled differential equations (equation of motion for each point-like parts of the extended object). One can find a drastic simplification of this rather involved mathematical problem by choosing a different strategy based on the concept of angular momentum which is discussed in this section.

2.1 Angular momentum of a point-like particle and system of point-like particles

2.1.1 Areal velocity and the angular momentum of a point-like particle

Let us start our discussion on the angular moment with Kepler's second law of planetary motion which states that a planet sweeps out equal areas in equal times, that is, the area divided by time, called the areal velocity, is constant. In other words, a line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

In a small time Δt the planet sweeps out a small triangle having base line $\mathbf{r}$ and height $\Delta\mathbf{r} = \mathbf{v}\Delta t$ and the area is $\Delta A \approx \frac{1}{2}|\mathbf{r}\times\mathbf{v}\Delta t|$. Then the magnitude of the areal velocity is given by

$$\dot{A} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\frac{1}{2}\,|\mathbf{r}\times\mathbf{v}\Delta t| = \frac{1}{2}|\mathbf{r}\times\mathbf{v}|\,. \tag{140}$$

The vector $\mathbf{r}\times\mathbf{v}$ is perpendicular to the plane determined by the vectors $\mathbf{r}$ and $\mathbf{v}$. The planet moves in this plane, thus, one can write

$$\dot{\mathbf{A}} = \frac{1}{2}\mathbf{r}\times\mathbf{v} \tag{141}$$

which is the areal velocity vector. If one proves that the areal acceleration is zero then the areal velocity should be constant,

$$\frac{\mathrm{d}\dot{\mathbf{A}}}{\mathrm{d}t} = \frac{1}{2}(\mathbf{v}\times\mathbf{v}+\mathbf{r}\times\mathbf{a}) = \frac{1}{2}\,(\mathbf{r}\times\mathbf{a}) = \frac{1}{2m}\,(\mathbf{r}\times\mathbf{F}) = 0\,. \tag{142}$$

where we used that the force is central, thus, it is parallel to the position vector.

What if the force is not central? In the very general case one finds,

$$\frac{\mathrm{d}\dot{\mathbf{A}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{r}\times\mathbf{v}\right) = \frac{1}{2m}\,(\mathbf{r}\times\mathbf{F}) \qquad \to \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{r}\times m\mathbf{v}) = (\mathbf{r}\times\mathbf{F}) \tag{143}$$

and if the right hand side of the above equation is zero, the following quantity is conserved,

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{144}$$

which is the angular momentum of a point particle where $\mathbf{p}$ is the momentum of the particle and $\mathbf{r}$ is its position vector from the O reference point, a fixed point of the inertial reference frame. Thus angular momentum depends on the reference point and $\dim L = \mathrm{ML}^2\mathrm{T}^{-1}$ and its SI unit is $[L] = \mathrm{kg{\cdot}m^2/s}$. In addition, one can define another new quantity, the torque,

$$\mathbf{M} \equiv \mathbf{r} \times \mathbf{F} \tag{145}$$

where $\dim M = \mathrm{ML}^2\mathrm{T}^{-2}$ and its SI unit is $[M] = \mathrm{kg{\cdot}m^2/s^2}$. Thus, the angular momentum theorem reads as,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{M} \tag{146}$$

which is a collorary of Newton's second law. In the case when there are only central forces, for instance, in the case of motion of planets, the position vector from the Sun is parallel to the force and thus its torque is zero. As a result the angular momentum is conserved.

It is useful to introduce the moment of inertia

$$L = \Theta\omega\,, \tag{147}$$

where

$$\Theta = mr^2 \tag{148}$$

is a scalar quantity, the moment of inertia of a point-like particle. By using the definition of the angular acceleration, $\beta = d\omega/dt$, one can write $\Theta\boldsymbol{\beta} = \mathbf{M}$.

As an application of the angular momentum theorem let us consider the mathematical pendulum. The angular momentum of the object is $\mathbf{L} = \mathbf{r} \times (m\vec{\omega} \times \mathbf{r})$, where the length of the pendulum is r and the angular velocity is ω where $\omega = \dot{\varphi}$. Since the vectors $\mathbf{r}$ and $\vec{\omega} \equiv \omega\mathbf{k}$ are perpendicular to each other, the magnitude of the angular momentum is $L = mr^2\omega$, and its direction $\vec{\omega}||\mathbf{k}$. The force exerted by the rope is parallel to the position vector thus its torque is zero. Therefore, only the gravitational force has a non-vanishing torque, $\mathbf{r} \times (m\mathbf{g})$ with the following magnitude $M = rmg\sin\varphi$, and its direction is $\mathbf{M} = -M\mathbf{k}$. According to the angular momentum the theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t} mr^2\omega = -rmg\sin\varphi \tag{149}$$

which is exactly the known equation of motion obtained by Newton's second law.

2.1.2 Angular momentum of system of particles

Let us generalise the angular momentum for a system of particle

$$\mathbf{L} = \sum_{\mathrm{i}} \mathbf{L}_{\mathrm{i}} = \sum_{\mathrm{i}} (\mathbf{r}_{\mathrm{i}} \times m_{\mathrm{i}}\mathbf{v}_{\mathrm{i}})\,. \tag{150}$$

The equation of motion for the angular momentum of the individual particles of the system is given by

$$\frac{\mathrm{d}\mathbf{L}_{\mathrm{i}}}{\mathrm{d}t} = \mathbf{r}_{\mathrm{i}} \times \mathbf{F}_{\mathrm{i}} \tag{151}$$

where the net force $\mathbf{F}_{\mathrm{i}}$ can be decomposed into external $\mathbf{F}_{\mathrm{i}}^{(\mathrm{k})}$ and internal $\mathbf{F}_{\mathrm{ij}}^{(\mathrm{b})}$ forces,

$$\mathbf{F}_{\mathrm{i}} = \mathbf{F}_{\mathrm{i}}^{(\mathrm{k})} + \sum_{\mathrm{j}\neq\mathrm{i}} \mathbf{F}_{\mathrm{ij}}^{(\mathrm{b})} \tag{152}$$

which results in

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_{\mathrm{i}} \mathbf{r}_{\mathrm{i}} \times \mathbf{F}_{\mathrm{i}}^{(\mathrm{k})} + \sum_{\mathrm{i}} \sum_{\mathrm{j}\neq\mathrm{i}} \mathbf{r}_{\mathrm{i}} \times \mathbf{F}_{\mathrm{ij}}^{(\mathrm{b})}\,, \tag{153}$$

where the first torque on the right hand side is the torque of external forces and the second is that of internal ones.

According to Newton's third law of motion, $\mathbf{F}_{\rm ij}^{(\rm b)} = -\mathbf{F}_{\rm ji}^{(\rm b)}$,

$$\sum_{\rm i}\sum_{j\neq i}\mathbf{r}_{\rm i}\times\mathbf{F}_{\rm ij}^{(\rm b)} = \sum_{\rm i}\sum_{j>i}\left(\mathbf{r}_{\rm i}\times\mathbf{F}_{\rm ij}^{(\rm b)} + \mathbf{r}_j\times\mathbf{F}_{\rm ji}^{(\rm b)}\right) = \sum_{\rm i}\sum_{j>i}(\mathbf{r}_{\rm i}-\mathbf{r}_j)\times\mathbf{F}_{\rm ij}^{(\rm b)}\,, \tag{154}$$

where we used the following identity,

$$\begin{aligned}\sum_{\rm i=1}^{\rm n}\sum_{\substack{\rm j=1\\ \rm j\neq i}}^{\rm n}X_{\rm ij} &= \sum_{\rm i=1}^{\rm n}\left(\sum_{\rm j=1}^{\rm i-1}+\sum_{\rm j=i+1}^{\rm n}\right)X_{\rm ij} = \sum_{\rm i=1}^{\rm n}\sum_{\rm j=1}^{\rm i-1}X_{\rm ij}+\sum_{\rm i=1}^{\rm n}\sum_{\rm j=i+1}^{\rm n}X_{\rm ij} = \sum_{\rm j=1}^{\rm n}\sum_{\rm i=1}^{\rm j-1}X_{\rm ji}+\sum_{\rm i=1}^{\rm n}\sum_{\rm j=i+1}^{\rm n}X_{\rm ij}\\ &= \sum_{\rm i=1}^{\rm n}\sum_{\rm j=i+1}^{\rm n}X_{\rm ji}+\sum_{\rm i=1}^{\rm n}\sum_{\rm j=i+1}^{\rm n}X_{\rm ij} = \sum_{\rm i=1}^{\rm n}\sum_{\rm j=i+1}^{\rm n}\left(X_{\rm ij}+X_{\rm ji}\right)\end{aligned} \tag{155}$$

which holds for any quantity $X_{\rm ij}$ which depends on the indices i and j.

So, if the internal forces are central, i.e., they are parallel to the vectors $(\mathbf{r}_{\rm i}-\mathbf{r}_j)$, as is often the case, their contribution is zero and the angular momentum theorem for a system of particle reads as,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_{\rm i}\mathbf{M}_{\rm i}^{(\rm k)}\,, \tag{156}$$

where $\mathbf{M}_{\rm i}^{(\rm k)} = \mathbf{r}_{\rm i}\times\mathbf{F}_{\rm i}^{(\rm k)}$ is the torque of the external forces. However, there are forces that are not central, for instance, two magnets exert on each other equal and opposite, but non-aligned forces. Nevertheless, it is an empirical fact that when the net external torque acting on a system of particles is zero, then the total angular momentum of the system remains a constant, even if the internal forces are not central. As a result, we conclude that the net internal torque is always zero, and the net external torque acting on a system of particles is equal to the time rate of change of the total angular momentum of the system.

The angular momentum and the torque are defined with respect to a reference point O. Let us examine this dependence by choosing another point O′ and then the position vector can be written $\mathbf{r} = \mathbf{r}_{\rm O'}+\mathbf{r}'$ and

$$\sum_{\rm i}\mathbf{r}_{\rm i}'\times\mathbf{X}_{\rm i} = \sum_{\rm i}\mathbf{r}_{\rm i}\times\mathbf{X}_{\rm i} - \mathbf{r}_{\rm O'}\times\sum_{\rm i}\mathbf{X}_{\rm i}\,. \tag{157}$$

which clearly indicates that if one considers a centre of mass frame the torque and the angular momentum is independent of the particular choice of the reference point. Similarly if the vector sum of all external forces is zero, for example in the case of two parallel forces which have identical magnitude but different directions, the torque depends on the difference of the position vectors,

$$\mathbf{M}_1+\mathbf{M}_2 = \mathbf{r}_1\times\mathbf{F}_1+\mathbf{r}_2\times\mathbf{F}_2 = (\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{F}_1\,. \tag{158}$$

Another interesting case when one can define a torque but no force can be associated to this torque. One example is the torque of the torsion pendulum, $M_z = -c^*\varphi$ or the torque caused by a magnetic field on a magnetic dipole, $\mathbf{M} = \mathbf{m}\times\mathbf{B}$.

2.2 Angular momentum of rigid bodies

Angular momentum of a system of particles is defined as $\mathbf{L} = \sum_{\rm i}(\mathbf{r}_{\rm i}\times m_{\rm i}\mathbf{v}_{\rm i})$. If a rigid body is considered as a system of point-like particles then rigidity represents a constraint for these particles. Is it possible to use this constraint to simplify the rather general expression for the angular momentum? Let us examine this possibility.

2.2.1 Angular momentum with respect to a fixed axis

Assume that a rigid object is spinning around a fixed axis where the axis of rotation is chosen to be the z-axis, thus, the angular velocity ω has only a z-component ω_z. Let us calculate the z-component of the angular momentum vector,

$$L_z = \mathbf{L}\cdot\mathbf{k} = \sum_{\mathrm{i}}(\mathbf{r}_{\mathrm{i}}\times m_{\mathrm{i}}\mathbf{v}_{\mathrm{i}})\cdot\mathbf{k} = \sum_{\mathrm{i}}(\mathbf{r}_{\mathrm{i}z}\times m_{\mathrm{i}}\mathbf{v}_{\mathrm{i}})\cdot\mathbf{k} + \sum_{\mathrm{i}}(\mathbf{r}_{\mathrm{i}\perp}\times m_{\mathrm{i}}\mathbf{v}_{\mathrm{i}})\cdot\mathbf{k}\,, \tag{159}$$

where the position vector $\mathbf{r}_{\mathrm{i}}$ is decomposed into a perpendicular $\mathbf{r}_{\mathrm{i}\perp}$ and a parallel $\mathbf{r}_{\mathrm{i}z}$ combination, $\mathbf{r}_{\mathrm{i}} = \mathbf{r}_{\mathrm{i}\perp} + \mathbf{r}_{\mathrm{i}z}$. The first term of the right hand side of (159) vanishes because the vectors $(\mathbf{r}_{\mathrm{i}z}\times m_{\mathrm{i}}\mathbf{v}_{\mathrm{i}})$ and $\mathbf{k}$ are perpendicular to each other and consequently their scalar product is zero. The vectors $\mathbf{r}_{\mathrm{i}\perp}$ and $\mathbf{v}_{\mathrm{i}} = \vec{\omega}\times\mathbf{r}_{\mathrm{i}\perp}$ are perpendicular to each other and the magnitude of their vector product is parallel to $\mathbf{k}$ with the magnitude $r_{\mathrm{i}\perp}^2\omega$, thus one can write,

$$L_z = \Theta^{(z)}\omega_z\,, \tag{160}$$

where

$$\Theta^{(z)} = \sum_{\mathrm{i}} m_{\mathrm{i}} r_{\mathrm{i}\perp}^2 \tag{161}$$

which is the moment of inertia of the rigid body with respect to the z-axis.
The expression (161) can be generalised for the continuous case

$$\Theta = \int_m r_\perp^2\,\mathrm{d}m = \int_V \rho(\mathbf{r}) r_\perp^2\,\mathrm{d}V \tag{162}$$

where the lower index m and the lower index V indicate that the integration is understood over the total mass and the whole volume of the object.
The unit of moment of inertia is $\dim\Theta = \mathrm{ML}^2$ and it has a general expression $\Theta = Cmr^2$ where m is the total mass of the object and r is some geometric parameter perpendicular to the axis of rotation, for example the radius of a sphere or cylinder. C is a dimensionless number which can be calculated as

$$C = \frac{1}{r^2 V}\int_V r_\perp^2\,\mathrm{d}V\,. \tag{163}$$

where we assumed a homogeneous mass distribution, $m = \rho V$.

As an example let us calculate the moment of inertia of a cylinder with respect to its symmetry axis where we assume homogeneous mass distribution. Instead of using cartesian coordinate x, y and z where $r_\perp^2 = x^2 + y^2$, it is more convenient to switch to cylindrical coordinates r, φ and z where $r_\perp^2 = r^2$. In this case the integration measure is $\mathrm{d}V = r\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\varphi$ with the following bounderies

$$0 \le r \le R\,; \qquad 0 \le \varphi \le 2\pi\,; \qquad 0 \le z \le l\,, \tag{164}$$

where R is the radius and l is the length of the cylinder. Its total volume is $V = R^2\pi\, l$, thus,

$$C = \frac{1}{R^4\pi\, l}\int_0^R r^3\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\varphi\int_0^l \mathrm{d}z = \frac{2\pi l}{R^4\pi\, l}\int_0^R r^3\,\mathrm{d}r = \frac{2\pi l}{R^4\pi\, l}\left[\frac{r^4}{4}\right]_0^R = \frac{2\pi l}{R^4\pi\, l}\frac{R^4}{4} = \frac{1}{2}. \tag{165}$$

Similar procedure can be applied to determine the moment of inertia for other object. For example, the moment of inertia of a ball with respect to one of its symmetry axis is $\frac{2}{5}mR^2$ where R is its radius and m is its total mass. Important to note that the moment of inertia is an additive scalar, so, for example the moment of inertia of a half ball is $\frac{1}{5}mR^2$.

2.2.2 Parallel axis theorem

The parallel axis theorem, also known as Huygens-Steiner theorem, or just as Steiner's theorem, named after Christiaan Huygens and Jakob Steiner, can be used to determine the moment of inertia or the second moment of area of a rigid body about any axis, given the body's moment of inertia about a parallel axis through the object's center of gravity and the perpendicular distance between the axes.

Suppose a body of mass m is rotated about an axis z passing through the body's center of mass. The body has a moment of inertia Θ_{c} with respect to this axis. The parallel axis theorem states that if the body is made to rotate instead about a new axis z', which is parallel to the first axis and displaced from it by a distance s, then the moment of inertia Θ_P with respect to the new axis is related to Θ_c by $\Theta_\mathrm{P} = \Theta_\mathrm{c} + ms^2$. Its derivation is

$$\begin{aligned}\Theta_\mathrm{P} &= \sum_\mathrm{i} m_\mathrm{i}(x_\mathrm{i}^2 + y_\mathrm{i}^2) = \sum_\mathrm{i} m_\mathrm{i}((x'_\mathrm{i} + s)^2 + y'^2_\mathrm{i}) \\ &= \sum_\mathrm{i} m_\mathrm{i}(x'^2_\mathrm{i} + y'^2_\mathrm{i}) + \sum_\mathrm{i} m_\mathrm{i} s^2 + \sum_\mathrm{i} m_\mathrm{i} 2x'_\mathrm{i} s = \Theta_\mathrm{c} + ms^2 \end{aligned} \tag{166}$$

where it is assumed, without loss of generality, that in a Cartesian coordinate system the perpendicular distance between the axes lies along the x-axis and that the center of mass lies at the origin. In addition, x' and y' are coordinates of the center of mass system and $x = x' + s$. We used that $\left(\sum_\mathrm{i} m_\mathrm{i} x'_\mathrm{i}\right)/m$ is the coordinate of the position vector to the center of mass in the center of mass frame which is zero. By using the Stener's theorem, one can calculate the moment of inertia of a cylinder about an axis going through the cylinder wall (and being parallel to its symmetry axis), $\frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2$.

According ot Eq. (156) if the torque is zero the angular moment is conserved, $\mathbf{L}$ = constant, thus,

$$\Theta^{(z)}\omega_z = \text{const.} \tag{167}$$

Therefore, if the moment of inertia has been changed for some reason like in case of collapsing stars, the new angular velocity should fulfil the following expression,

$$\Theta^{(z)}_\mathrm{old}\ \omega_\mathrm{old} = \Theta^{(z)}_\mathrm{new}\ \omega_\mathrm{new} \tag{168}$$

2.2.3 Angular momentum of a rigid body with respect to a reference point

While velocity and linear momentum always show in the same direction, angular velocity and angular momentum are, in general, not parallel. As an example, let us consider a rigid body consisting of a vertical shaft rotating freely between two (an upper and a lower) bearings and an arm of length r' welded perpendicularly to the shaft. We assume both the shaft and the arm massless. At the end of the arm there is a pointlike particle of mass m. Let us fix the reference point O to the lower bearing that is at distance r from the particle. If the rigid body rotates, its angular velocity is parallel to the shaft, while the angular momentum is perpendicular to the position vector of the particle, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and thus, $\vec{\omega} = \omega_z\mathbf{k}$ and $\mathbf{L}$ are not parallel. To show this, let us calculate

$$\mathbf{v} = \vec{\omega} \times \mathbf{r} = \omega_z\mathbf{k} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -\omega_z y\mathbf{i} + \omega_z x\mathbf{j}\,. \tag{169}$$

and then $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ gives

$$L_x = -mz\omega_z x\,, \quad L_y = -mz\omega_z y\,, \quad L_z = m(x^2 + y^2)\omega_z \tag{170}$$

which demonstrates that the angular momentum has parallel $\mathbf{L}_z$ and perpendicular $\mathbf{L}_\perp$ components while the angular velocity is parallel to the z-axis. In addition, since the perpendicular component of the angular momentum vector depends on time, $\mathbf{L}_\perp = -mz\omega_z\mathbf{r}_\perp(t)$, a non-vanishing torque is required

$$\dot{\mathbf{L}} = \dot{\mathbf{L}}_\perp = \vec{\omega} \times \mathbf{L}_\perp = \vec{\omega} \times \mathbf{L}\,, \tag{171}$$

where we used $\dot{\mathbf{r}} = \vec{\omega} \times \mathbf{r}$.

We can extend the above argument to the discussion of rotation of an arbitrary rigid body. Le us assume that $\vec{\omega} = \omega_z\mathbf{k}$,

$$L^{(z)}_x = \Theta_{xz}\omega_z\,, \quad L^{(z)}_y = \Theta_{yz}\omega_z\,, \quad L^{(z)}_z = \Theta_{zz}\omega_z\,, \tag{172}$$

where the upper index indicates that the angular velocity is parallel to the z-axis and

$$\Theta_{xz} = -\sum_\mathrm{i} m_\mathrm{i} x_\mathrm{i} z_\mathrm{i}\,, \quad \Theta_{yz} = -\sum_\mathrm{i} m_\mathrm{i} x_\mathrm{i} y_\mathrm{i}\,, \quad \Theta_{zz} = \sum_\mathrm{i} m_\mathrm{i}(x_\mathrm{i}^2 + y_\mathrm{i}^2)\,. \tag{173}$$

Let us pay the attention of the reader that Θ_{zz} is the moment of inertia Eq. (161) introduced previously when we discussed the rotation of a rigid body around a fixed axis and we computed the moment of inertia with respect to the rotation axis. Θ_{xz} and Θ_{yz} are moments of inertia which show that the angular momentum and the angular velocity (which has only z-component) are not parallel. One can consider the most general case, where the angular velocity has three components, $\vec{\omega}_x$, $\vec{\omega}_y$ and $\vec{\omega}_z$. In this case the three components of the angular momentum reads as

$$L_x = L_x^{(x)} + L_x^{(y)} + L_x^{(z)} = \Theta_{xx}\omega_x + \Theta_{xy}\omega_y + \Theta_{xz}\omega_z \,, \tag{174}$$

$$L_y = L_y^{(x)} + L_y^{(y)} + L_y^{(z)} = \Theta_{yx}\omega_x + \Theta_{yy}\omega_y + \Theta_{yz}\omega_z \,, \tag{175}$$

$$L_z = L_z^{(x)} + L_z^{(y)} + L_z^{(z)} = \Theta_{zx}\omega_x + \Theta_{zy}\omega_y + \Theta_{zz}\omega_z \,, \tag{176}$$

which can be summarised in the following expression

$$\mathbf{L} = \hat{\Theta}\vec{\omega}. \tag{177}$$

where $\hat{\Theta}$ is the moment of inertia tensor. The moment of inertia tensor is a convenient way to summarise all moments of inertia of an object with one quantity. It may be calculated with respect to any point in space. The representation of the moment of inertia tensor in a particular frame of reference is the moment of inertia matrix,

$$\hat{\Theta} \to \begin{pmatrix} \Theta_{xx} & \Theta_{xy} & \Theta_{xz} \\ \Theta_{yx} & \Theta_{yy} & \Theta_{yz} \\ \Theta_{zx} & \Theta_{zy} & \Theta_{zz} \end{pmatrix} , \tag{178}$$

which is symmetric by the definition, thus,

$$\Theta_{xy} = -\sum_{\mathrm{i}} m_{\mathrm{i}} x_{\mathrm{i}} y_{\mathrm{i}} = -\sum_{\mathrm{i}} m_{\mathrm{i}} y_{\mathrm{i}} x_{\mathrm{i}} = \Theta_{yx} \,. \tag{179}$$

Therefore, it has only 6 (instead of nine) independent elements.

Since the moment of inertia matrix is real symmetric, thus, it can be diagonalised or in other words, a real symmetric matrix has the eigendecomposition into the product of a rotation matrix and a diagonal matrix. This means one can always find a reference frame where it is diagonal. This reference frame is called the principal axis frame where the the columns of the rotation matrix define the directions of the principal axes of the body, and the elements of the diagonal moment of inertia matrix, Θ_{11}, Θ_{22}, Θ_{33} are the principal moments of inertia. The highest and the lowest moments of inertia are among the principal moments. The principal axis with the highest moment of inertia is sometimes called the figure axis. When all principal moments of inertia are distinct, the principal axes through center of mass are uniquely specified. If two principal moments are the same, the rigid body is called a symmetrical top and there is no unique choice for the two corresponding principal axes. If all three principal moments are the same, the rigid body is called a spherical top (although it need not be spherical) and any axis can be considered a principal axis, meaning that the moment of inertia is the same about any axis. The principal axes are often aligned with the object's symmetry axes. If a rigid body has at l east two symmetry axes that are not parallel or perpendicular to each other, it is a spherical top.

The motion of vehicles is often described in terms of yaw, pitch, and roll which usually correspond approximately to rotations about the three principal axes. If the vehicle has bilateral symmetry then one of the principal axes will correspond exactly to the transverse (pitch) axis. A practical example of this mathematical phenomenon is the routine automotive task of balancing a tire, which basically means adjusting the distribution of mass of a car wheel such that its principal axis of inertia is aligned with the axle so the wheel does not wobble.

Let us denote the unit vectors of the principal axis frame by $\mathbf{e}_{\mathrm{i}}$ (i = 1, 2, 3) and the principal axis by $x_s 1$, x_2 and x_3 where the principal moments of inertia are Θ_{11}, Θ_{22}, Θ_{33}, thus one can write

$$L_{\mathrm{i}} = \Theta_{\mathrm{ii}}\omega_{\mathrm{i}} \,, \qquad i = 1,\ 2,\ 3 \,. \tag{180}$$

Let the body rotate around an arbitrarily chosen axis given by the Euler angles α_{i} ($i = 1, 2, 3$) and its unit vector is denoted by $\mathbf{e}_{\mathrm{t}}$. In this case, $\vec{\omega} = \omega_{\mathrm{t}}\mathbf{e}_{\mathrm{t}}$. One can write

$$\cos\alpha_{\mathrm{i}} = \mathbf{e}_{\mathrm{t}} \cdot \mathbf{e}_{\mathrm{i}}\,, \qquad \rightarrow \qquad \omega_{\mathrm{i}} = \vec{\omega} \cdot \mathbf{e}_{\mathrm{i}} = \omega_{\mathrm{t}} \cos\alpha_{\mathrm{i}}\,, \qquad \rightarrow \qquad \mathbf{L} = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3 \tag{181}$$

and the component of the angular momentum with respect to this arbitrary direction is

$$\begin{aligned} L_{\mathrm{t}} &= \mathbf{L} \cdot \mathbf{e}_{\mathrm{t}} = L_1 \cos\alpha_1 + L_2 \cos\alpha_2 + L_3 \cos\alpha_3 = \Theta_{11}\omega_1 \cos\alpha_1 + \Theta_{22}\omega_2 \cos\alpha_2 + \Theta_{33}\omega_3 \cos\alpha_3 \\ &= \left(\Theta_{11} \cos^2\alpha_1 + \Theta_{22} \cos^2\alpha_2 + \Theta_{33} \cos^2\alpha_3\right) \omega_t\,. \end{aligned} \tag{182}$$

Since $L_{\mathrm{t}} = \Theta^{(\mathrm{t})}\omega_{\mathrm{t}}$, one can conclude,

$$\Theta^{(\mathrm{t})} = \Theta_{11} \cos^2\alpha_1 + \Theta_{22} \cos^2\alpha_2 + \Theta_{33} \cos^2\alpha_3\,. \tag{183}$$

2.2.4 Solving the equation of motion for rigid bodies

The most general motion of a rigid body has six degrees of freedom: three of the translational motion of one of its points, let it be P, and three of the orientation of the body around this point P. It is natural to choose P the center of mass, denoted by C. The motion of the center of mass is described by Newton's second law that can be solved using the algorithm,

$$\begin{tikzcd} \mathbf{F} \arrow[r] & \Delta\mathbf{p} \arrow[r] & \mathbf{p} \arrow[r] & \mathbf{v} \arrow[r] \arrow[lll, bend left] & \Delta\mathbf{r} \arrow[r] & \mathbf{r} \arrow[lllll, bend left] \end{tikzcd} \tag{184}$$

The orientation around the center of mass can be described by solving the equation of motion for the angular momentum vector. It follows the same algorithm as that for a point particle,

$$\begin{tikzcd} \mathbf{M} \arrow[r] & \Delta\mathbf{L} \arrow[r] & \mathbf{L} \arrow[r] & \vec{\omega} \arrow[r] \arrow[lll, bend left] & \overrightarrow{\Delta\varphi} \arrow[r] & \tilde{\varphi} \arrow[lllll, bend left] \arrow[dl] \\ & & & & \hat{\Theta} \arrow[ull] & \end{tikzcd} \tag{185}$$

where $\tilde{\varphi}$ represents the angles that describe the orientation of the rigid body, the angular position. However, it is important to note that one cannot get the step $\mathbf{L} \rightarrow \vec{\omega}$ by simple division with the moment of inertia because in general it is a tensor. If the axis of rotation is fixed, or the body rotates around an axis that is an axis of rotational symmetry of the body, than this step is straightforward. In addition, one has to recalculate the moment of inertia matrix in every Δt steps of the algorithm, since it depends on the particular configuration.

2.2.5 Equilibrium of rigid bodies, equivalent substitution of forces

A pointlike object is in equilibrium (or in more general terms: it performs inertial motion) if the net external force acting on it is zero. Under the same condition the center of mass of an extended object also performs inertial motion. We now extend the notion of inertial motion to all other points of a rigid body: If there exists an inertial reference frame where each point of a rigid body is at rest and the forces acting on this body are such that all points remain at rest, then we say that the rigid body performs inertial motion. Thus, the inertial motion of a rigid body has a kinematical condition and two dynamical conditions, $\sum_{\mathrm{i}} \mathbf{F}_{\mathrm{i}} = 0$ and $\sum_{\mathrm{i}} \mathbf{M}_{\mathrm{i}} = 0$.

If a force acting on a body is represented (or replaced) by another force or a force-moment system (at a different point on the body) such that the resulting rigid-body effects (i.e., translation and rotation) remain unchanged, i.e.,

$$\sum_{\mathrm{i}} \mathbf{F}_{\mathrm{i}} = \sum_{j} \mathbf{F}'_j\,, \qquad \sum_{\mathrm{i}} \mathbf{M}_{\mathrm{i}} = \sum_{j} \mathbf{M}'_j\,, \tag{186}$$

the two systems are said to be equivalent. We are interested in this concept because in many problems, it may be more convenient to replace the existing force with another equivalent force or force-moment system. For example, the force can be shifted along its line of action if we consider rigid bodies (and do not interested in the deformation).

2.2.6 Equivalent substitution of weight

Let us consider the weight of an extended object. It has been discussed that for a pointlike particle with a mass m_i the weight is given by $m_i\mathbf{g}$ where $\mathbf{g}$ is the gravitational acceleration. Assume that the extended object is divided into pointlike parts and we wish to substitute the weight of these parts by a single force. According to the rules of equivalent substitution one can write,

$$\sum_{\mathrm{i}} \Delta m_i \mathbf{g} = m\mathbf{g} \tag{187}$$

where m is the total mass. In addition one should write,

$$\sum_{\mathrm{i}} \mathbf{r}_i \times \Delta m_i \mathbf{g} = \sum_{\mathrm{i}} \mathbf{r}_i \Delta m_i \times \mathbf{g} = \sum_{\mathrm{i}} \frac{\mathbf{r}_i \Delta m_i}{m} \times m\mathbf{g} = \mathbf{r}_c \times m\mathbf{g} \tag{188}$$

which indicates that the equivalent substitution is indeed $m\mathbf{g}$.

2.2.7 Equation of motion for of rotation

For a rigid body rotating around the z axis the equation of motion is

$$\Theta^{(z)}\beta_z = \mathbf{M}\cdot\mathbf{k} = M_z \tag{189}$$

which can be generalised for a rotation around an arbitrary axis given by the unit vector $\mathbf{e}_{\mathrm{t}}$,

$$\Theta^{(\mathrm{t})}\beta_{\mathrm{t}} = \mathbf{M}\cdot\mathbf{e}_{\mathrm{t}} = M_{\mathrm{t}} \tag{190}$$

It is useful to discuss the magnitude of the torque acting about an axis which is directly proportional to the distance of the force from the axis. It is defined as the product of the perpendicular component of the force ($F_\perp$) and the moment arm (k), so $M^{(\mathrm{t})} = \pm kF_\perp$ where the moment arm is the perpendicular distance between the line of action of the force and the center of moments,

$$\begin{aligned} M_{\mathrm{t}} =& [(\mathbf{r}_\perp + \mathbf{r}_{\mathrm{t}}) \times (\mathbf{F}_\perp + \mathbf{F}_{\mathrm{t}})]\cdot\mathbf{e}_{\mathrm{t}} \\ =& (\mathbf{r}_\perp \times \mathbf{F}_\perp)\cdot\mathbf{e}_{\mathrm{t}} + (\mathbf{r}_{\mathrm{t}} \times \mathbf{F}_\perp)\cdot\mathbf{e}_{\mathrm{t}} + (\mathbf{r}_\perp \times \mathbf{F}_{\mathrm{t}})\cdot\mathbf{e}_{\mathrm{t}} + (\mathbf{r}_{\mathrm{t}} \times \mathbf{F}_{\mathrm{t}})\cdot\mathbf{e}_{\mathrm{t}} = (\mathbf{r}_\perp \times \mathbf{F}_\perp)\cdot\mathbf{e}_{\mathrm{t}}\,, \end{aligned} \tag{191}$$

since the vector $\mathbf{r}_\perp \times \mathbf{F}_\perp$ is parallel to $\mathbf{e}_{\mathrm{t}}$

$$|M_{\mathrm{t}}| = |(\mathbf{r}_\perp \times \mathbf{F}_\perp)\cdot\mathbf{e}_{\mathrm{t}}| = |\mathbf{r}_\perp \times \mathbf{F}_\perp|\,. \tag{192}$$

In addition, $|\mathbf{r}_\perp \times \mathbf{F}_\perp| = r_\perp\ F_\perp \sin\alpha = kF_\perp$, with the moment arm $k = r_\perp \sin\alpha$ where α is the angle between the vectors $\mathbf{r}_\perp$ and $\mathbf{F}_\perp$ and $M^{(\mathrm{t})} = M_{\mathrm{t}}$, thus one finds,

$$\Theta^{(\mathrm{t})}\beta = M^{(\mathrm{t})} \tag{193}$$

which is the equation of motion for rotation around a fixed axis.

A simple application of the above equation of motion is the case of the torsion pendulum where the torque is given by $M_t = -c^*\varphi$, so, the angular acceleration $\beta = \ddot{\varphi}$ can be calculated by the equation of motion

$$\ddot{\varphi} = -\frac{c^*}{\Theta^{(\mathrm{t})}}\,\varphi \tag{194}$$

which is the well-known differential equation of the harmonic oscillator having the solution $\varphi(t) = \varphi_0\ \sin(\omega t + \phi)$ where ω is given

$$\omega = \sqrt{\frac{c^*}{\Theta^{(\mathrm{t})}}}\,. \tag{195}$$

Another example is the compound pendulum. A compound pendulum is a body formed from an assembly of particles of continuous shape that rotates rigidly around a pivot. Let us assume that a rigid object is rotating around a horizontal axis going through an arbitrary point of the rigid object which is not the centre of mass. The distance between this point and the centre of mass is denoted by s. In this case the torque caused by the gravitational force (weight) is given by $M_z = -mgs\sin\varphi \approx -mgs\,\varphi$ where M_z and φ has opposite signs. The equation of motion reads

$$\ddot{\varphi} = -\frac{mgs}{\Theta^{(\mathrm{t})}}\,\varphi \tag{196}$$

with a solution $\varphi(t) = \varphi_0\,\sin(\omega t + \phi)$ where the natural frequency of the compound pendulum is

$$\omega = \sqrt{\frac{mgs}{\Theta^{(\mathrm{t})}}}\,. \tag{197}$$

thus, $T = 2\pi\sqrt{\Theta^{(\mathrm{t})}/(mgs)}$.

Our last example is Atwood's machine (fixed pulley). Let us assume that the system is released from rest (assuming that the string does not stretch or slip) and that the friction of the pulley is negligible. Our goal is to find the linear acceleration of the blocks attached to both sides of the pulley and the angular acceleration of the pulley. In this case the set of equations of motion is given

$$m_1 a = m_1 g - K_1\,, \tag{198}$$

$$m_2 a = K_2 - m_2 g\,, \tag{199}$$

$$\Theta\beta = rK_1 - rK_2\,, \tag{200}$$

where Θ is the moment of inertia with respect to the rotational axis. Since we assume that the string does not stretch or slip, the kinematic condition $a = r\beta$ holds and the third equatin can be written as $K_1 - K_2 = \Theta a/r^2$. Then the solution of the differential equations is $K_1 = m_1(g-a)$, $K_2 = m_2(g+a)$, ahol $a = g(m_1 - m_2)/(m_1 + m_2 + \Theta/r^2)$.

2.2.8 Planar rigid body dynamics; orbital and spin angular momenta

If a system of particles (or rigid body) moves parallel to a fixed plane, the system is said to be constrained to planar movement. In this case, the dynamics for a rigid system of N particles, simplify because there is no movement in the direction perpendicular to the plane. To study this simplified motion let us first decompose the position vector $\mathbf{r}_\mathrm{i} = \mathbf{r}_\mathrm{c} + \mathbf{r}'_\mathrm{i}$, and the velocity vector $\mathbf{v}_\mathrm{i} = \mathbf{v}_\mathrm{c} + \mathbf{v}'_\mathrm{i}$ where the vectors $\mathbf{r}'_\mathrm{i}$ and $\mathbf{v}'_\mathrm{i}$ are understood as the position and velocity in the center of frame. In this case, the total angular momentum of the system is

$$\mathbf{L} = \mathbf{r}_\mathrm{c} \times \left(\sum_\mathrm{i} m_\mathrm{i}\right)\mathbf{v}_\mathrm{c} + \left(\sum_\mathrm{i} m_\mathrm{i}\mathbf{r}'_\mathrm{i}\right) \times \mathbf{v}_\mathrm{c} + \mathbf{r}_\mathrm{c} \times \left(\sum_\mathrm{i} m_\mathrm{i}\mathbf{v}'_\mathrm{i}\right) + \sum_\mathrm{i}\left(\mathbf{r}'_\mathrm{i} \times m_\mathrm{i}\mathbf{v}'_\mathrm{i}\right)\,. \tag{201}$$

where only the first and the last terms are non-vanishing this is because $\sum_\mathrm{i} m_\mathrm{i}\mathbf{r}'_\mathrm{i}$ is the position vector and $\sum_\mathrm{i} m_\mathrm{i}\mathbf{v}'_\mathrm{i}$ is the total momentum of the center of mass in the center of mass frame which are zero. The first term simplifies as

$$\mathbf{L}_\mathrm{p} = \mathbf{r}_\mathrm{c} \times m\mathbf{v}_\mathrm{c}\,. \tag{202}$$

which is the orbital angular momentum. The last term reads

$$\mathbf{L}_\mathrm{s} = \sum_\mathrm{i}\left(\mathbf{r}'_\mathrm{i} \times m_\mathrm{i}\mathbf{v}'_\mathrm{i}\right)\,. \tag{203}$$

which is the spin angular momentum. The total angular momentum is the sum of these,

$$\mathbf{L} = \mathbf{L}_\mathrm{p} + \mathbf{L}_\mathrm{s}\,. \tag{204}$$

The advantage of the decomposition of the total angular momentum into orbital and spin angular momenta is the equation of motion can also be separated. Let us first consider the equation of motion for the orbital angular momentum,

$$\frac{\mathrm{d}\mathbf{L}_\mathrm{p}}{\mathrm{d}t} = \mathbf{r}_\mathrm{c} \times m\frac{\mathrm{d}\mathbf{v}_\mathrm{c}}{\mathrm{d}t} = \mathbf{r}_\mathrm{c} \times \mathbf{F}^{(\mathrm{e})}\,, \tag{205}$$

which is basically Newton's second law for the center of mass of the rigid body. The equation of motion for the total momentum reads,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_{\mathrm{i}} (\mathbf{r}_{\mathrm{c}} + \mathbf{r}'_{\mathrm{i}}) \times \mathbf{F}_{\mathrm{i}}^{(\mathrm{k})} = \mathbf{r}_{\mathrm{c}} \times \mathbf{F}_{\mathrm{i}}^{(\mathrm{e})} + \sum_{\mathrm{i}} \mathbf{r}'_{\mathrm{i}} \times \mathbf{F}_{\mathrm{i}}^{(\mathrm{k})}\,, \tag{206}$$

and one can also write

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{L}_{\mathrm{p}}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{L}_{\mathrm{s}}}{\mathrm{d}t} = \mathbf{r}_{\mathrm{c}} \times \mathbf{F}^{(\mathrm{e})} + \frac{\mathrm{d}\mathbf{L}_{\mathrm{s}}}{\mathrm{d}t}\,. \tag{207}$$

thus, one obtains

$$\frac{\mathrm{d}\mathbf{L}_{\mathrm{s}}}{\mathrm{d}t} = \sum_{\mathrm{i}} \mathbf{r}'_{\mathrm{i}} \times \mathbf{F}_{\mathrm{i}}^{(k)} \equiv \mathbf{M}_{\mathrm{c}} \tag{208}$$

which is the equation of motion for the spin angular momentum where where $\mathbf{M}_{\mathrm{c}}$ is the torque about an axis perpendicular to the movement of the rigid system and through the center of mass. Important to note that the above expression holds also if the center of mass has a non-inertial motion.

2.2.9 Motion on a slope

In case of the motion of rigid bodies in a plane there are three degrees of freedom, which are two coordinates of the center of mass in the plane and an angle describing the rotation around an axis perpendicular to the plane. The three equations of motion are

$$ma_x = \sum_i F_{i\,x} \tag{209}$$

$$ma_y = \sum_i F_{i\,y} \tag{210}$$

$$\Theta^{(t_C)}\beta_z = \sum_i M_{i\,z}^{(t_C)}\,. \tag{211}$$

The first two equations describe the motion of the center of mass and the third one is the equation of motion for the rotation around the t_C axis. It can be shown that the motion can always be considered for a short time interval as a pure rotation around the *instantaneous axis of rotation* t_I. For instance, if a wheel rolls on a surface without slipping, then the instantaneous axis of rotation is the line where the surface touches the wheel. If we are not interested in the forces of constraints, then for the description of the motion, we can solve the single equation

$$\Theta^{(t_I)}\beta_z = \sum_i M_{i\,z}^{(t_I)}\,. \tag{212}$$

As an example, let us consider the motion of a cylinder on a slope (or incline) described by the angle α. An incline (or slope) is an ideal arrangement to realize accelerated rolling motion. Let us use a frame where the x-axis is parallel and the y-axis is perpendicular to the incline. Force due to gravity acts through the center of mass of the rolling body. When a body rolls down, it has linear acceleration in downward direction. The friction, therefore, acts upward to counter sliding tendency. This friction constitutes an anticlockwise torque,

$$ma = mg\sin\alpha - F_{\mathrm{S}}\,, \tag{213}$$

$$0 = mg\cos\alpha - F_{\mathrm{n}}\,, \tag{214}$$

$$\Theta\beta = RF_{\mathrm{S}}\,, \tag{215}$$

where the moment of inertia is $\Theta = CmR^2$, where C is a number and it depends on whether we consider a solid cylinder or a ring. The condition of rolling (without slipping) is $a = R\beta$, thus

$$a = \frac{g\sin\alpha}{1+C}\,, \tag{216}$$

$$F_{\mathrm{S}} = mg\sin\alpha\,\frac{C}{1+C}\,. \tag{217}$$

Let us compare the case of the solid cylinder $C_{\text{cylinder}} = 1/2$ and the ring $C_{\text{ring.}} = 1$ which gives $a_{\text{cylinder}}/a_{\text{ring.}} = (1 + C_{\text{ring}})/(1 + C_{\text{cylinder}}) = 4/3$. Once can also calculate the condition of rolling (without slipping) which is

$$mg \sin\alpha\, C/(1 + C) \leq \mu_0 mg \cos\alpha\,, \tag{218}$$

which gives $\mathrm{tg}\,\alpha \leq \mu_0(1 + 1/C)$.

2.2.10 Spinning top

A typical example of a rotating rigid body is a spinning top. The top has three degrees of freedom. It has a fixed point and the orientation of the top around that point is arbitrary.

Let us first consider the case of the torqueless spinnig top with the following principal moments of inertia $\Theta_{11} = \Theta_{22}$, and Θ_{33}. Let the spinning top rotates around an arbitrary axis where the angular moment can be decomposed into a parallel and a perpendicular combination according to the symmetry axis,

$$L_3 = \Theta_{33}\omega_3 \qquad L_\perp = \Theta_{11}\omega_\perp\,, \tag{219}$$

where $\omega_\perp$ is the perpendicular component of the angular velocity. Important observation is that the angular velocity, the symmetry axis and the angular momentum are in the same plane. Assume, $\Theta_{33} > \Theta_{11}$, in this case

$$\mathrm{tg}\,\beta = \frac{L_3}{L_\perp} = \frac{\Theta_{33}\omega_3}{\Theta_{11}\omega_\perp} = \frac{\Theta_{33}}{\Theta_{11}}\mathrm{tg}\,\alpha > \mathrm{tg}\,\alpha. \tag{220}$$

Since the spinning top is torqueless, the angular momentum is constant, thus, the symmetry axis and the angular velocity are orbiting around the angular momentum. This is called the nutation.

Let us consider the case of the torque induced precession. In this case the axis of rotation, which coincides with the symmetry axis of the top, rotates relatively slowly around the z axis. This motion is called precession. The top interacts with two external bodies. One is the Earth exerting the gravitational pull, the other is the table on which the top resides. Choosing the reference point to be the tip of the top, the only extrenal torque is exerted by gravity,

$$\boldsymbol{M} = \sum_i \boldsymbol{r}_i \times \Delta m_i \boldsymbol{g} = (\sum_i \boldsymbol{r}_i \Delta m_i) \times \boldsymbol{g} = \boldsymbol{r}_c \times m\boldsymbol{g}\,. \tag{221}$$

The magnitude of the torque is

$$M = mgr \sin\alpha\,, \tag{222}$$

where α is the angle between the z axis and the symmetry axis of the top. In a short time interval Δt, the change in the angular momentum of the top is

$$\Delta\boldsymbol{L} = \boldsymbol{M}\Delta t\,. \tag{223}$$

Furthermore, if the angular velocity of the precession is ω_P, then in a short time interval

$$\Delta\phi = \omega_P \Delta t\,, \tag{224}$$

and, using geometric considerations, the same angle can be expressed as

$$\Delta\phi = \frac{\Delta L}{L \sin\alpha}\,. \tag{225}$$

From these equations we obtain ω_P as

$$\omega_P = \frac{mgr}{L} \approx \frac{mgr}{\Theta_{33}\omega_3}\,. \tag{226}$$

Considering the directions of the vectors, it is easy to show that

$$\boldsymbol{M} = \vec{\omega}_P \times \boldsymbol{L}\,. \tag{227}$$

3 Energy

3.1 Collisions

There certain situations when it is difficult to represent the effect of the environment on a body by forces. An example is collisions, when the the duration of the interaction (the collision) is seemingly instantaneous. Careful examination of the process shows that actually the forces acting between colliding bodies can be considered elastic elastic forces, but the quantitative description of the force is difficult. Without knowing the forces explicitly, all we know so far is that in pure two-body interactions the total momentum is conserved. However, that is not sufficient to predict the velocities of the colliding bodies after collision expressed as functions of the velocities before collision.

If we describe the collision in the centre-of-mass (cm) reference frame of the two bodies, then we know that the total momentum is zero: $m_\mathrm{A}\mathbf{w}_{\mathrm{A},b} + m_\mathrm{B}\mathbf{w}_{\mathrm{B},b} = 0$. (In the following, we shall denote velocities by $\mathbf{v}$ in the laboratory frame V, and by $\mathbf{w}$ in the cm frame. Furthermore, we use index b to denote velocities *before* and a to denote those *after* collision.) As total momentum is conserved, we know that $m_\mathrm{A}\mathbf{w}_{\mathrm{A},a} + m_\mathrm{B}\mathbf{w}_{\mathrm{B},a} = 0$. Clearly, from these equations we can only predict the ratio of the speeds,

$$\frac{w_{\mathrm{A},b}}{w_{\mathrm{B},b}} = \frac{m_\mathrm{B}}{m_\mathrm{A}} = \frac{w_{\mathrm{A},a}}{w_{\mathrm{B},a}}\,. \tag{228}$$

From these equations it follows that

$$\frac{w_{\mathrm{A},a}}{w_{\mathrm{A},b}} = \frac{w_{\mathrm{B},a}}{w_{\mathrm{B},b}} \equiv r\,, \tag{229}$$

meaning that the speeds increase ($r > 1$), decrease ($r < 1$) or remain unchanged ($r = 1$) with the same ratio. If the speeds remain unchanged we talk about elastic collision, if change, then inelastic collision. According to our daily experience, speeds usually decrease during collisions, but it is also possible to prepare collision such that the speeds increase.

Let us first consider the case of elastic collision. In this case

$$w_{\mathrm{A},a} = w_{\mathrm{A},b}\,, \quad \text{and} \quad w_{\mathrm{B},a} = w_{\mathrm{B},b}\,. \tag{230}$$

We obtain the speeds in the laboratory frame V by using the transformation rule

$$\mathbf{v}_i = \mathbf{v}_\mathrm{c} + \mathbf{w}_i\,, \qquad i = \mathrm{A},\,\mathrm{B}\,, \tag{231}$$

where $\mathbf{v}_\mathrm{c}$ means the velocity of the centre of mass in the laboratory frame. As a result, for both bodies

$$w_i = \sqrt{v^2 + v_\mathrm{c}^2 - 2\mathbf{v}_i\cdot\mathbf{v}_\mathrm{c}}$$

Substituting into the Eqn. (230), we obtain

$$v_{i,a}^2 + v_\mathrm{c}^2 - 2\mathbf{v}_{i,a}\cdot\mathbf{v}_\mathrm{c} = v_{i,b}^2 + v_\mathrm{c}^2 - 2\mathbf{v}_{i,b}\cdot\mathbf{v}_\mathrm{c}\,, \qquad i = \mathrm{A},\,\mathrm{B}\,. \tag{232}$$

We then multiply each equation by the mass of the corresponding body and find the following two equations:

$$m_\mathrm{A}v_{\mathrm{A},a}^2 - 2m_\mathrm{A}\mathbf{v}_{\mathrm{A},a}\cdot\mathbf{v}_\mathrm{c} = m_\mathrm{A}v_{\mathrm{A},b}^2 - 2m_\mathrm{A}\mathbf{v}_{\mathrm{A},b}\cdot\mathbf{v}_\mathrm{c} \tag{233}$$

and

$$m_\mathrm{B}v_{\mathrm{B},a}^2 - 2m_\mathrm{B}\mathbf{v}_{\mathrm{B},a}\cdot\mathbf{v}_\mathrm{c} = m_\mathrm{B}v_{\mathrm{B},b}^2 - 2m_\mathrm{B}\mathbf{v}_{\mathrm{B},b}\cdot\mathbf{v}_\mathrm{c}\,. \tag{234}$$

Adding these two equations and dividing by two and using the conservation of total momentum,

$$m_\mathrm{A}\mathbf{v}_{\mathrm{A},b} + m_\mathrm{B}\mathbf{v}_{\mathrm{B},b} = m_\mathrm{A}\mathbf{v}_{\mathrm{A},a} + m_\mathrm{B}\mathbf{v}_{\mathrm{B},a}\,, \tag{235}$$

we obtain that in elastic collisions the quantity

$$E_\mathrm{k} = \frac{1}{2}m_\mathrm{A}v_\mathrm{A}^2 + \frac{1}{2}m_\mathrm{B}v_\mathrm{B}^2 \tag{236}$$

conserved, as well: $E_{\mathrm{k\,a}} = E_{\mathrm{k\,b}}$. We already know that any conserved quantity is very useful even if there are conditions of conservation. Therefore, we give a name for the new quantity $E_\mathrm{k} \equiv \frac{1}{2}mv^2$: it is the kinetic energy of body of mass m moving with speed v. The kinetic energy is a derived physical quantity. Its physical dimension is $\dim E_\mathrm{k} = \mathrm{ML^2T^{-2}}$, its unit in SI is $[E_\mathrm{k}] = \mathrm{kg{\cdot}m^2/s^2}$, that is called joule, $1\,\mathrm{J} \equiv 1\,\mathrm{joule} = 1\,\mathrm{kg{\cdot}m^2/s^2}$.

3.1.1 Collisions in one dimensions

We can use the conservation of kinetic energy for solving the complete dynamical problem: with conservation of momentum we can write sufficient number of equations for computing the speeds after collision. Let us consider elastic collisions in one dimension first. Conservation of momentum can be written as

$$m_\mathrm{A} v_{\mathrm{A},b} + m_\mathrm{B} v_{\mathrm{B},b} = m_\mathrm{A} v_{\mathrm{A},a} + m_\mathrm{B} v_{\mathrm{B},a}\,, \tag{237}$$

while conservation of kinetic energy reads

$$\frac{1}{2} m_\mathrm{A} v_{\mathrm{A},b}^2 + \frac{1}{2} m_\mathrm{B} v_{\mathrm{B},b}^2 = \frac{1}{2} m_\mathrm{A} v_{\mathrm{A},a}^2 + \frac{1}{2} m_\mathrm{B} v_{\mathrm{B},a}^2\,, \tag{238}$$

The best strategy to solve these equations for the unkown quantities $v_{\mathrm{A},a}$ and $v_{\mathrm{B},a}$ is to rearrange them such that the quantities of body A and those of body B appear on different sides of the equations:

$$m_\mathrm{A}(v_{\mathrm{A},b} - v_{\mathrm{A},a}) = m_\mathrm{B}(v_{\mathrm{B},a} - v_{\mathrm{B},b})\,, \tag{239}$$

$$m_\mathrm{A}(v_{\mathrm{A},b}^2 - v_{\mathrm{A},a}^2) = m_\mathrm{B}(v_{\mathrm{B},a}^2 - v_{\mathrm{B},b}^2)\,. \tag{240}$$

Dividing the second equation with the first, we find

$$v_{\mathrm{A},b} + v_{\mathrm{A},a} = v_{\mathrm{B},a} + v_{\mathrm{B},b}\,, \tag{241}$$

which yields $v_{\mathrm{B},a} = v_{\mathrm{A},b} + v_{\mathrm{A},a} - v_{\mathrm{B},b}$. Substituting this expression into the equation on momenta, we can express $v_{\mathrm{A},a}$ easily:

$$v_{\mathrm{A},a} = v_{\mathrm{A},b} + \frac{2m_\mathrm{B}}{m_\mathrm{A} + m_\mathrm{B}}(v_{\mathrm{B},b} - v_{\mathrm{A},b})\,. \tag{242}$$

The starting equations are symmetric with respect to the interchange A $\leftrightarrow$ B, so the solution for $v_{\mathrm{B},a}$ is obtained from that for $v_{\mathrm{A},a}$ by this interchange,

$$v_{\mathrm{B},a} = v_{\mathrm{B},b} + \frac{2m_\mathrm{A}}{m_\mathrm{A} + m_\mathrm{B}}(v_{\mathrm{A},b} - v_{\mathrm{B},b})\,. \tag{243}$$

3.1.2 Special cases of one-dimensional collisions

It is instructive to discuss some special cases.

(a) For collisions of bodies of equal masses, we find that the speeds interchange, $v_{\mathrm{A},a} = v_{\mathrm{B},b}$ and $v_{\mathrm{B},a} = v_{\mathrm{A},b}$.

(b) If one body, called target, is at rest, $v_{\mathrm{B},b} = 0$, then

$$v_{\mathrm{A},a} = \frac{m_\mathrm{A} - m_\mathrm{B}}{m_\mathrm{A} + m_\mathrm{B}} v_{\mathrm{A},b}\,, \qquad v_{\mathrm{B},a} = \frac{2m_\mathrm{A}}{m_\mathrm{A} + m_\mathrm{B}} v_{\mathrm{A},b} \tag{244}$$

(c) If the target is much heavier than the projectile, for instance, a car crashing into a truck, $m_\mathrm{B} \gg m_\mathrm{A}$, then $v_{\mathrm{A},a} \approx -v_{\mathrm{A},b} + 2v_{\mathrm{B},b}$ and $v_{\mathrm{B},a} \approx v_{\mathrm{B},b}$. As a result, the acceleration of the car will be very large, leading to severe damage, while that of the truck is small. The same physics is used by soccer players when stopping the ball flying to him. Choosing $v_{\mathrm{B},b}$ properly, he can reach $v_{\mathrm{A},a} = 0$. If the target is a wall, then $v_{\mathrm{B},b} = 0$, and the projectile has the same speed after collision, then before. We can use the same result for colliding particles into a wall away from right angles, provided the surface of the wall is slippery (no friction during collision), so it does not exert a force tangent to the wall. In such circumstances the tangent (to the wall) component of the velocity does not change, while the perpendicular component changes sign, keeping the original magnitude. Therefore, the particle is reflected from the wall at the same angle as it arrived to it.

(d) If the projectile is much faster than the target, for instance a car hit a pedestrian, $m_\mathrm{A} \gg m_\mathrm{B}$, then the speed of the car hardly changes, $v_{\mathrm{A},a} \approx v_{\mathrm{A},b}$, but the pedestrian is shot off, $v_{\mathrm{B},a} \simeq 2v_{\mathrm{A},b} - v_{\mathrm{B},b}$. For example, a car of speed 30 km/h hitting a pedestrian, the initial speed of the latter will be 60 km/h!

If the collision is inelastic, then the kinetic energy is not conserved. In order to write two independent equations, we have to specify the ratio (the collision number) r. The case of completely inelastic collision means $r = 0$. In such a case, we only have one equation, conservation of momentum, but also only one unknown, as the bodies move on with common velocity. Then conservation of momentum is sufficient to determine this common velocity in any dimensions:

$$m_\mathrm{A}\mathbf{v}_{\mathrm{A},b} + m_\mathrm{B}\mathbf{v}_{\mathrm{B},b} = (m_\mathrm{A} + m_\mathrm{B})\mathbf{v}_a \tag{245}$$

leads to

$$\mathbf{v}_a = \frac{m_\mathrm{A}}{m_\mathrm{A} + m_\mathrm{B}}\mathbf{v}_{\mathrm{A},b} + \frac{m_\mathrm{B}}{m_\mathrm{A} + m_\mathrm{B}}\mathbf{v}_{\mathrm{B},b}\,. \tag{246}$$

For instance, if the target is at rest, $\mathbf{v}_{\mathrm{B},b} = 0$, then the common final velocity is smaller than the initial one by the ratio of the masses, $\mathbf{v}_a = \frac{m_\mathrm{A}}{m_\mathrm{A}+m_\mathrm{B}}\mathbf{v}_{\mathrm{A},b}$. The same can be used in blow-up into two pieces. In the cm frame of the original single body

$$0 = m_\mathrm{A}\mathbf{w}_{\mathrm{A},a} + m_\mathrm{B}\mathbf{w}_{\mathrm{B},a}\,, \tag{247}$$

so $w_{\mathrm{A},a}/w_{\mathrm{B},a} = m_\mathrm{B}/m_\mathrm{A}$. The velocity in the laboratory frame V can be obtained from transformation of velocites.

3.2 Work-energy theorem

3.2.1 Work-energy theorem for point-like particle, power

The conservation of kinetic energy is more limited than conservation of momentum for two reasons: (i) it is valid only for elastic collisions, and (ii) the kinetic energy is actually not a constant: it decreases for a short time (during the actual interaction of the colliding bodies) and it attains its original value only after the collision is over. The bodies in interaction act by forces on one another. It is conceivable that these forces cause the change of kinetic energy. Let us examine why and how. As before, we denote the velocities before and after collision by indices b and a. Let us assume first that the force acting on the body is a constant (if not so, let us consider a sufficiently short time interval, so that it is true approximately). Thus

$$\begin{aligned}\frac{1}{2}mv_a^2 - \frac{1}{2}mv_b^2 &= \frac{1}{2}m\left(\mathbf{v}_a^2 - \mathbf{v}_b^2\right) = \frac{1}{2}m(\mathbf{v}_a - \mathbf{v}_b)\cdot(\mathbf{v}_a + \mathbf{v}_b) = \\ &= m\frac{\mathbf{v}_a - \mathbf{v}_b}{\Delta t}\cdot\frac{\mathbf{v}_a + \mathbf{v}_b}{2}\Delta t = m\frac{\Delta\mathbf{v}}{\Delta t}\cdot\langle\mathbf{v}\rangle\Delta t \approx m\cdot\mathbf{a}\Delta\mathbf{r}\,.\end{aligned} \tag{248}$$

According to Newton's 2nd law $m\mathbf{a} = \mathbf{F}$, therefore,

$$\frac{1}{2}mv_a^2 - \frac{1}{2}mv_b^2 = \mathbf{F}\cdot\Delta\mathbf{r}\,. \tag{249}$$

Here we see the reason for introducing the factor 1/2 in the definition of kinetic energy. Otherwise, a factor of two would appear on the right hand side. We name the new physical quantity on the right hand side: *the scalar product of force and displacement,*

$$\Delta W = \mathbf{F}\cdot\Delta\mathbf{r} \tag{250}$$

is called (mechanical) work, abbreviated by W, while the result in Eq. (249) is called work-energy theorem. In words: *the change of kinetic energy of a body is equal to the work done by the forces acting on the body,*

$$W = F\Delta s_F = F_s\Delta s = F\Delta s\cos\alpha\,. \tag{251}$$

Let us note, that work is a derived physical quantity of physical dimension $\dim W = \mathrm{ML}^2\mathrm{T}^{-2}$. It has the following SI unit $[W] = \mathrm{N{\cdot}m} = \mathrm{kg{\cdot}m^2/s^2} = \mathrm{J}$.

So far we have defined work of constant force and straight path. The generalization is of course easy: we partition the path into sufficiently small sections that can be considered straight lines and the force can be considered constant along the sections. Then we simply sum up the individual contributions as numbers,

$$W = \lim_{\Delta_\mathrm{i}\to 0} \sum_\mathrm{i} \mathbf{F}_\mathrm{i} \cdot \Delta\mathbf{r}_\mathrm{i} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{s_1}^{s_2} F_s(s)\mathrm{d}s \equiv W(s_1 \to s_2)\,. \tag{252}$$

In the last line of this equation the notation $W(s_1 \to s_2)$ is meant to emphasize that work depends on the path and not only on the starting and ending positions. On the graph of the component of force tangent to the path as a function of the measured path, $F_s(s)$, the work is equal to the area under this function. Of course, we can also compute the work using

$$W = \int_{x_1}^{x_2} F_x(x)\mathrm{d}x + \int_{y_1}^{y_2} F_y(y)\mathrm{d}y + \int_{z_1}^{z_2} F_z(z)\mathrm{d}z\,. \tag{253}$$

Let us now write the work energy theorem for each section of the partition of path:

$$E_\mathrm{k}(1) - E_\mathrm{k}(0) \quad = F_{s1}\Delta s_1 \tag{254}$$

$$E_\mathrm{k}(2) - E_\mathrm{k}(1) \quad = F_{s2}\Delta s_2 \tag{255}$$

$$\ldots \tag{256}$$

$$E_\mathrm{k}(n) - E_\mathrm{k}(n-1) \quad = F_{s\mathrm{n}}\Delta s_\mathrm{n}\,. \tag{257}$$

Adding these equations, on the left hand side we always obtain the total change of kinetic energy, $E_\mathrm{k}(n) - E_\mathrm{k}(0)$. Thus we can refine infinitely the partition and obtain

$$E_\mathrm{k}(s_1) - E_\mathrm{k}(s_0) = \lim_{\Delta s_\mathrm{i}\to 0} \sum_\mathrm{i} F_{s\mathrm{i}}\Delta s_\mathrm{i} = \int_{s_0}^{s_1} F_s \mathrm{d}s\,, \tag{258}$$

i.e. the total change in kinetic energy is equal to the work done by the force on the body (cf. with Eq. (252)). Then the work-energy theorem can then be written in the form

$$E_\mathrm{k}(s) - E_\mathrm{k}(0) = W(s_0 \to s) \tag{259}$$

Work can be positive, negative (when the angle between the direction of displacement and force is at an angle larger than right angle), or zero even when neither the force, nor the path is zero, but these are perpendicular to one another. For instance, in the case of uniform circular motion, the displacement element is tangent to the path, while the force is radial, therefore, these are perpendicular and the work is zero. As a result the kinetic energy and the speed are constant.

If there are more than one forces acting on the body, the work of the net force can be computed as a simple sum of the work of each individual force, as the scalar product is distributive,

$$\left(\sum_\mathrm{i} \mathbf{F}_\mathrm{i}\right) \cdot \Delta\mathbf{r} = \sum_\mathrm{i} \mathbf{F}_\mathrm{i} \cdot \Delta\mathbf{r} = \sum_\mathrm{i} W_\mathrm{i}\,. \tag{260}$$

The work-energy theorem can be used easily when the forces do not depend on the relative velocities and we are interested in the path-dependence of the speed.

For example, if we choose the starting point of the coordinate system to the equlibrium position, then the force law of the spring is $F_x = -cx$. Its work on the body belonging to the displacement $x_1 \to 0$ is

$$W = \int_{x_1}^{0} F_x\,\mathrm{d}x = -c\int_{x_1}^{0} x\,\mathrm{d}x = \frac{1}{2}cx_1^2\,. \tag{261}$$

The work-energy theorem can be used to find the requested starting position: $\frac{1}{2}mv^2 = \frac{1}{2}cx_1^2$ implies $x_1 = v\sqrt{m/c}$.

We often use the "speed" of work: the ratio of work and the time interval needed for that work, in the limit when the interval shrinks to zero is called power,

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} \tag{262}$$

Its physical dimension is $\dim P = \mathrm{ML}^2\mathrm{T}^{-3}$, SI unit is $[P] = \mathrm{J/s} = \mathrm{N{\cdot}m/s} = \mathrm{kg{\cdot}m^2/s^3}$, which has its own name: $1\,\mathrm{watt} \equiv 1\,\mathrm{W} = 1\,\mathrm{J/s}$. Using the watt unit of power, work os often measured in units of Ws or Wh, especially in electromagnetism.

Power can also be computed as

$$P = \lim_{\Delta t \to 0} \frac{\mathbf{F} \cdot \Delta \mathbf{r}}{\Delta t} = \mathbf{F} \cdot \mathbf{v}\,. \tag{263}$$

3.2.2 Work-energy theorem for extended bodies

The kinetic energy of a system of particles reads as

$$E_\mathrm{k} = \sum_\mathrm{i} E_\mathrm{k,i} = \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} v_\mathrm{i}^2\,, \tag{264}$$

and the work-energy theorem includes internal and external forces

$$\Delta E_\mathrm{k} = \sum_\mathrm{i} \mathbf{F}_\mathrm{i}^{(\mathrm{k})} \cdot \Delta \mathbf{r}_\mathrm{i} + \sum_\mathrm{i} \sum_{\mathrm{j} \neq \mathrm{i}} \mathbf{F}_\mathrm{ij}^{(\mathrm{b})} \cdot \Delta \mathbf{r}_\mathrm{i}\,. \tag{265}$$

Thus, for system of particles the work of the internal forces matters!

However, in case of rigid bodies, the net work of the internal forces are zero. This holds for central and non-central internal forces. If the internal forces are central one can argue. For translational motion the work of the opposite internal forces has opposite signs. For rotational motion the relative displacement of two points of the rigid object is perpendicular to the vector which connects the two points, thus the work done by central forces is zero.

If the internal forces are not central one can argue as follows. Let us calculate the total kinetic energy of a rotating rigid object,

$$E_\mathrm{k} = \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} v_\mathrm{i}^2 = \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} (\omega_z r_{\mathrm{i}\perp})^2 = \left(\sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} r_{\mathrm{i}\perp}^2 \right) \omega_z^2 = \frac{1}{2} \Theta^{(z)} \omega_z^2\,. \tag{266}$$

where $r_{\mathrm{i}\perp}$ is the distance of the point-like particle i from the axis of rotation which is the z-axis. Then,

$$\Delta E_\mathrm{k} = \frac{1}{2} \Theta^{(z)} \left(\omega_{z,2}^2 - \omega_{z,1}^2 \right) = \Theta^{(z)} \frac{\omega_{z,2} - \omega_{z,1}}{\Delta t} \frac{\omega_{z,2} + \omega_{z,1}}{2} \Delta t = \Theta^{(z)} \beta_z \Delta \varphi\,, \tag{267}$$

where $\Delta\varphi$ is the angle displacement over Δt. By using $\Theta^{(z)}\beta_z = M_z$, one finds

$$\Delta E_\mathrm{k} = M_z \Delta \varphi\,, \qquad \to \qquad \Delta E_\mathrm{k} = \mathbf{M} \cdot \overrightarrow{\Delta \varphi}\,. \tag{268}$$

The expression (268) is equal to the work done by the external forces. For example in case of two forces which are not central (but have the same magnitude and opposite direction) the work done is $F_s \Delta s = F_s r \Delta \varphi = M_z \Delta \varphi$, thus

$$\Delta W = \sum_\mathrm{i} M_{\mathrm{i},z} \Delta \varphi = \left(\sum_\mathrm{i} M_{\mathrm{i},z} \right) \Delta \varphi = M_z \Delta \varphi \tag{269}$$

where $\Delta\varphi$ is the same for all cases. Thus, we showed that the work of the internal forces is zero in case of rigid bodies, so, for rotating motion one can write the work-energy theorem as

$$\frac{1}{2} \Theta^{(z)} \omega_{z,2}^2 - \frac{1}{2} \Theta^{(z)} \omega_{z,1}^2 = M_z \Delta \varphi \tag{270}$$

3.2.3 Work-energy theorem for planar motion of rigid bodies

Let us apply the work-energy theorem for planar motion of rigid bodies. The velocity vector of an arbitrary point-like part of the rigid object is written as the sum of the velocity of the center of mass and the relative velocity in the center of mass frame $\mathbf{v}_\mathrm{i} = \mathbf{v}_\mathrm{c} + \mathbf{v}'_\mathrm{i}$. Then the kinetic energy reads

$$E_\mathrm{k} = \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} v_\mathrm{i}^2 = \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} \left(\mathbf{v}_\mathrm{c} + \mathbf{v}'_\mathrm{i}\right)^2 = \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} v_\mathrm{c}^2 + \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} v_\mathrm{i}'^2 + \mathbf{v}_\mathrm{c} \cdot \sum_\mathrm{i} m_\mathrm{i} \mathbf{v}'_\mathrm{i} . \tag{271}$$

Since the last term is zero, one finds

$$E_\mathrm{k} = \frac{1}{2} m v_\mathrm{c}^2 + \sum_\mathrm{i} \frac{1}{2} m_\mathrm{i} v_\mathrm{i}'^2 . \tag{272}$$

thus the kinetic energy consists of a two terms, the translational which is the first one and a rotational which is the last one. The latter can be written as $\frac{1}{2}\Theta_\mathrm{c}\omega^2$, thus by choosing the z-axis as the rotational axis, one finds

$$E_\mathrm{k} = E_\mathrm{k,tr} + E_\mathrm{k,rot} = \frac{1}{2} m v_\mathrm{c}^2 + \frac{1}{2} \Theta_\mathrm{c}^{(z)} \omega_z^2 . \tag{273}$$

In order to apply the work-energy theorem the dispalcement of an arbitrary point-like part of the rigid object is $\mathbf{\Delta r}_\mathrm{i} = \mathbf{\Delta r}_\mathrm{c} + \overrightarrow{\Delta\varphi} \times \mathbf{r}'_i$, thus the work can be calculated

$$\begin{aligned}
\Delta W &= \sum_\mathrm{i} \mathbf{\Delta r}_\mathrm{i} \cdot \mathbf{F}_\mathrm{i} = \sum_\mathrm{i} \left(\mathbf{\Delta r}_\mathrm{c} + \overrightarrow{\Delta\varphi} \times \mathbf{r}'_i\right) \cdot \mathbf{F}_\mathrm{i} \\
&= \mathbf{\Delta r}_\mathrm{c} \cdot \sum_\mathrm{i} \mathbf{F}_\mathrm{i} + \sum_\mathrm{i} \left(\overrightarrow{\Delta\varphi} \times \mathbf{r}'_i\right) \cdot \mathbf{F}_\mathrm{i} \\
&= \mathbf{\Delta r}_\mathrm{c} \cdot \sum_\mathrm{i} \mathbf{F}_\mathrm{i} + \overrightarrow{\Delta\varphi} \cdot \sum_\mathrm{i} \mathbf{r}'_i \times \mathbf{F}_\mathrm{i} \\
&= \mathbf{\Delta r}_\mathrm{c} \cdot \sum_\mathrm{i} \mathbf{F}_\mathrm{i} + \overrightarrow{\Delta\varphi} \cdot \sum_\mathrm{i} \mathbf{M}_\mathrm{c,i} ,
\end{aligned} \tag{274}$$

which gives

$$\frac{1}{2} m v_\mathrm{c,2}^2 - \frac{1}{2} m v_\mathrm{c,1}^2 + \frac{1}{2} \Theta_\mathrm{c,2}^{(z)} \omega_z^2 - \frac{1}{2} \Theta_\mathrm{c,1}^{(z)} \omega_z^2 = \mathbf{\Delta r}_\mathrm{c} \cdot \sum_\mathrm{i} \mathbf{F}_\mathrm{i} + \overrightarrow{\Delta\varphi} \cdot \sum_\mathrm{i} \mathbf{M}_\mathrm{c,i} . \tag{275}$$

which is the work-energy theorem for planar motion of rigid bodies. Important to note that this theorem holds separately for the translational and rotational parts,

$$\frac{1}{2} m v_\mathrm{c,2}^2 - \frac{1}{2} m v_\mathrm{c,1}^2 = \mathbf{\Delta r}_\mathrm{c} \cdot \sum_\mathrm{i} \mathbf{F}_\mathrm{i} , \tag{276}$$

$$\frac{1}{2} \Theta_\mathrm{c,2}^{(z)} \omega_z^2 - \frac{1}{2} \Theta_\mathrm{c,1}^{(z)} \omega_z^2 = \overrightarrow{\Delta\varphi} \cdot \sum_\mathrm{i} \mathbf{M}_\mathrm{c,i} . \tag{277}$$

In order to prove the above separation one should consider the translational part,

$$\frac{1}{2} m v_\mathrm{c,2}^2 - \frac{1}{2} m v_\mathrm{c,1}^2 = \frac{1}{2} m \left(\mathbf{v}_\mathrm{c,2}^2 - \mathbf{v}_\mathrm{c,1}^2\right) = \frac{1}{2} m (\mathbf{v}_\mathrm{c,2} - \mathbf{v}_\mathrm{c,1}) \cdot (\mathbf{v}_\mathrm{c,2} + \mathbf{v}_\mathrm{c,1}) \tag{278}$$

$$= m \frac{\mathbf{v}_\mathrm{c,2} - \mathbf{v}_\mathrm{c,1}}{\Delta t} \cdot \frac{\mathbf{v}_\mathrm{c,2} + \mathbf{v}_\mathrm{c,1}}{2} \Delta t =\approx m \mathbf{a}_\mathrm{c} \cdot \mathbf{\Delta r}_\mathrm{c} = \mathbf{\Delta r}_\mathrm{c} \cdot \sum_\mathrm{i} \mathbf{F}_\mathrm{i} , \tag{279}$$

which demonstrates that the above separation of the work-energy theorem holds.

For the sake of completeness, the expression of the power can also be written as

$$P = \mathbf{F} \cdot \mathbf{v} = \mathbf{F} \cdot (\vec{\omega} \times \mathbf{r}) = \vec{\omega} \cdot (\mathbf{r} \times \mathbf{F}) = \vec{\omega} \cdot \mathbf{M} . \tag{280}$$

3.3 Potential energy

In general, the work of a force depends on the path over which the work is done. If the force is a central force (always pointing to the same centre) then the work is independent of the path and depends only on the starting and ending position. Examples are the gravitational and the spring forces. On any small path segment, labelled by i, the work is $\Delta W_i = \pm F \Delta r_i$, where Δr_i is the component of the displacement in the radial direction, and the sign is positive if the direction of the force and that of the radial displacement are the same, otherwise negative. Thus, the work does not depend on the actual path, only on the change in the radial distance from the centre. Then the total work over any path starting at a distance r and ending at r_0 is equal to the work done over a straight radial line starting at r_1 and ending at r_2:

$$\begin{aligned} W(r_1 \to r_2) = \int_{r_1}^{r_2} F(r)\,\mathrm{d}x = -c\int_{r_1}^{r_2}(r-r_0)\,\mathrm{d}r = -c\left[\frac{1}{2}(r-r_0)^2\right]_{r_1}^{r_2} \\ =\frac{1}{2}c(r_1-r_0)^2 - \frac{1}{2}c(r_2-r_0)^2\,. \end{aligned} \tag{281}$$

where r_0 is the unstretched length of the spring. We can define a function $U(r) = W(r \to r_0)$, which depends only on r and the work done by the force between any staring position r_1 and ending position r_2 is

$$W(r_1 \to r_2) = U(r_1) - U(r_2) \equiv U_1 - U_2\,. \tag{282}$$

Using the work-energy theorem, $E_{\mathrm{kin}}(2) - E_{\mathrm{kin}}(1) = W(r_1 \to r_2)$ and Eq. (282), we find $E_{\mathrm{kin}}(2) - E_{\mathrm{kin}}(1) = U_1 - U_2$, which yields

$$U + E_{\mathrm{kin}} = \mathrm{const} \tag{283}$$

Although we obtained Eq. (281) assuming linear force law, the result in Eq. (282) and consequently in Eq. (283) is clearly more general. If the work of a force does not depend on the path, we can define a function $U(r)$ such that the work done by the force between any staring position r_1 and ending position r_2 can be obtained by simple difference as in Eq. (282). In such cases of force laws, the quantity in Eq. (283), called the total mechanical energy, is conserved. As always, we welcome conserved quantities, as those can be used for making predictions, even if the validity of consarvation is limited. We call those forces for which the total mechanical energy is conserved conservative forces. The function $U(r)$ is called potential energy and it is the work done by the force on the body during the latter moves from the starting position r to the arbitrarily chose ending position r_0. (Observe that the work being a difference is independent of r_0.) Strictly speaking the potential energy characterizes the force, but having Eq. (283) in mind, we say that it is the potential energy of the body under consideration. Thus we interpret Eq. (283) such that *if only onservative forces act on a body, then its total mechanical energy, i.e. the sum of its potential and kinetic energy, is conserved.*

Friction or drag are examples of forces when the work depends also on the path, as those are always opposite to the relative displacement (as compared to the surface or medium). For such forces potential energy cannot be defined and the total mechanical energy is not conserved. Such forces are called dissipative.

Let us give two examples for the potential energy. The first is the potential energy of a spring

$$U(r) = \frac{1}{2}c(r-r_0)^2\,, \tag{284}$$

the second example is the potential energy of the torsion spring, ($M_z = -c^*\varphi$),

$$U(r) = \frac{1}{2}c^*\varphi_0^2\,. \tag{285}$$

3.3.1 Interaction potential energy

Up to now we have been discussing the potential energy of a single particle attached to a spring. Let us look at the energy of two particles attached to each edges of an extended spring. The two particles are considered as isolated ones, which means no external forces interacting with each other. Particle B, at position $\mathbf{r}_B$ from the origin, exerts a force $\mathbf{F}_{AB}$ on particle A, while the particle 1 at position $\mathbf{r}_A$ exerts an equal and opposite force $\mathbf{F}_{BA} = -\mathbf{F}_{AB}$ on particle 2. The displacements of the two particles are $\Delta\mathbf{r}_\mathrm{A}$ and $\Delta\mathbf{r}_\mathrm{B}$. In this case one can write,

$$\mathbf{F}_{AB}\Delta\mathbf{r}_\mathrm{A} + \mathbf{F}_{BA}\Delta\mathbf{r}_\mathrm{B} = \mathbf{F}_{AB}\Delta\mathbf{r}_\mathrm{A} - \mathbf{F}_{AB}\Delta\mathbf{r}_\mathrm{B} = \mathbf{F}_{AB}(\Delta\mathbf{r}_\mathrm{A} - \Delta\mathbf{r}_\mathrm{B}) = \mathbf{F}_{AB}\Delta\mathbf{r}_\mathrm{AB}\,, \tag{286}$$

thus, the works done in the lab frame and in the frame attached to one of the particles are the same. According to the work-energy theorem,

$$W(1 \to 2) = U_1 - U_2 = E_\mathrm{kin,A2} + E_\mathrm{kin,B2} - (E_\mathrm{kin,A1} + E_\mathrm{kin,B1})\,, \tag{287}$$

which means the the total mechanical energy is conserved, $U + E_\mathrm{kin,A} + E_\mathrm{kin,B} = \mathrm{constant}$.

Thus, potential energy is the energy held by an object because of its position relative to other objects, stresses within itself, its electric charge, or other factors. There are various types of potential energy, each associated with a particular type of force. Common types of potential energy include the gravitational potential energy of an object that depends on its mass and its distance from the center of mass of another object, the elastic potential energy of an extended spring, and the electric potential energy of an electric charge in an electric field. The total work done by these (gravitational, elastic and Coulomb) forces on the body depends only on the initial and final positions of the body in space. These forces are called conservative forces.

There are, however, non-conservative forces such as the frictional force where the work does depend on the path and not just the initial and final positions. In this case the total mechanical energy is not conserved, or in other words, strictly speaking one cannot define potential energy.

3.3.2 Potential energy of a force field

A conservative forces conserve mechanical energy. We showed that this can also be recognised by the closed path test which means that the work done by the force is independent of the path, thus, it depends on the initial and the final position only. So, if one considers the work done by the force over a closed path it should be zero. Any force that passes the closed path test for all possible closed paths is classified as a conservative force.

What if multiple conservative forces act on a particular point-like object? In this case one can take the net force, so, thus it can be considered as a single (conservative) force problem. In addition, the concept of field force can be introduced. Only the actual magnitude and direction of the net force at a given space point matters if we wish to determine the work. In other words, various force laws standing behind the actual value of the net force are irrelevant if one is interested in the work done by the net force. A force field $\mathbf{F}$, defined everywhere in space is called a conservative force or conservative vector field if passes the closed path test. Therefore, the potential energy of this conservative force field is defined as $U(\mathbf{r}) = W(\mathbf{r} \to \mathbf{r}_0)$ where $\mathbf{r}_0$ is the point where the potential energy is assumed to be zero. Note that there is an arbitrary constant of integration in the definition of the potential energy, showing that any constant can be added to the potential energy. Practically, this means that you can set the zero of potential energy at any point which is convenient,

$$U'(\mathbf{r}) = W(\mathbf{r} \to \mathbf{r}'_0) = W(\mathbf{r} \to \mathbf{r}_0) + W(\mathbf{r}_0 \to \mathbf{r}'_0) = U(\mathbf{r}) + \mathrm{constant}. \tag{288}$$

The term conservative force comes from the fact that when a conservative force exists, it conserves mechanical energy, so, if the potential energy function is known then the work can be calculated, $W(1 \to 2) = U_1 - U_2$. Introducing this into the work-energy theorem one finds, $U_1 - U_2 = E_\mathrm{kin,2} - E_\mathrm{kin,1}$ which clearly shows the conservation of the mechanical energy,

$$U + E_\mathrm{kin} = \mathrm{constant} \tag{289}$$

The most familiar conservative forces are gravity, the electric force and the spring force. Up to now we considered the potential energy of the spring force. Let us now turn to gravitational potential energy.

A well known central force is gravitation. Let us compute the potential energy of a body under the influence of gravitational force of the Earth only. The force law is central again, so the work is independent of the path, we can compute its work along any path, say along a straight radial line alined with the x axis:

$$U(r) = W(r - r_0) = -GmM \int_r^{r_0} \frac{\mathrm{d}x}{x^2} = -GmM \left[-\frac{1}{x}\right]_r^{r_0} = -GmM \left(\frac{1}{r} - \frac{1}{r_0}\right) . \tag{290}$$

There are two natural choices for the zero point. If the motion is close to the surface of the Earth, i.e. the distance h is much smaller than the radius of Earth R, $h \ll R$, then the potential energy at height h is

$$U(r) = -GmM \left(\frac{1}{R+h} - \frac{1}{R}\right) = m\frac{GM}{R^2\left(1+\frac{h}{R}\right)}h \approx m\frac{GM}{R^2}h = mgh . \tag{291}$$

The second natural choice is at infinity for arbitrary motions, $r_0 \to \infty$. Then

$$U(r) = -G\frac{mM}{r} . \tag{292}$$

Let us calculate how fast we should start the spacecraft vertically, if we want it to leave Earth forever. The gravitational force is conservative, so the total mechanical energy is a constant,

$$\frac{1}{2}mv^2 - G\frac{mM}{R} = E = \text{constant} , \tag{293}$$

where R is the radius of Earth and we used Eq. (292). If the initial speed of the spacecraft is smaller than necessary, then it will turn back at a distance $r_{\max}$. At the moment of turn its speed and consequently, kinetic energy is zero, so

$$-G\frac{mM}{r_{\max}} = E . \tag{294}$$

We see that the total energy is negative (and it was so at any moment due to our choice of zero-energy point at infinity). Thus the spacecraft turns back as long as we start it with negative total energy ($E < 0$). If we start it with positive total energy ($E > 0$), it will never turn back. The smallest starting speed of path with no return is obtained from the condition

$$\frac{1}{2}mv_2^2 - G\frac{mM}{R} = E = 0 , \tag{295}$$

i.e.,

$$v_2 = \sqrt{2\frac{GM}{R}} = \sqrt{2\frac{GM}{R^2}R} = \sqrt{2gR} \approx 11\,\frac{\mathrm{km}}{\mathrm{s}} . \tag{296}$$

This is called escape speed or second cosmic speed (hence the subscript 2).

As a second example, let us proove Kepler's III. law of planetary motion. Kepler's three laws state that: (i) the orbit of a planet is an ellipse with the Sun at one of the two foci, (ii) A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time, (iii) The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit. Let us consider the total mechanical energy at two distinguished points of the ellipse, the perihelion (minimum distance to the sun) and the aphelion (maximum distance to the sun),

$$\frac{1}{2}mv_1^2 - G\frac{mM}{a-c} = \frac{1}{2}mv_2^2 - G\frac{mM}{a+c} , \tag{297}$$

from which one finds,

$$v_1^2 - v_2^2 = 2GM\left(\frac{1}{a-c} - \frac{1}{a+c}\right) = GM\frac{4c}{a^2-c^2} . \tag{298}$$

As a next step let us apply Kepler's second law, again for perihelion and for aphelion,

$$\frac{1}{2}(a-c)v_1 = \frac{1}{2}(a+c)v_2 = \frac{ab\pi}{T}\,, \tag{299}$$

where T is the orbital period, and $ab\pi$ is the circumference of the ellipse. Express v_1 and v_2,

$$v_1 = \frac{2ab\pi}{T(a-c)}\,, \qquad v_2 = \frac{2ab\pi}{T(a+c)}\,. \tag{300}$$

Substituting them back to Eq. (298), one finds

$$\frac{4a^2b^2\pi^2}{T^2}\left(\frac{1}{(a-c)^2} - \frac{1}{(a+c)^2}\right) = \frac{4a^2b^2\pi^2}{T^2}\frac{4ac}{(a^2-c^2)^2} = GM\frac{4c}{a^2-c^2} \tag{301}$$

By using the expression $a^2 - c^2 = b^2$ valid for any ellipse, one finds

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2} = \text{constant} \tag{302}$$

which is Kepler's third law of planetary motion.

3.4 Forces and potential energy

3.4.1 Force derived from the potential energy function

Potential energy is closely linked with forces. If the work done by a force on a body that moves from A to B does not depend on the path between these points, then the work of this force measured from A assigns a scalar value to every other point in space and defines a scalar potential field. If the force law is given one can compute the corresponding potential energy by calculating the work done by the force. This can be obtained by integrating out the scalar product of the force and displacement vector which serves as the integration measure. What if the potential energy function is known and we are interested in the corresponding force? Let us show that in this case, the force can be defined as the negative of the vector gradient of the potential field.

Let us consider the work over the so called virtual displacement $\delta\mathbf{r}$ which is a mathematical construction where we assume that physical quantities, like the force remains unchanged over the displacement. In other words, over virtual displacement one assumes time-independence, i.e. the displacement is happened to be instanteneous. In this case the corresponding work is the virtual work and it reads as,

$$\delta W = \mathbf{F}\cdot\delta\mathbf{r} = U(\mathbf{r}) - U(\mathbf{r}+\delta\mathbf{r})\,. \tag{303}$$

In order to express the force, one has to evaluate the scalar product and then one can divide by the magnitude of $\delta\mathbf{r}$,

$$F_{\delta r} = -\frac{U(\mathbf{r}+\delta\mathbf{r}) - U(\mathbf{r})}{|\delta\mathbf{r}|}\,. \tag{304}$$

In order to complete our task we need to define the direction of the force. Indeed, one can evaluate the virtual work for various virtual displacement vectors with various directions and let us choose the one which gives the largest component $F_{\delta r}$, and this direction is chosen to be the direction of the force component. Finally, with a particular choice for the reference frame, the x-component reads as

$$F_x = -\lim_{\delta x\to 0}\frac{U(x+\delta x) - U(x)}{\delta x} \equiv -\frac{\partial U}{\partial x}\,, \tag{305}$$

and similarly the two other components y and z are

$$F_y = -\frac{\partial U}{\partial y}\,, \qquad F_z = -\frac{\partial U}{\partial z}\,, \tag{306}$$

which gives

$$\mathbf{F} = -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j} - \frac{\partial U}{\partial z}\mathbf{k}\,. \tag{307}$$

Let us choose another way to discuss the relationship between work and potential energy. The details of this derivation will be discussed in advanced studies, however, it is insightful to summarise the major steps. The line integral that defines work along a curve takes a special form if the force $\mathbf{F}$ is related to a scalar field $U(\mathbf{r})$ so that

$$\mathbf{F} = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) U(\mathbf{r}) \equiv -\boldsymbol{\nabla} U(\mathbf{r}), \tag{308}$$

where the vector $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is introduced. In this case, the work along the curve is given by

$$W = \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F}\, d\mathbf{r} = -\int_{\mathbf{r_1}}^{\mathbf{r_2}} \boldsymbol{\nabla} U(\mathbf{r})\, d\mathbf{r} = \int_{\mathbf{r_2}}^{\mathbf{r_1}} \boldsymbol{\nabla} U(\mathbf{r})\, d\mathbf{r} \tag{309}$$

which can be evaluated using the so called gradient theorem (which is discussed in advanced studies) to obtain

$$W = U(\mathbf{r_1}) - U(\mathbf{r_2}). \tag{310}$$

Thus, we recovered the usual relationship between the work and the potential energy and our result implies that $\mathbf{F}$ must be a conservative vector field and the potential $U(\mathbf{r})$ defines a force $\mathbf{F}$ at every point $\mathbf{r}$ in space, so the set of forces is called a force field. Another usual notation is, $\mathbf{F} = -\mathrm{grad}\, U$ where the so called gradient is given by the vector $\boldsymbol{\nabla}$. The force vector is indeed perpendicular to the curve where each point has the same potential energy, so, they are equipotential.

3.5 Equilibrium positions

In mechanics, equilibrium is the condition of a system when its state of motion remains unchanged with time. A simple mechanical body is said to be in equilibrium if it experiences neither linear acceleration nor angular acceleration; unless it is disturbed by an outside force, it will continue in that condition indefinitely. For a single particle, equilibrium arises if the vector sum of all forces acting upon the particle is zero. A rigid body is considered to be in equilibrium if, in addition to the states listed for the particle above, the vector sum of all torques acting on the body equals zero so that its state of rotational motion remains constant.

There are three types of equilibrium: stable, unstable, and neutral. An equilibrium is said to be stable if small, externally induced displacements from that state produce forces that tend to oppose the displacement and return the body or particle to the equilibrium state. An example is a ball placed in the potential well. An equilibrium is unstable if the least departure produces forces that tend to increase the displacement. An example is a ball bearing balanced on the potential hill. A system is in neutral equilibrium if its equilibrium is independent of displacements from its original position. A ball on a flat horizontal surface (equipotential points) is an example.

3.6 Gravitational field, gravitational potential

In classical mechanics gravitational field is a model used to explain the influence that a massive body extends into the space around itself, producing a force on another massive body. In its original concept, gravity was a force between point masses. In a field model, rather than two particles attracting each other, a gravitational field is used to explain gravitational phenomena. We have already introduced the gravitational potential energy which contains the mass of the test particle, too. In the concept of the gravitational field and gravitational potential the mass of the test particle is removed.

Thus, gravitational field can be defined using Newton's law of universal gravitation. Determined in this way, the gravitational field $\mathbf{g}$ around a single particle of mass M is a vector field consisting at every point of a vector pointing directly towards the particle. The magnitude of the field at every point is calculated applying the universal law, and represents the force per unit mass on any object at that point in space. Because the force field is conservative, there is a scalar potential energy per unit mass, at each point in space associated with the force fields; this is called gravitational potential. The gravitational field can be constructed from the gravitational force

$$\mathbf{F}(\mathbf{r}) = m\mathbf{g}(\mathbf{r})\,, \tag{311}$$

which acts an the test particle with mass m. Let us consider the most general case where $\mathbf{g}(\mathbf{r})$ is the field around multiple particles with various masses m_i which is the vector sum of the fields around each individual particle and it is determined by the position vectors ($\mathbf{r}_\mathrm{i}$) of these multiple particles,

$$\mathbf{g}(\mathbf{r}) = -\sum_\mathrm{i} G\frac{m_\mathrm{i}}{(\mathbf{r}-\mathbf{r}_\mathrm{i})^2}\frac{\mathbf{r}-\mathbf{r}_\mathrm{i}}{|\mathbf{r}-\mathbf{r}_\mathrm{i}|}\,. \tag{312}$$

Similarly, the gravitational potential can be constructed from the gravitational potential energy,

$$U(\mathbf{r}) = mu(\mathbf{r})\,, \tag{313}$$

where the gravitational potential reads as

$$u(\mathbf{r}) = -\sum_\mathrm{i} G\frac{m_\mathrm{i}}{|\mathbf{r}-\mathbf{r}_\mathrm{i}|}\,. \tag{314}$$

Thus, $u(\mathbf{r})$ is the gravitational potential which is the function of the position and related to the gravitational field,

$$\mathbf{g}(\mathbf{r}) = -\mathrm{grad}\, u(\mathbf{r})\,. \tag{315}$$

If the source is a single particle with the mass M then

$$\mathbf{g}(\mathbf{r}) = -G\frac{M}{r^2}\mathbf{r}^0\,, \qquad u(\mathbf{r}) = -G\frac{M}{r}\,. \tag{316}$$

As an example, let us calculate the gravitational potential inside and outside of a spherical shell. It is important since a spherical mass can be thought of as built up of many infinitely thin spherical shells, each one nested inside the other. We will consider the gravitational attraction that such a shell exerts on a particle at a distance r from the center of the shell. The total mass of the shell is m and its radius is R. To do this, consider cutting the shell into rings. Every point on the ring is a distance $r(\vartheta_\mathrm{i})$ from the test particle, and the ring has width $R\Delta\vartheta_\mathrm{i}$ and radius $R\sin\vartheta_\mathrm{i}$ where ϑ_i is the angle corresponds to the ring. The surface area of the ring equals

$$\Delta A_\mathrm{i} = 2\pi R^2 \sin\vartheta_\mathrm{i}\Delta\vartheta_\mathrm{i}\,, \tag{317}$$

The total mass of the shell, m, is evenly distributed over the surface, so the mass of the ring is given by the fraction of the total surface area

$$\Delta m_\mathrm{i} = \frac{\Delta A_\mathrm{i}}{4\pi R^2}m = \frac{1}{2}m\sin\vartheta_\mathrm{i}\Delta\vartheta_\mathrm{i}\,. \tag{318}$$

Since $r(\vartheta_\mathrm{i}) = \sqrt{r^2+R^2-2rR\cos\vartheta_\mathrm{i}}$ then,

$$u = -G\sum_\mathrm{i}\frac{m\sin\vartheta_\mathrm{i}\Delta\vartheta_\mathrm{i}}{2\sqrt{r^2+R^2-2rR\cos\vartheta_\mathrm{i}}}\,, \tag{319}$$

which results in the following integral,

$$\begin{aligned}
u &= -G\frac{m}{4rR}\int_0^\pi \frac{2rR\sin\vartheta\,\mathrm{d}\vartheta}{\sqrt{r^2+R^2-2rR\cos\vartheta}} \\
&= -G\frac{m}{2rR}\left[\sqrt{r^2+R^2-2rR\cos\vartheta}\right]_0^\pi \\
&= -G\frac{m}{2rR}\left(\sqrt{(r+R)^2}-\sqrt{(r-R)^2}\right) \\
&= -G\frac{m}{2rR}\left(r+R-|r-R|\right)\,.
\end{aligned} \tag{320}$$

Outside of the shell one finds $r > R$, thus $|r - R| = r - R$, and

$$u = -G\frac{m}{r}. \tag{321}$$

This result mirrors the result we would receive if all the mass had been concentrated at the center of the shell. This similarity holds true for all shells, and since a sphere is composed of such shells, it must be true for a sphere too. The phenomenon holds even if the different shells are not of equal mass density–that is, if the density is a function of the radius. We can conclude that the gravitational force exerted by one planet on another acts as if all the mass of each planet were concentrated at its center. The corresponding gravitational field reads as

$$\mathbf{g}(\mathbf{r}) = -G\frac{m}{r^2}\mathbf{r}^0\,. \tag{322}$$

Now let us consider the potential for a particle inside such a shell, $|r - R| = R - r$,

$$u = -G\frac{m}{R} \tag{323}$$

which is constant. Thus the potential inside the sphere is independent of position, i.e., it is constant in r. Since the force is the gradient (derivative) of the potential, we can infer that the shell exerts no force on the particle inside it. For a solid sphere this means that for a particle, the only gravitational force it feels will be due to the matter closer to center of the sphere (below it). The matter above it (since it is inside its shell) exerts no influence on it, clearly illustrates this fact.

4 Mechanics of elastic bodies, liquids and gases

4.1 Elastic bodies

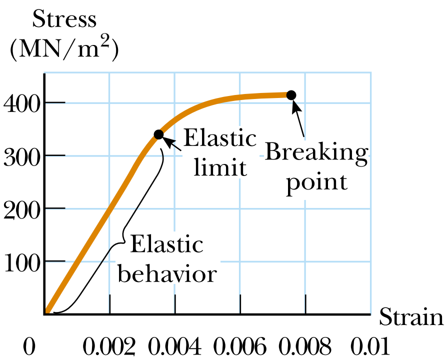
All rigid bodies are to some extent elastic: their dimensions change slightly by pulling, pushing, twisting or compressing them. For instance, let us hang a car on a steel rod that is 1 m long and 1 cm in diameter. The rod stretches 0.5 mm, but returns to its original size if the car is removed. If we hang two cars on the same rod, then the rod will be permanently stretched after the removal of the cars. Hanging three cars will break the rod. Just before rupture, the elongation will be 20 mm.

4.1.1 Differential and integral Hook's law in one dimension

There are three ways of changing the dimensions of a solid by acting forces on them: (i) tensile stress, (ii) shearing stress and (iii) uniform compression. All three ways produce some kind of deformation of the solid, which is called strain. Any stressed material strains uniformly, therefore, it is useful to consider relative strain, a dimensionless quantity. This means that,

$$\epsilon = \frac{\Delta\ell_1}{\ell_1} = \frac{\Delta\ell_2}{\ell_2} = \frac{\Delta\ell_i}{\ell_i} \tag{324}$$

where $\Delta\ell_i$ is the deformation on the length ℓ_i.



Consider a bar of a cross sectional area A being subjected to equal and opposite forces $\mathbf{F}$ pulling at the ends so the bar is under tension. Strain is a description of deformation in terms of relative displacement of particles in the body that excludes rigid-body motions. An experimental fact is that the material is experiencing a stress where the strain is a monotonic function of the force and depends on the inverse of the cross sectional area of the bar,

$$\epsilon = f\left(\frac{F}{A}\right) \tag{325}$$

where **F** is perpendicular to the cross sectional area. Based on this observation one can define the (longitudinal) stress

$$\sigma_l = \lim_{\Delta A \to 0} \frac{\Delta F_\perp}{\Delta A} \tag{326}$$

where the strain is a unique function of stress $\epsilon = f(\sigma)$. If the strain is small, the dependence of the stress on the relative strain is linear,

stress = modulus × relative strain.

If the stress is tensile one, then the modulus is called Young's modulus E. Let us note that strains are classified as either normal or shear. A normal strain is perpendicular to the face of an element, and a shear strain is parallel to it. These definitions are consistent with those of normal stress and shear stress. If the stress is shearing one, then the modulus is the shear modulus G. Finally, if the stress is uniform compression, then the modulus is called compression modulus, κ. For uniform compression the relative strain is negative, therefore, in order to keep κ positive, we have $\sigma = -\kappa \Delta V/V$ which will be discussed later.

Increasing strain we find eventually that stress and strain are no longer proportional. Instead, we find that equal increments in stress cause smaller and smaller change in strain: we say that the elastic material becomes more rigid. However, beyond some stress the material becomes more fluid like, and small increase in stress produces large increase in strain. Finally, at some ultimate stress the material breaks. In physics, we are usually interested in the region of linear response. In the case of tensile stress, the linear response is described by what is usually called Hooke's law:

$$\sigma_l = E\epsilon\,, \qquad \rightarrow \qquad \frac{\Delta \ell}{\ell} = \frac{1}{E}\frac{F_\perp}{A} \tag{327}$$

which are also called the differential (first expression) and integral (second expression) forms of the one-dimensional Hooke's Law.

4.1.2 Transverse contraction

When analyzing more than one dimension, interaction between all directions needs to be considered. This is done through Poisson's ratio. Basically, Poisson's ratio is the amount of transverse contraction, or negative strain, when strained in a given direction. For a basic object pulled or strained in the x-direction, the Poisson's Ratio is defined as

$$-\mu = \frac{\epsilon_y}{\epsilon_x} \qquad \rightarrow \qquad \frac{\Delta \ell_x}{\ell_x} = -\frac{1}{\mu}\frac{\Delta \ell_y}{\ell_y} \tag{328}$$

Thus, when a member is pulled in the x-direction, there is a contraction strain in the y-direction (and z-direction). If it is pulled in the y-direction, then the contraction strain will be in the x-direction (and z-direction). For a three dimensional object, Poisson's ratio will occur in equally in both perpendicular directions. If the load is in the x-direction, then strain in the y- and z-direction will be

$$\epsilon_y = \epsilon_z = -\mu\epsilon_x. \tag{329}$$

Therefore, the volume deformation can be written as

$$\frac{\Delta V}{V} = \frac{(\ell_x + \Delta\ell_x)(A + \Delta A) - \ell_x A}{\ell_x A} \tag{330}$$

where A is the cross sectional area perpendicular to the x-direction. If one neglects the term $\Delta A \Delta \ell_x$ it reduces to

$$\frac{\Delta V}{V} \approx \frac{\Delta \ell_x A + \Delta A \ell_x}{\ell_x A} = \frac{\Delta \ell_x}{\ell_x} + \frac{\Delta A}{A}. \tag{331}$$

which can be furthered simplified by using $\Delta A/A \approx \Delta\ell_y/\ell_y + \Delta\ell_z/\ell_z$,

$$\frac{\Delta V}{V} \approx \frac{\Delta \ell_x}{\ell_x} + \frac{\Delta \ell_y}{\ell_y} + \frac{\Delta \ell_z}{\ell_z} = \epsilon_x + \epsilon_y + \epsilon_z = \epsilon_x(1 - 2\mu). \tag{332}$$

Since the volume deformation is typically positive, $\Delta V/V \geq 0$, the following relation holds $(1 - 2\mu) \geq 0$ which results in a constraint for the Poisson coefficient, $0.5 \geq \mu \geq 0$.

4.1.3 Stress tensor

In continuum mechanics, the physical quantity which describes the stress and denoted by σ is a second order tensor and called the stress tensor. In general, the tensor consists of nine components σ_{ij} that completely define the state of stress at a point inside a material in the deformed state, placement, or configuration. It is used for stress analysis of material bodies experiencing small deformations which is the central concept of the linear theory of elasticity. If one considers the stress over a given plane defined by its normal vector $\mathbf{n}$, the so called stress vector can be derived from the stress tensor.

$$\boldsymbol{\sigma}(\mathbf{n},\mathbf{r}) = \lim_{\Delta A\to 0}\frac{\mathbf{F}}{\Delta A} \tag{333}$$

Depending on the orientation of the plane under consideration, the stress vector may not necessarily be perpendicular to that plane, i.e. parallel to $\mathbf{n}$, and can be resolved into two components. One is normal to the plane, called normal or longitudinal (or radial) stress,

$$\sigma_l = \lim_{\Delta A\to 0}\frac{F_\perp}{\Delta A} \tag{334}$$

where $F_\perp$ is perpendicular to the cross section area. The other is parallel to this plane, called the shear stress

$$\sigma_s = \lim_{\Delta A\to 0}\frac{F_\parallel}{\Delta A}. \tag{335}$$

where $F_\parallel$ is in the plane of the cross section area.

4.1.4 Elastic potential energy

Strained material posseses elastic potential energy. For instance, in the case of Hooke's law, this energy can be obtained by observing that the elastic rod under tensile stress behaves like a spring, with force law

$$F = -\sigma A = -\frac{EA}{\ell}\Delta\ell \equiv -k\Delta\ell\,, \tag{336}$$

i.e. the "spring constant" $k = EA/\ell$. The stored elastic energy is

$$E_{\text{pot}} = \frac{1}{2}k(\Delta\ell)^2 = \frac{1}{2}\frac{EA}{\ell}(\Delta\ell)^2 = \frac{1}{2}E\epsilon^2(A\ell)\,, \tag{337}$$

where in the parenthesis we recognize the volume of the rod, therefore, the elastic energy density is

$$u_p = \frac{E_{\text{pot}}}{V} = \frac{1}{2}E\epsilon^2\,. \tag{338}$$

4.1.5 Volume deformation, compression

Volume deformation is due to either tension or compression. In case of compression, we define compressive stress,

$$p = \lim_{\Delta A\to 0}\frac{\Delta F_\perp}{\Delta A} \tag{339}$$

where $p = -\sigma_\ell$ is the pressure which has the SI unit [Pa] where $1\text{Pa} = 1\text{N/m}^2$. For uniform compression the volume deformation reads

$$\frac{\Delta V}{V} = -\frac{1}{\kappa}p \tag{340}$$

Just like 1D or 2D, Hooke's Law can also be applied to material undergoing three dimensional stress (triaxial loading). The development of 3D equations is similar to 2D, sum the total normal strain in one direction due to loads in all three directions. For the x-direction, this gives,

$$\epsilon_{x,\text{total}} = \epsilon_{x,\text{due to }\sigma_\text{x}} + \epsilon_{x,\text{due to }\sigma_\text{y}} + \epsilon_{x,\text{due to }\sigma_\text{z}} = \frac{1}{E}\left(\frac{F_x}{A} - \mu\frac{F_y}{A} - -\mu\frac{F_z}{A}\right) = \frac{1}{E}\left(\sigma_{xx} - \mu\sigma_{yy} - \mu\sigma_{zz}\right). \tag{341}$$

In case of uniform compression, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, one finds

$$\epsilon_{x,\text{total}} = -\frac{(1-2\mu)}{E} p. \tag{342}$$

Since $\Delta V/V = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ our final results is

$$\frac{\Delta V}{V} = -\frac{3(1-2\mu)}{E} p = -\frac{1}{\kappa} p \tag{343}$$

which shows that E, κ and μ are not independent of each other.

4.1.6 Shear stress, Torsion spring

If a material is isotropic (homogenous in all directions, such as a solid metal) and is pulled in two directions, then due to Poisson's ratio, the overall normal strain will be the total of the two strains. However, Hooke's Law also relates shear strain and shear stress. If the shear stress and strain occurs in a plane then the stress and strain are related as

$$\sigma_s = \lim_{\Delta A \to 0} \frac{\Delta F_{\|}}{\Delta A} = G\gamma \tag{344}$$

where G is the shear modulus (a material property) and γ is the shear strain. The shear strain is defined as the angle (radians) caused by the shear stress. The shear modulus is related to Young modulus and Poisson's ratio,

$$G = \frac{E}{2(1+\mu)}. \tag{345}$$

Let us use the relation between the shear stress and shear strain in order to consider the dependence of the torsion spring constant on the length and width of the torsion fibre. A torsion spring is a spring that works by twisting its end along its axis. When it is twisted, it exerts a torque in the opposite direction, proportional to the amount (deflection angle) it is twisted.

$$M = c\varphi \tag{346}$$

A torsion bar is a straight bar of metal or rubber that is subjected to twisting (shear stress) about its axis by torque applied at its ends. Let us consider the case when the deflection angle is φ and the corresponding shear strain is γ where one finds a geometric relation between these angles and the length ℓ and the radius R of the bar,

$$r\varphi = \ell\gamma. \tag{347}$$

In order to calculate the torque and the spring constant one assume the bar as a solid cylinder and it has to be divided into symmetric rings each of which has the width Δr. Assume two parallel forces with a magnitude F acting on this ring with a moment arm r which generate a torque,

$$\Delta M = r\,F = r\,\sigma_s \Delta A = r\,G\,\gamma 2\pi r \Delta r = 2\pi \frac{\varphi G}{\ell} r^3 \Delta r \tag{348}$$

and if one summs up the torques of the individual rings and takes the limit $\Delta r \to 0$ one finds,

$$M = 2\pi \frac{\varphi G}{\ell} \int_0^R r^3 dr = \frac{\pi G R^4}{2\ell} \varphi \tag{349}$$

which results in the following expression for the torsion spring constant, $c = \frac{\pi G R^4}{2\ell}$.

4.2 Fluid statics – hydrostatics

In a mechanical view, a fluid is a substance that does not support shear stress which means that a fluid at rest has no shear stress at all. Fluid mechanics is the branch of physics concerned with the mechanics of fluids. It can be divided into fluid statics (hydrostatics) which is the study of fluids at rest and fluid dynamics which is the study of the effect of forces on fluid motion.

Consider a bucket holding some viscous, homogeneous and incompressible fluid (e.g. water). The bucket starts to rotate with constant angular frequency ω around a vertical axis passing through the center of its bottom surface. Due to friction with the walls, the fluid is influenced by the rotation of the bucket and at some point will also be rotating with angular frequency ω. Let us determine the shape of the fluid surface.

The easiest way to solve this problem is by using a non-inertial frame of reference which rotates with the bucket. The vertical axis z of this rotating frame is chosen to be the axis of rotation of the bucket. At the point at which the fluid is rotating with the bucket with angular frequency ω the fluid is seen as being at rest in this rotating frame. We are interested in the $z(r)$ function which determine the shape of the fluid surface where r is the radial distance from the axis of rotation. Let us consider the forces acting on a fluid element of mass m on the surface of the fluid (assumed to be already rotating with the bucket). There are only two forces experienced by the fluid element in the non-rotating frame: the gravitational force mg and the force exerted on the fluid element by the surrounding fluid. This latter force is orthogonal to the surface of the fluid. This is because a fluid at rest (which is the case in the non-inertial frame) has no shear stress. In the non-inertial frame one has to incorporate the centrifugal force, which is perpendicular to the rotation axis and its magnitude is $mr\omega^2$ where r is the radial distance of the fluid element from the rotation axis. Thus, the centrifugal force is horizontal. The gravitational force is vertical. The force exerted on the fluid element by the surrounding fluid has both horizontal and vertical components since it is orthogonal to the surface. Let us denote by α the angle between the horizontal line and the tangent to the surface. In this case one finds,

$$\tan\alpha = \frac{dz(r)}{dr} = \frac{mr\omega^2}{mg} = \frac{r\omega^2}{g}, \qquad \rightarrow \qquad z(r) = \frac{\omega^2}{2g}r^2 + z_0 \tag{350}$$

where the solution gives us the required shape function of the rotating surface.

4.2.1 Pascal's law

Pascal's law (also Pascal's principle) is a principle in fluid mechanics given by Blaise Pascal that states that a pressure change at any point in a confined incompressible fluid is transmitted throughout the fluid such that the same change occurs everywhere. In other words, Pascal's principle is defined as a change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid. Alternate definition: The pressure applied to any part of the enclosed liquid will be transmitted equally in all directions through the liquid.

Let us prove Pascal's law. Assume an element in the interior of a fluid at rest. The element is an arbitrary right-angled prismatic triangle in the liquid. Since the prismatic element is very small, every point is considered to be at the same depth from the liquid surface. The effect of gravity is also the same at all these points. Since the shear stress is zero, forces exerted on various surfaces of the prismatic element by the surrounding fluid are always perpendicular to the surfaces.

Let ΔA is the area of the largest surface of the element which corresponds to the hypotenuse of the right triangle where pressure exerts a force which is normal to this surface and denoted by ΔF. The area of the vertical surface of the element is denoted by $\Delta A_1 = \Delta A \sin\alpha$ and the corresponding force is given by ΔF_1. The are of the horizontal surface of the element is $\Delta A_2 = \Delta A \cos\alpha$ and the corresponding force is given by ΔF_2. The net force on the prism will be zero since the prism is in equilibrium. The horizontal components give,

$$\Delta F_1 = \Delta F \sin\alpha \qquad \rightarrow \qquad \frac{\Delta F_1}{\Delta A \sin\alpha} = \frac{\Delta F}{\Delta A} \qquad \rightarrow \qquad p_1 = p. \tag{351}$$

The vertical components give,

$$\Delta F_1 = \Delta F \cos\alpha + \Delta mg \qquad \rightarrow \qquad \frac{\Delta F_1}{\Delta A \cos\alpha} = \frac{\Delta F}{\Delta A} + \frac{\Delta mg}{\Delta A \cos\alpha} \qquad \rightarrow \qquad p_2 = p + \Delta h \rho g \tag{352}$$

where the mass of the element $\Delta m = \rho \Delta V$ is related to its volume $\Delta V = \Delta A \cos\alpha \Delta h$ where Δh is the height of the element. In order to fulfil the conditions of Pascal's law, one has to take the limit $\Delta h \to 0$, which results in $p_2 = p$. Therefore, the pressure exerted is the same in all directions in the fluid, which is at rest.

We can say that pressure is not a vector quantity. No direction can be assigned to it. The force against any area within (or bounding) a fluid at rest and under pressure is normal to the area, regardless of the orientation of the area.

4.2.2 Hydrostatic pressure

Let us consider the variation of pressure with depth. Consider a fluid at rest in a container. The z-axis is chosen to be vertical, i.e., orthogonal to the surface of the fluid and the we set $z = 0$ at the surface with a pressure p_0. Consider a cylindrical element of fluid having an area of base ΔA and height Δz. Assume that the bottom of the cylindrical element of fluid is at the depth $h = -z$ from the fluid surface. Since the fluid is at rest, the resultant horizontal forces should be zero along with the resultant vertical forces balancing the weight of the element. The forces, which are acting in the vertical direction, are due to the fluid pressure at the top $p(z + \Delta z)\Delta A$ acting downward and at the bottom $p(z)\Delta A$ acting upward. If mg is the weight of the fluid in the cylinder then we can say that,

$$p(z)\Delta A = p(z + \Delta z)\,\Delta A + g\rho\,\Delta A\,\Delta z \qquad \to \qquad \frac{p(z + \Delta z) - p(z)}{\Delta z} = -\rho g \tag{353}$$

where ρ is the mass density of the fluid, so, $m = \rho \Delta A \Delta z$. If one takes the limit $\Delta z \to 0$ one finds the following differential equation,

$$\frac{dp(z)}{dz} = -\rho g \tag{354}$$

Since fluids has a constant mass density (it does not depend on the pressure), the solution is

$$p(z) = -z\rho g + p_0 \qquad \to \qquad p(h) = h\rho g + p_0 \tag{355}$$

where we used the identification $h = -z$.

4.2.3 Archimedes' principle

Archimedes' principle states that the upward buoyant force that is exerted on a body immersed in a fluid, whether fully or partially submerged, is equal to the weight of the fluid that the body displaces. Archimedes' principle is a law of physics fundamental to fluid mechanics. It was formulated by Archimedes of Syracuse. In other words, Archimedes suggested that any object, totally or partially immersed in a fluid or liquid, is buoyed up by a force equal to the weight of the fluid displaced by the object.

Archimedes' principle allows the buoyancy of any floating object partially or fully immersed in a fluid to be calculated. The downward force on the object is simply its weight. The upward, or buoyant, force on the object is that stated by Archimedes' principle, above. Thus, the net force on the object is the difference between the magnitudes of the buoyant force and its weight. If this net force is positive, the object rises; if negative, the object sinks; and if zero, the object is neutrally buoyant (it remains in place without either rising or sinking).

Consider a cuboid immersed in a fluid, its top and bottom faces orthogonal to the direction of gravity (assumed constant across the cube's stretch). The fluid will exert a normal force on each face, but only the normal forces on top and bottom will contribute to buoyancy. The pressure difference between the bottom and the top face is directly proportional to the height (difference in depth of submersion). Multiplying the pressure difference by the area of a face gives a net force on the cuboid equaling in size the weight of the fluid displaced by the cuboid. By summing up sufficiently many arbitrarily small cuboids this reasoning may be extended to irregular shapes, and so, whatever the shape of the submerged body, the buoyant force is equal to the weight of the displaced fluid,

$$\text{weight of displaced fluid} = \text{weight of object in vacuum} - \text{weight of object in fluid} \tag{356}$$

The weight of the displaced fluid is directly proportional to the volume of the displaced fluid (if the surrounding fluid is of uniform density). The weight of the object in the fluid is reduced, because of the force acting on it, which is called upthrust. In simple terms, the principle states that the buoyant force F_b on an object is equal to the weight of the fluid displaced by the object, or the density ρ of the fluid multiplied by the submerged volume V times the gravity g. We can express this relation in the equation,

$$F_b = \rho g V \tag{357}$$

where F_b denotes the buoyant force applied onto the submerged object, ρ denotes the density of the fluid, V represents the volume of the displaced fluid and g is the acceleration due to gravity. Thus, among completely submerged objects with equal masses, objects with greater volume have greater buoyancy.

4.3 Aerostatics

A subfield of fluid statics, aerostatics is the study of gases that are not in motion with respect to the coordinate system in which they are considered. The corresponding study of gases in motion is called aerodynamics. Aerostatics studies density allocation, especially in air. The mass density of air or atmospheric density is approximately $\rho = 1.225$ kg/m^3. A lifting gas or lighter than air gas is a gas that has a lower density than the normal atmospheric density and rises above them as a result.

4.3.1 Torricelli's experiment

Torricelli's experiment was invented in Pisa in 1643 by the Italian scientist Evangelista Torricelli (1608-1647). The experiment uses a simple barometer to measure the pressure of air, filling it with mercury. Any air bubbles in the tube must be removed by inverting several times. After that, a clean mercury is filled once again until the tube is completely full. The barometer is then placed inverted on the dish full of mercury. This causes the mercury in the tube to fall down until the difference between mercury on the surface and in the tube is about 760 mm. Even when the tube is shaken or tilted, the difference between the surface and in the tube is not affected due to the influence of atmospheric pressure. Torricelli concluded that the mercury fluid in the tube is aided by the atmospheric pressure that is present on the surface of mercury fluid on the dish. Thus the atmospheric pressure is equal to the hydrostatical pressure of the mercury,

$$p_{\text{air}} = h\rho g, \qquad \rightarrow \qquad 1\text{ atm} = 0.76\text{ m} \times 13.59\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} = 1.013 \times 10^5\text{Pa} \tag{358}$$

and 1 Bar $= 10^5$ Pa. He also stated that the changes of liquid level from day to day are caused by the variation of atmospheric pressure. The empty space in the tube is called the Torricellian vacuum.

4.3.2 Barometric formula

One of the applications of aerostatics is the barometric formula. The barometric formula, sometimes called the exponential atmosphere or isothermal atmosphere, is a formula used to model how the pressure (or density) of the air changes with altitude. Treatment of the equations of gaseous behaviour at rest is generally taken, as in hydrostatics. Assuming that all pressure is hydrostatic, one can write

$$\frac{dp}{dz} = -\rho(p) g. \tag{359}$$

However, the presence of a non-constant density ($\rho = \rho(p)$) as is found in gaseous fluid systems (due to the compressibility of gases) requires the inclusion of the ideal gas law which makes connection between the pressure (p) and the mass density (ρ).

Boyle's law, also referred to as the Boyle–Mariotte law is an experimental gas law that describes how the pressure of a gas tends to increase as the volume of the container decreases. A modern statement of Boyle's law is: The absolute pressure exerted by a given mass of an ideal gas is inversely proportional to the volume it occupies if the temperature and amount of gas remain unchanged within a closed system. Mathematically, Boyle's law can be stated as,

$$p\ V = \text{constant}, \quad \rightarrow \quad \mathrm{p}\ \frac{\mathrm{V}}{\rho} = \text{constant}, \quad \rightarrow \quad \frac{\mathrm{p}}{\rho} = \text{constant}, \quad \rightarrow \quad \frac{\mathrm{p}}{\rho} = \frac{\mathrm{p}_0}{\rho_0}, \tag{360}$$

which results in $\rho(p) = p\ \rho_0/p_0$. Thus, one finds the following equation,

$$\frac{dp}{dz} = -p\ \frac{p_0}{\rho_0} g \qquad \rightarrow \qquad p(z) = p_0 \exp\left(-\frac{p_0}{\rho_0} g z\right) \tag{361}$$

where p_0 is the pressure at the surface (sea level) and z is the altitude. According to the barometric formula the pressure drops approximately by 11.3 pascals per meter in first 1000 meters above sea level and it drops to half of its sea level value at the altitude $z = 4.5$ km.

4.4 Fluid dynamics - inviscid flow

Fluid dynamics is a subdiscipline of fluid mechanics that describes the flow of fluids (i.e., liquids and gases). It has several subdisciplines, including aerodynamics (the study of air and other gases in motion) and hydrodynamics (the study of liquids in motion). All fluids are compressible to an extent; that is, changes in pressure cause changes in density, however, liquids are considered as incompressible by definition.

The solution to a fluid dynamics problem typically involves the calculation of various properties of the fluid, such as flow velocity $\mathbf{v}(\mathbf{r}, t)$, pressure $p(\mathbf{r}, t)$, density $\rho(\mathbf{r}, t)$ as functions of space $\mathbf{r}$ and time t. By using the so called Euler's method the time-derivative of the flow velocity is given as

$$\begin{aligned} \frac{d\mathbf{v}(\mathbf{r}, t)}{dt} &= \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} + \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} + v_x \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial x} + v_y \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial y} + v_z \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial z}. \end{aligned} \tag{362}$$

A flow that is not a function of time is called **steady flow**. Steady-state flow refers to the condition where the fluid properties at a point in the system do not change over time. Time dependent flow is known as unsteady (also called transient). Whether a particular flow is steady or unsteady, can depend on the chosen frame of reference. For instance, laminar flow over a sphere is steady in the frame of reference that is stationary with respect to the sphere. In a frame of reference that is stationary with respect to a background flow, the flow is unsteady.

Flow in which randomness (turbulence) is not exhibited is called **laminar**. Laminar flow is characterised by fluid particles following smooth paths in layers, with each layer moving smoothly past the adjacent layers with little or no mixing. At low velocities, the fluid tends to flow without lateral mixing, and adjacent layers slide past one another like playing cards. There are no cross-currents perpendicular to the direction of flow, nor eddies or swirls of fluids. In laminar flow, the motion of the particles of the fluid is very orderly with particles close to a solid surface moving in straight lines parallel to that surface.

Streamlines and **pathlines** are field lines in a fluid flow. They differ only when the flow changes with time, that is, when the flow is not steady. Streamlines are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction in which a massless fluid element will travel at any point in time. Pathlines are the trajectories that individual fluid particles follow. These can be thought of as "recording" the path of a fluid element in the flow over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time. In steady flow (when the velocity vector-field does not change with time), the streamlines and pathlines coincide. Knowledge of the streamlines can be useful in fluid dynamics,

- different streamlines at the same instant in a flow do not intersect, because a fluid particle cannot have two different velocities at the same point.
- streamlines have no origin and end points.

Imagine a set of streamlines starting at points that form a closed loop. These streamlines form a tube that is impermeable since the walls of the tube are made up of streamlines, and there can be no flow normal to a streamline (by definition). This tube is called a **streamtube**.

4.4.1 Continuity equation

Since streamtube is the flow element which is bounded by streamlines, the flow is possible only through the tube and not across the tube. Therefore, the definition of the streamtube is particularly useful for deriving the so called continuity equation for fluid flow. For deriving the continuity equation, consider an elemental streamtube. Since there is no mass flow across the surface of the streamtube, by the law of conservation of mass, we can say mass entering the streamtube per second is equal to the mass leaving the tube per second. Thus the mass current is constant,

$$J^{(m)} \equiv \lim_{\Delta t \to 0} \frac{\Delta t A \rho v}{\Delta t} = A\rho v = \text{constant}, \qquad \to \qquad J_1^{(m)} = J_2^{(m)} \tag{363}$$

where A is the surface area of the cross-section of the streamtube, ρ is the density of the fluid (liquid and gas) and v is the flow velocity. This can be integrated and written for a finite streamtube with average values at the entrance and exit of the streamtube.

For incompressible steady flow the equation for conservation of mass is reduced to the continuity equation which means that the volume current is constant in a streamtube,

$$J^{(V)} \equiv \lim_{\Delta t \to 0} \frac{\Delta t A v}{\Delta t} = Av = \text{constant}, \qquad \to \qquad J_1^{(V)} = J_2^{(V)}. \tag{364}$$

4.4.2 Bernoulli's equation

In fluid dynamics, Bernoulli's principle states that an increase in the speed of a fluid occurs simultaneously with a decrease in static pressure or a decrease in the fluid's potential energy. Bernoulli's principle can be derived from the principle of conservation of energy (work-energy theorem). This states that, in a steady, incompressible and ideal flow, the sum of all forms of energy in a fluid along a streamline is the same at all points on that streamline. This formula is Bernoulli's equation. Consider a pipe (reservoir) with varying diameter and height through which an incompressible fluid (liquid) is flowing. In figure Fig. 3 one can see the relationship between the areas of cross-sections A, the flow speed v, height from the ground z, and pressure p at two different points 1 and 2 is given in the Let us assume that (i) the density of the incompressible fluid remains constant at both points, (ii) the energy of the fluid is conserved as there are no viscous forces in the fluid. Therefore, the work done on the fluid is given as,

$$\begin{aligned} \Delta W &= F_1 \Delta s_1 - F_2 \Delta s_2, \\ \Delta W &= p_1 A_1 \Delta s_1 - p_2 A_2 \Delta s_2, \\ \Delta W &= p_1 \Delta V - p_2 \Delta V = (p_1 - p_2)\Delta V \end{aligned} \tag{365}$$

where the continuity equation, $\Delta V_1 = A_1 v_1 \Delta t = \Delta V_2 = A_2 v_2 \Delta t \equiv \Delta V$ has been used. The change in kinetic energy of the fluid is given as,

$$\Delta E_{\text{kin}} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = \frac{1}{2} \rho \Delta V (v_2^2 - v_1^2). \tag{366}$$

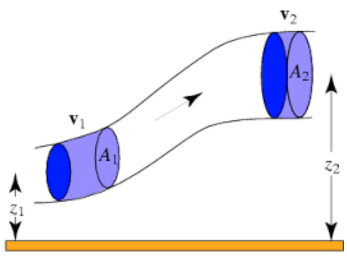


Figure 3:

The change in potential energy is given as,

$$\Delta E_{\text{pot}} = mgz_1 - mgz_2 = \rho \Delta V g (z_1 - z_2). \tag{367}$$

Therefore, the work-energy equation is given as,

$$\Delta E_{\rm kin} = \Delta E_{\rm pot} + \Delta W, \quad \rightarrow \quad \frac{1}{2}\rho\Delta V(v_2^2 - v_1^2) = \rho\Delta V g(z_1 - z_2) + (p_1 - p_2)\Delta V$$
$$\frac{1}{2}\rho v_1^2 + p_1 + \rho g z_1 = \frac{1}{2}\rho v_2^2 + p_2 + \rho g z_2 \quad \rightarrow \quad \frac{1}{2}\rho v^2 + p + \rho g z = \text{constant}. \tag{368}$$

Let us consider some applications of Bernoulli's principle or Bernoulli's equation which states that $\frac{1}{2}\rho v^2 + p + \rho g z$ is constant for any two points in the liquid.

- Torricelli's law states that the speed v of efflux of a fluid through a sharp-edged hole at the bottom of a tank filled to a depth h is $v = \sqrt{2gh}$, where g is the acceleration due to gravity. If we need to calculate the speed of efflux we can use Bernoulli's formula where the first point can be taken at the liquid's surface, and the second just outside the opening, so one finds $p_0 + \rho g h = \frac{1}{2}\rho v^2 + p_0$ which results in $v = \sqrt{2gh}$.

- Venturi meter is a device that is based on Bernoulli's theorem and is used for measuring the rate of flow of liquid through the pipes. The Venturi effect is the reduction in fluid pressure that results when a fluid flows through a constricted section (or choke) of a pipe. Using Bernoulli's equation in the special case of steady, incompressible, inviscid flows along a streamline, the theoretical pressure drop at the constriction is given by, $p_1 - p_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_2^2$. If one incorporates the continuity equation, $A_1 v_1 = A_2 v_2$ then the volume current can be given as $J^{(V)} = A_1 A_2 \sqrt{p_1 - p_2}/\sqrt{\frac{1}{2}\rho(A_1^2 - A_2^2)}$.

- The Pitot-Prandtl tube is a flow measurement device used to measure fluid flow velocity based on Bernoulli's equation. The pitot tube is widely used to determine the airspeed of an aircraft. The basic pitot tube consists of a tube pointing directly into the fluid flow. As this tube contains fluid, a pressure can be measured; the moving fluid is brought to rest (stagnates) as there is no outlet to allow flow to continue. This pressure is the stagnation pressure of the fluid, also known as the total pressure or (particularly in aviation) the pitot pressure. The measured stagnation pressure cannot itself be used to determine the fluid flow velocity (airspeed in aviation). However, Bernoulli's equation states: Stagnation pressure = static pressure + dynamic pressure. Which can also be written $p_t = p_s + \frac{1}{2}\rho v^2$ which results in $v = \sqrt{2(p_t - p_s)/\rho}$.

4.5 Fluid dynamics - viscous flow

As we have already discussed, fluid is a substance that does not support shear stress which means that a fluid *at rest* has no shear stress at all. What if we consider fluid dynamics? In this case, it is not anymore true that shear stress does not appear. Indeed, viscosity is the physical property that characterizes the flow resistance of simple fluids. A fluid that has no resistance to shear stress is known as an ideal or inviscid fluid.

4.5.1 Newton's law of viscosity

Newton's law of viscosity defines the relationship between the shear stress and shear rate of a fluid subjected to a mechanical stress. The ratio of shear stress to shear rate is a constant, for a given (temperature) and pressure, and is defined as the viscosity or coefficient of viscosity. Newtonian fluids obey Newton's law of viscosity. The viscosity is independent of the shear rate. Non-Newtonian fluids do not follow Newton's law and, thus, their viscosity (ratio of shear stress to shear rate) is not constant and is dependent on the shear rate.

Viscosity can be conceptualized as quantifying the internal frictional force that arises between adjacent layers of fluid that are in relative motion. For instance, a fluid is trapped between two infinitely large plates, one fixed and one in parallel motion at constant speed $v_{x,max}$. If the speed of the top plate is low enough (to avoid turbulence), then in steady state the fluid particles move parallel to it, and their speed varies from $v_x = 0$ at the bottom to $v_x = v_{x,max}$ at the top. Each layer of fluid (in the z-direction) moves faster (in the x-direction) than the one just below it, thus one can define an angle which characterises the velocity difference of each layers,

$$\tan(\Delta\gamma) \approx \Delta\gamma = \frac{[v_x(z+\Delta z) - v_x(z)]\Delta t}{\Delta z} \quad \rightarrow \quad \frac{\Delta\gamma}{\Delta t} = \frac{v_x(z+\Delta z) - v_x(z)}{\Delta z} \quad \rightarrow \quad \frac{d\gamma}{dt} = \frac{dv_x}{dz} \tag{369}$$

and friction between them gives rise to a force resisting their relative motion, thus one can define a shear stress,

$$\sigma_{\text{shear}} = \eta \; |\frac{dv_x}{dz}|, \tag{370}$$

where η is Newton's viscosity constant and the corresponding force is given as

$$F_{\text{shear}} = \sigma_{\text{shear}} A = \eta \; A \; |\frac{dv_x}{dz}|, \tag{371}$$

where A is the surface area.

4.5.2 Hagen-Poiseuille law

In nonideal fluid dynamics, the Hagen-Poiseuille equation, also known as the Hagen-Poiseuille law, is a physical law that gives the pressure drop in an incompressible and Newtonian fluid in laminar flow flowing through a long cylindrical pipe of constant cross section. When a fluid is forced through a tube, it flows more quickly near the tube's axis than near its walls. In such a case, experiments show that some stress such as a pressure difference between the two ends of the tube is needed to sustain the flow through the tube. This is because a force is required to overcome the friction between the layers of the fluid which are in relative motion: the strength of this force is proportional to the viscosity. Hagen-Poiseuille equation describes the pressure drop due to the viscosity of the fluid.

A cylindrical fluid element (fluid parcel) with the radius r and length l is considered. Forces act on the end faces of this volume element which are considered constant over the entire cross-section of the pipe. These forces result from the pressures in the fluid acting on the faces of the volume element. At the point x_1 the pressure p_1 acts and at the point x_2 a lower pressure p_2 acts which results in the force $(p_1 - p_2)r^2\pi$.

On the other hand, the speed of the flow within the pipe is not constant over the cross-section. There is friction between the fluid and the pipe, which is why the flow velocity near the wall is lower than in the middle of the pipe. At the wall, the fluid even adheres to the wall due to the adhesive forces. Thus, there is no relative speed between fluid and wall, i.e., $v(r = R) = 0$ which is also called no-slip condition where the radius of the pipe is R. The individual fluid layers also have friction between each other due to the viscosity of the fluid. This leads to the formation of a certain velocity profile, the course of which is yet to be determined. It should be noted, that the flow velocity is maximum in the middle of the pipe i.e., $v(r = 0) = v_0$. Thus, one can define a shear (frictional) force as $\eta 2\pi l \frac{dv}{dr}$.

For a steady flow, in which the flow velocities no longer change over time, a balance of forces applies to the volume element under consideration. The pressure force and the counteracting shear (frictional) force can thus be equated,

$$(p_1 - p_2)r^2\pi + \eta 2\pi l \frac{dv}{dr} = 0, \quad \rightarrow \quad \frac{dv}{dr} = -\frac{p_1 - p_2}{2\eta l} r, \quad \rightarrow \quad v(r) = -\frac{p_1 - p_2}{4\eta l} r^2 + v_0. \tag{372}$$

Since $v(R) = 0$ one finds $v_0 = \frac{p_1 - p_2}{4\eta l} R^2$ and the radial dependence of the velocity profile reads as

$$v(r) = \frac{p_1 - p_2}{4\eta l} R^2 \left(1 - \frac{r^2}{R^2}\right) = v_0 \left(1 - \frac{r^2}{R^2}\right). \tag{373}$$

Since the velocity profile is now known, the volume flow rate or volume current can now be determined by the cross-section of a pipe. To derive the flow rate, we consider a ring with the infinitesimal thickness Δr_i at any distance r_i from the center of the pipe. The area of the considered ring ΔA_i can be derived from the "length" of the ring $2\pi r_i$ (circumference) and the "height" of the ring Δr_i (thickness), $\Delta A_i = 2\pi r_i \Delta r_i$. The volume current is then given by

$$\begin{aligned} J^{(V)} &= \sum_{i=0}^{R} v(r_i)\Delta A_i = \sum_{i=0}^{R} v_0 \left(1 - \frac{r_i^2}{R^2}\right) 2\pi r_i \Delta r_i \\ &= 2\pi v_0 \int_0^R \left(1 - \frac{r^2}{R^2}\right) r dr = \frac{R^2 \pi v_0}{2} = \frac{R^4 \pi (p_1 - p_2)}{8\eta l}. \end{aligned} \tag{374}$$

4.6 Turbulent flows - Aerodynamics

In fluid dynamics, turbulence or turbulent flow is fluid motion characterized by chaotic changes in pressure and flow velocity. It is in contrast to a laminar flow, which occurs when a fluid flows in parallel layers, with no disruption between those layers. Turbulence is commonly observed in everyday phenomena such as surf, fast flowing rivers, billowing storm clouds, or smoke from a chimney, and most fluid flows occurring in nature or created in engineering applications are turbulent. In general terms, in turbulent flow, unsteady vortices appear of many sizes which interact with each other, consequently drag due to friction effects increases.

The onset of turbulence can be predicted by the dimensionless Reynolds number. Laminar flow occurs at lower Reynolds number, below a threshold at which the flow becomes turbulent. Turbulent flow is a less orderly flow regime that is characterised by randomness which result in lateral mixing. The Reynolds number depends on the viscosity and density of the fluid and dimensions of the channel,

$$Re = \frac{Rv\rho}{\eta} \tag{375}$$

where R is a geometric parameter (e.g., radius of a ball), v is the velocity, ρ is the mass density of the fluid and η is the viscosity. A typical value for the critical Reynolds number is around $Re \approx 1160$ where the flow becomes turbulent.

In 1851, George Gabriel Stokes derived an expression, now known as Stokes law, for the frictional force - also called drag force - exerted on spherical objects with very small Reynolds numbers in a viscous fluid. The force of viscosity on a small sphere moving through a viscous fluid is given by

$$F_I = 6\pi\eta R v, \tag{376}$$

thus the constant of the drag force for low velocities can be determined. For large velocities one finds,

$$F_{II} = \text{constant}\frac{1}{2}\rho R v^2. \tag{377}$$

The Magnus effect is an observable phenomenon that is commonly associated with a spinning object moving through air or another fluid. The path of the spinning object is deflected in a manner that is not present when the object is not spinning. The deflection can be explained by the difference in pressure of the fluid on opposite sides of the spinning object. The Magnus Effect depends on the speed of rotation. An intuitive understanding of the phenomenon comes from Newton's third law, that the deflective force on the body is a reaction to the deflection that the body imposes on the air-flow. The body "pushes" the air in one direction, and the air pushes the body in the other direction. In particular, a lifting force is accompanied by a downward deflection of the air-flow.

5 Wave motion - elastic waves

Wave motion is very common in nature. We have even evolved receptors capable of detecting waves. In this chapter we develop the verbal and mathematical description of waves.

Mechanical waves can travel far, but the particles of the *elastic medium*, where the waves travel do not make that journey. For instance, ocean waves travel thousands of kilometers, without transporting any water. What kind of physical quantity is transported in waves then? We can produce mechanical waves by displacing some portion of a continuous elastic medium, which is called the center of the wave. For instance, two of us can stretch a long elastic rope; one can make a disturbance at one end by suddenly displacing the end. In a short time the person at the other end will feel a sudden pull at his hand indicating that the elastic medium exerts force on it, therefore, momentum is transported in the wave. Energy is also transported. When displacing the medium, it becomes stretched, therefore, at the disturbance the medium has elastic potential energy. The particles also move with some velocity, therefore, kinetic energy is also transported.

We can distinguish two kind of mechanical waves by considering the direction of the motion of particles of the elastic medium and the direction of the propagation of waves. If these are perpendicular, then the wave is called a transverse wave, if the motion of particles is aligned with the direction of wave propagation, the wave is called a longitudinal wave. The waves on the surface of water are complicated combination of both types. Waves can be further classified according to the type of motion of the particles at the center of the wave. If it is a single displacement followed by a return, then we talk about a pulse. If the center continues to move back and forth, then it produces a train of waves. If the motion at the center is periodic, then it produces a periodic train of waves — the simplest special case being a harmonic wave.

5.1 Speed of travelling waves

Wave motion can occur in any physical dimensions. For the sake of simplicity, let us consider wave motion in one dimension. It is an experimental fact that the speed of the wave, c is to a good approximation does not depend on the waveform, or on the frequency of the periodic wave. Let us investigate the of speed of travelling waves for three cases, (i) transverse wave travelling along a stretched string, (ii) longitudinal wave travelling along a string, (iii) longitudinal wave travelling in a fluid.

5.1.1 Transverse wave travelling along a stretched string

Let us consider a stretched horizontal string where two horizontal forces with the same magnitude F_0 act on each edges of the string which has a cross-section surface A, so, the stress is given as $\sigma_0 = F_0/A$. Let us calculate the speed of a transverse wave travelling along the stretched string. Assume that the deformation is caused by a constant vertical force (F) acting on a very small line segment of the horizontal stretched string which creates a V-shaped deformation or in other words a triangle-shaped pulse. The points of this deformed part have a constant vertical velocity (v) and the deformation propagates horizontally with the speed (c) towards both left and right directions. The half of the V-shaped deformation is a right triangle where θ is the angle between the horizontal line and the deformation, thus

$$\tan\theta = \frac{v\,\Delta t}{c\,\Delta t} = \frac{v}{c}. \tag{378}$$

The V-shaped deformation carries a momentum $\mathbf{\Delta p} = 2\rho A\, c\,\Delta t\,\mathbf{v}$ which is related to the vertical force as

$$F = \frac{|\Delta\mathbf{p}|}{\Delta t} = 2\rho A\, c\, v \tag{379}$$

On the other hand, the following relations holds betweem the vertical and horizontal components of forces,

$$F = 2F_1\sin\theta, \qquad F_0 = F_1\cos\theta, \quad \rightarrow \quad F = 2F_0\tan\theta = 2F_0\frac{v}{c} \tag{380}$$

where F_1 is the magnitude of the forces exerted by the deformed string. Thus, one finds

$$2\rho A\, c\, v = 2F_0\frac{v}{c} \quad \rightarrow \quad c = \sqrt{\frac{F_0}{A\rho}} = \sqrt{\frac{\sigma_0}{\rho}} = \sqrt{\epsilon}_0\sqrt{\frac{E}{\rho}} \tag{381}$$

where $\sigma_0 = E\epsilon_0$. We showed that the speed of the wave, c depends only on the mechanical properties of the elastic medium, namely on its density ρ and elasticity, characterized by Young's modulus E.

5.1.2 Longitudinal wave travelling along a string

Let us repeat the previous analysis for a longitudinal pulse where the strain is given by

$$\epsilon = \frac{\Delta \ell}{\ell} = \frac{v\Delta t}{c\Delta t} = \frac{v}{c} \tag{382}$$

and the corresponding stress and the force can be written as

$$\sigma = E\epsilon = E\frac{v}{c} \qquad \rightarrow \qquad F = \sigma A = EA\frac{v}{c}. \tag{383}$$

Since $\Delta p = \rho A \, c \, \Delta t \, v$ and the force is $F = \Delta p/\Delta t$, thus one finds

$$\rho A \, c \, v = EA\frac{v}{c}. \qquad \rightarrow \qquad c = \sqrt{\frac{E}{\rho}}. \tag{384}$$

Therefore, we found the general formula

$$c = k\sqrt{\frac{E}{\rho}}\,. \tag{385}$$

where k is a dimensionless constant and for longitudinal pulse $k = 1$ and for a transverse pulse is $k = \sqrt{\epsilon_0}$, where ϵ_0 is the relative deformation of the string without the wave on it. This shows that transverse waves can travel only if there is already stress in the medium without the wave.

5.1.3 Longitudinal wave travelling in a fluid

In fluids (liquids and gases) one observes only longitudinal travelling waves. This is because transverse waves can travel only if there is already stress in the medium and one finds no shear stress in fluids at rest. Longitudinal waves are waves in which the displacement of the medium is in the same direction as, or the opposite direction to, the direction of propagation of the wave. Mechanical longitudinal waves are also called compressional or compression waves, because they produce compression and rarefaction when traveling through a medium, and pressure waves, because they produce increases and decreases in pressure. Although liquids are considered as incompressible fluids this does not mean that a very small deformation is not allowed. Thus, in case of longitudinal travelling waves in fluids one should start from the volume deformation caused by the mechanical longitudinal wave,

$$\frac{\Delta V}{V} = -\frac{1}{\kappa}\Delta p \tag{386}$$

where κ is the compression modulus. Based on this, the speed of the travelling longitudinal wave is written as

$$F = A\Delta p = A\kappa\frac{(-\Delta V)}{V} = A\kappa\frac{v}{c}, \quad F = \rho A \, c \, v, \quad \rightarrow \quad \rho A \, c \, v = \kappa A\frac{v}{c} \quad \rightarrow \quad c = \sqrt{\frac{\kappa}{\rho}}. \tag{387}$$

5.2 Wave function and wave equation

If the motion of the particles at the center of the wave, where we choose the origin of our coordinate system ($x = 0$), can be described by a function $f(t)$ than the same motion will appear at position $x \neq 0$ at a later time. The time delay is determined by the speed of the wavefront. Thus the motion of the particles at position x is described by a function

$$\xi(x,t) = f\left(t \mp \frac{x}{c}\right)\,, \tag{388}$$

which is called the wave function. The sign in the argument of f is negative if the wave travels in positive x direction and vice versa. The wave function is a function of both position and time.

The wave function has the special property that its two arguments are related as shown by Eq. (388). As a result the time and space derivatives are also related:

$$\frac{\partial \xi}{\partial t} = \mp c \frac{\partial \xi}{\partial x} \,. \tag{389}$$

Eq. (389) depends on the direction of wave propagation. If we take the second derivatives, this ambiguity disappears,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} \,, \tag{390}$$

which is called the wave equation. It is identical to the equation of motion for the particles of the elastic medium at position x, therefore, it is a consequence of Newton's 2nd law (therefore, can be derived from it).

5.3 Energy transport in travelling waves

As we have already discussed, apart from momentum, energy is also transported in wave motion. If one creates a disturbance (transverse or longitudinal), the medium has elastic potential energy. In additon, the particles of the medium in case of disturbance also move with some velocity, therefore, kinetic energy is also transported. Let us consider the relation between the kinetic and potential energy carried by the mechanical wave in two cases, when (i) transverse wave is travelling along a stretched string, (ii) longitudinal wave is travelling along a string.

5.3.1 Energy transport in a transverse wave travelling along a stretched string

If we have a triangle-shape transverse wave in a string, one can relate kinetic energy to each individual bit of the disturbed part of the string, so, a kinetic energy density is written as,

$$u_k = \frac{1}{2}\rho v^2 \tag{391}$$

where v is the transverse velocity of each individual bit of the disturbed part of the string and ρ is the mass density. The potential energy depends on how stretched the string is. Indeed, the potential energy density,

$$u_p = \frac{1}{2}E\epsilon^2 \tag{392}$$

depends on the relative deformation $\epsilon = (\ell' - \ell_0)/\ell_0$. Of course, having a string with some tension $\epsilon_0 = (\ell - \ell_0)/\ell_0$ automatically has some potential energy due to the stretching, even if there are no waves passing through the string and one finds $\ell/\ell_0 = 1 + \epsilon_0$. We are instead interested in the potential energy stored in the string as it is stretched further from ℓ to ℓ' due to the propagation of transverse waves.

The half of the triangle-shape disturbance is a right triangle and the amount the string is stretched at point x is given by the difference between the length of the hypotenuse ℓ' of this triangle and its base ℓ,

$$\epsilon = \frac{\ell' - \ell_0}{\ell_0} = \frac{\sqrt{\ell^2 + d^2} - \ell_0}{\ell_0} = \frac{\ell\sqrt{1 + d^2/\ell^2} - \ell_0}{\ell_0} \tag{393}$$

where d is the height of the triangle. Since the string is close to equilibrium $d << \ell$, so we can Taylor expanding the square-root,

$$\sqrt{1 + d^2/\ell^2} \approx 1 + \frac{1}{2}\frac{d^2}{\ell^2} \qquad \rightarrow \qquad \epsilon = \frac{\ell(1 + \frac{1}{2}\frac{d^2}{\ell^2}) - \ell_0}{\ell_0} = \epsilon_0 + \frac{1}{2}\frac{d^2}{\ell^2}\frac{\ell}{\ell_0} = \epsilon_0 + \frac{1}{2}\frac{d^2}{\ell^2}(1 + \epsilon_0) \tag{394}$$

Since $\epsilon_0 << 1$ and $d << \ell$ one can further approximate the relative deformation and its squared value

$$\epsilon \approx \epsilon_0 + \frac{1}{2}\frac{d^2}{\ell^2} = \epsilon_0 + \frac{1}{2}\frac{v^2}{c^2} \qquad \rightarrow \qquad \epsilon^2 = \left(\epsilon_0 + \frac{1}{2}\frac{v^2}{c^2}\right)^2 \approx \epsilon_0^2 + \epsilon_0\frac{v^2}{c^2} \tag{395}$$

Thus, by dropping all subleading terms, the potential energy density reads as

$$u_p = \frac{1}{2}E\epsilon_0^2 + \frac{1}{2}E\epsilon_0\frac{v^2}{c^2} \qquad \rightarrow \qquad u_p = \frac{1}{2}E\epsilon_0\frac{v^2}{c^2} \tag{396}$$

where the constant term $\frac{1}{2}E\epsilon_0^2$ is set to zero by choosing appropriately the zero point of the potential energy density. Inserting $c = \sqrt{\epsilon_0}\sqrt{E/\rho}$ into the expression derived for the potential energy density, one finds

$$u_p = \frac{1}{2}E\epsilon_0\frac{v^2\rho}{\epsilon_0 E} = \frac{1}{2}\rho v^2 = u_k \tag{397}$$

that the kinetic and the potential energy densities are the same for all points of a travelling transverse wave.

5.3.2 Energy transport in a longitudinal wave travelling along a string

Let us repeat the above derivation for a longitudinal wave which is travelling along a string. In this case the relative deformation is directly related to the ratio v/c,

$$\epsilon = \frac{\Delta\ell}{\ell} = \frac{v\Delta t}{c\Delta t} = \frac{v}{c} \qquad \rightarrow \qquad \epsilon^2 = \frac{v^2}{c^2} \qquad \rightarrow \qquad u_p = \frac{1}{2}E\epsilon^2 = \frac{1}{2}E\frac{v^2}{c^2} = \frac{1}{2}E\frac{v^2\rho}{E} = u_k \tag{398}$$

where we used the expression $c = \sqrt{E/\rho}$ for the speed of the longitudinal wave.

Therefore, it is always true that for travelling waves the kinetic and the potential energy densities are the same for all points. Thus, the total energy density is given as

$$u = u_k + u_p = 2u_k = 2u_p. \tag{399}$$

From practical point of view, waves are created to transport energy without transporting particles. The total energy transported in time Δt over the cross section A of the elastic medium is $c\Delta t A u$. Accordingly, the energy current and the energy density current are written as

$$J^{(E)} = \frac{c\,\Delta t\,A\,u}{\Delta t} = c\,A\,u, \qquad \rightarrow \qquad j^{(E)} = \frac{J^{(E)}}{A} = c\,u. \tag{400}$$

where the latter is also called the rate of flow of energy density, or the transmitted power density.

5.4 Superposition, reflection

It is an experimental fact that linear superposition holds for simple mechanical waves. This means that energy and momentum are conserved if two wave fronts or pulses meet. In other words, if we require energy and momentum conservation then we can rely on linear superposition of wave fronts and pulses.

From dimensional analysis we could conclude that n different ξ_i waves traveling in the same region of space could merge to give a wave

$$\xi = \left(\sum_i^n \xi_i^n\right)^{1/n}, \tag{401}$$

with any n. The principle of linear superposition asserts that $n = 1$, which means that displacements, and therefore, velocities of particles in the elastic medium, add linearly. Observing simple triangle pulses of waves, it is easy to prove that energy and momentum conservation requires linear superposition.

Thus, for example if two identical triangle-shape transverse pulses travel towards each other and they do not overlap, the total energy density of the two waves is $u = 4u_p = 4u_k$. If they fully overlap, the height of the superposed wave front ($2d$) is twice as much large as those of the individual wave fronts (d) but its vertical velocity is zero $v = 0$, thus,

$$u = \frac{1}{2}E\frac{(2d)^2}{\ell^2} = 4\left(\frac{1}{2}E\frac{d^2}{\ell^2}\right) = 4u_p \tag{402}$$

which is in agreement of energy conservation in case of linear superposition.

Let us assume a wave pulse on a string moving from left to right towards the end which is rigidly clamped. As the wave pulse approaches the fixed end, the internal restoring forces which allow the wave to propagate exert an upward force on the end of the string. But, since the end is clamped, it cannot move. According to Newton's third law, the wall must be exerting an equal downward force on the end of the string. This new force creates a wave pulse that propagates from right to left, with the same speed and amplitude as the incident wave, but with opposite polarity (upside down).

Let us assume a wave pulse on a string moving from left to right towards the end which is free to move vertically (imagine the string tied to a massless ring which slides frictionlessly up and down a vertical pole). The net vertical force at the free end must be zero. This boundary condition is mathematically equivalent to requiring that the slope of the string displacement be zero at the free end. The reflected wave pulse propagates from right to left, with the same speed and amplitude as the incident wave, and with the same polarity (right-side up).

Therefore, observational rules of reflection of mechanical waves from hard and soft boundaries are the following:

- at a fixed (hard) boundary, the displacement remains zero and the reflected wave changes its polarity (undergoes a 180° phase change),
- at a free (soft) boundary, the restoring force is zero and the reflected wave has the same polarity (no phase change) as the incident wave.

When a wave encounters a boundary which is neither rigid (hard) nor free (soft) but instead somewhere in between, part of the wave is reflected from the boundary and part of the wave is transmitted across the boundary. The exact behavior of reflection and transmission depends on the material properties on both sides of the boundary.

5.5 Harmonic waves

We state without proof a mathematical theorem: *any periodic function $f(t)$ can be written as an infinite sum of harmonic functions.* Therefore, studying harmonic waves gives results that are valid generally. Harmonic waves are produced if the motion of the wave center is sinusoidal,

$$\xi(x=0,t) = A\sin(\omega t + \phi)\,. \tag{403}$$

According to Eq. (388), the wave function is

$$\xi(x,t) = A\sin\left[\omega\left(t \mp \frac{x}{c}\right) + \phi\right]\,. \tag{404}$$

We can rewrite the argument in a more symmetric way,

$$\omega\left(t \mp \frac{x}{c}\right) = 2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda}\right)\,, \tag{405}$$

where $\lambda = cT$ is called the wavelength, which is the distance that the wavefront covers during one complete period of the oscillation at the center (time period T). Introducing the wave number $k = 2\pi/\lambda$, which gives the number of waves in 2π units of length, we can write the argument as $\omega t - kx$, which is the most common way of representation.

5.6 Energy transport in waves with differentiable wave function

We have already proved that for traveling transverse and longitudinal wave pulses the kinetic (u_k) and potential (u_p) energy densities transported by the pulse are equal at any fixed position at any instant. Let us consider the most general case, where we do not restrict our considerations to a specific shape of a wave pulse but we investigate arbitrary wave fronts and pulses requiring only the differentiability of its wave function.

5.6.1 Energy transport in a transverse differentiable wave

We can prove this proposition for transverse waves using arguments similar to the case of the specific triangle-shape situation. Our starting point is that the relative deformation $\epsilon(x)$ can be obtained by considering a small part of the medium of length Δx at position x. In this case the total deformation is given as

$$\epsilon(x) = \frac{\Delta x' - \Delta x_0}{\Delta x_0} = \frac{\sqrt{\Delta x^2 + (\frac{\partial \xi}{\partial x}\Delta x)^2} - \Delta x_0}{\Delta x_0} = \frac{\Delta x\sqrt{1 + (\frac{\partial \xi}{\partial x})^2} - \Delta x_0}{\Delta x_0} \tag{406}$$

where $(\Delta x')^2 = \Delta x^2 + \Delta\xi^2 = \Delta x^2 + (\frac{\partial \xi}{\partial x}\Delta x)^2$. After dropping all subleading terms one finds,

$$\epsilon(x) \approx \epsilon_0 + \frac{1}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 , \tag{407}$$

where $\epsilon_0 = (\Delta x - \Delta x_0)/\Delta x_0$. Therefore, the potential energy density transported by the wave is

$$u_p = \frac{1}{2}E\epsilon(x)^2 - \frac{1}{2}E\epsilon_0^2 \simeq \frac{1}{2}\epsilon_0 E\left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{1}{2}\varrho c^2\left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{1}{2}\varrho\left(\frac{\partial \xi}{\partial t}\right)^2 = u_k\,, \tag{408}$$

where we used the square of eq. (389) and the expression for the speed of transverse wave propagation, $c^2 = \epsilon_0 E/\rho$.

5.6.2 Energy transport in a longitudinal differentiable wave

The potential energy of the elastic medium is due to the relative deformation in the wave, its density is given by

$$u_p(x) = \frac{1}{2}E\epsilon(x)^2\,. \tag{409}$$

The relative deformation $\epsilon(x)$ can be obtained by considering a small part of the medium of length Δx at position x. The difference of the wave function at $x + \Delta x$ and x and dividing by the length Δx, gives the relative deformation of this portion of the medium. If we take the limit $\Delta x \to 0$, we obtain

$$\epsilon(x) = \lim_{\Delta x\to 0}\frac{\Delta x + \xi(x+\Delta x, t) - \xi(x,t) - \Delta x}{\Delta x} \equiv \frac{\partial \xi}{\partial x}\,, \tag{410}$$

which is the derivative of the wave function with respect to x at fixed t. The kinetic energy density is

$$u_k(x) = \frac{1}{2}\rho v(x)^2\,, \tag{411}$$

where $v(x)$ is the velocity of the particles in the medium at position x. It is the time derivative of the displacement of the particle,

$$v(x) = \frac{\partial \xi}{\partial t}\,, \tag{412}$$

which is the derivative of the wave function with respect to time at fixed position. Taking the square of eq. (389) and using the expression for the speed of wave propagation, $c^2 = E/\rho$, we find that

$$v(x)^2 = \frac{E}{\rho}\epsilon(x)^2\,, \tag{413}$$

which proves that $u_k(x) = u_p(x)$. Thus the total energy density is twice the kinetic, or potential energy density.

5.6.3 Energy density current for longitudinal harmonic waves

For waves propagating in more than one dimension, the transmitted power density is the energy current transmitted across a unit area normal to the direction of the wave propagation. The intensity of the wave is the average power density over one period in time. We can explicitly compute the intensity of a harmonic wave from the total energy density transported by the wave.

Consider a longitudinal harmonic wave described by Eq. (404) and for simplicity, choose $\phi = 0$. The kinetic energy density is

$$u_k(x) = \frac{1}{2}\rho A^2 \omega^2 \cos^2(\omega t - kx)\,, \tag{414}$$

and the potential energy density is

$$u_p(x) = \frac{1}{2}EA^2 k^2 \cos^2(\omega t - kx)\,. \tag{415}$$

Using $c = \omega/k = \sqrt{E/\rho}$, we see explicitly that $u_p = u_k$. Then the energy density current is

$$j = uc = 2u_k c = \rho A^2 \omega^2 \cos^2(\omega t - kx) \tag{416}$$

and the intensity (of a harmonic wave) is

$$\langle j \rangle = \frac{1}{T}\int_t^{t+T} j(t)dt = \frac{1}{2}c\rho A^2 \omega^2\,. \tag{417}$$

5.7 Standing waves

Let us now imagine two harmonic waves of the same amplitude and frequency moving in *opposite* directions, $\xi_1(x,t) = A\sin(\omega t - kx)$, $\xi_2(x,t) = A\sin(\omega t + kx + \Delta\phi)$ where $\Delta\phi$ is the phase difference. The resulting wave is

$$\xi(x,t) = \xi_1(x,t) + \xi_2(x,t) = 2A\sin(\omega t + \Delta\phi/2)\cos(kx + \Delta\phi/2)\,. \tag{418}$$

We see that the dependence on time and position is factored. The particles of the medium oscillate in the same phase at every point of the medium. Such a wave pattern is called standing wave. The amplitudes of these oscillations vary with position. There are certain points, called nodes, where the displacement is always zero. Between the nodes, there are antinodes, where the displacement oscillates with the largest amplitude. Since the nodes never move, therefore, energy is not transported through those points, so energy is also standing in the string, although it alternates between vibrational kinetic energy and elastic potential energy.

From practical point of view, there are two interesting cases. Let us imagine a string with one fixed end. If a traveling wave arrives at the fixed end of the string, then it returns with a phase shift of $\Delta\phi = \pi$, which is the only way of ensuring the fixed position and energy conservation. Then the superposition of the incoming and reflected waves is $\xi(x,t) = 2A\cos(\omega t)\sin(kx)$. The position of antinodes is determined by the condition $kx = (2n+1)\frac{\pi}{2}$, where n is a positive integer. Thus, the antinodes are at $x = (2n+1)\frac{\lambda}{4}$, they are separated one-half wavelengths apart. The nodes are between antinodes at $x = n\frac{\lambda}{2}$

If a longitudinal wave arrives at the unfixed end of the medium, then it returns with phase shift $\Delta\phi = 0$, which is the only way of ensuring energy and momentum conservation. Then the superposition of the incoming and reflected waves is $\xi(x,t) = 2A\sin(\omega t)\cos(kx)$, which restricts the position of nodes to $x = (2n+1)\frac{\lambda}{4}$, and that of the antinodes to $x = n\frac{\lambda}{2}$. Thus we see that standing waves cannot have arbitrary frequency.

Next, consider a string with fixed ends at $x = 0$ and $x = L$. The string will have some damping as it is stretched by traveling waves, but assume the damping is very small. Suppose that at the $x = 0$ fixed end a sinusoidal force is applied that drives the string up and down in the y-direction with a small amplitude at some frequency ν. In this situation, the driving force produces a right-traveling wave. That wave reflects off the right fixed end and travels back to the left, reflects again off the left fixed end and travels back to the right, and so on. Eventually, a steady state is reached where the string has identical right- and left-traveling waves as in the infinite-length case and the power dissipated by damping in the string equals the power supplied by the driving force so the waves have constant amplitude. Since L is given, the boundary condition restricts the wavelength of the standing waves to,

$$L = \frac{\lambda}{2}n, \qquad n = 1, 2, 3, ... \tag{419}$$

Waves can only form standing waves on this string if they have a wavelength that satisfies this relationship with L. If waves travel with speed c along the string, then equivalently the frequency of the standing waves is restricted to

$$\nu_n = \frac{cn}{2L}, \qquad n = 1, 2, 3, ... \tag{420}$$

The standing wave with $n = 1$ oscillates at the fundamental frequency and has a wavelength that is twice the length of the string. Higher integer values of n correspond to modes of oscillation called higher harmonics or overtones. Any standing wave on the string will have $n + 1$ nodes including the fixed ends and n anti-nodes.

Let us consider the same string of length L, but this time it is only fixed at $x = 0$. At $x = L$, the string is free to move in the y-direction. Since at $x = L$ the string is not fixed, L should be an anti-node. If we consider the same string of length L where non of the edges are fixed, then both $x = 0$ and $x = L$ are anti nodes.

5.8 Multi-dimensional waves

If the propagation of waves occur in more than one dimension, for instance, waves on the surface of a lake, or sound in the air, there are points in the medium that can be connected by a continuous line (in 2 dimensions) or surface (in three dimensions) and the particles in these points are in the same state of motion. These are called wavefronts. Wavefronts can have many shapes. A point source at the center of the wave causes circular or spherical wavefronts. Such a wave is called a spherical wave. A stick at the center causes a plane wave in two dimensions, the three dimensional analogue is produced by the periodic motion of a plane at the center of the wave. The wave function of multi-dimensional waves have as many spatial arguments as the number of dimensions. However, in the case of spherical waves the dependence on the spatial coordinates is reduced to the dependence on r, the distance from the point source,

$$\xi(x, y, \ldots, t) = \xi(r, t)\,. \tag{421}$$

Similarly, in the case of a plane wave, the dependence on the spatial coordinates is reduced to the distance from the source in the coordinate that is perpendicular to the plane and is the direction of propagation,

$$\xi(x, y, \ldots, t) = \xi(x, t)\,. \tag{422}$$

Propagation of sound wave in the air is an example for a three-dimensional longitudinal spherical wave. Let us assume a sound speaker as a source of longitudinal spherical sound waves which has a constant power, P_0. The energy density current is related to the constant power of the source as

$$P_0 = j^{(E)} 4\pi r^2 \quad \rightarrow \quad \langle P_0 \rangle = \langle j^{(E)} \rangle 4\pi r^2 \tag{423}$$

where r is the radial distance from the source. Let us substitute the expression obtained previously for the average energy density current $\langle j^{(E)} \rangle = \frac{1}{2} c\rho A^2 \omega^2$ into the above equation

$$\langle P_0 \rangle = \frac{1}{2} c\rho A^2 \omega^2 4\pi r^2 \quad \rightarrow \quad A(r) = \sqrt{\frac{\langle P_0 \rangle}{2\pi c\rho\omega^2 r^2}} = \sqrt{\frac{\langle P_0 \rangle}{2\pi c\rho\omega^2}}\frac{1}{r} = \frac{A_0}{r}, \quad A_0 = \sqrt{\frac{\langle P_0 \rangle}{2\pi c\rho\omega^2}}, \tag{424}$$

where the amplitude $A(r)$ of the wave function $\xi(r, t) = A(r)\sin(\omega t - kr)$ should depend on the radial distance.

5.9 Wave phenomena

In Nature waves move in more than one dimensions usually, such as the ripples on the surface of a lake. Studying those ripples systematically, we can observe three phenomena that are characteristic to waves only, thus their observation signals the presence of wave motion.

A plane wave is a kind of wave whose value varies only in one spatial direction. That is, its value is constant on a plane that is perpendicular to that direction or in other word wave fronts (surfaces with constant phase) are planes. Although plane waves are multi-dimensional waves but they have an interesting property, namely their amplitude does not depend on the distance from the source. Thus, they are ideal to consider various types of wave phenomena fo example in a in a ripple tank

5.9.1 Interference

If two or more waves combine at a particular point, they are said to interfere. The resultant waveform depends on the relative phases of the interfering waves. Consider two harmonic waves moving into the same direction with the same amplitude and frequency, $\xi_i(x,t) = A\sin(\omega t - kx + \phi_i)$, $i = 1, 2$. The resulting wave is

$$\xi(x,t) = \xi_1(x,t) + \xi_2(x,t) = 2A\cos\Delta\phi\sin(\omega t - kx + \phi')\,, \tag{425}$$

where

$$\Delta\phi = \frac{1}{2}(\phi_1 - \phi_2)\,, \qquad \phi' = \frac{1}{2}(\phi_1 + \phi_2)\,. \tag{426}$$

Thus we see that if the waves meet in phase, then the amplitudes add up, we call this constructive interference. If the waves meet with a phase difference close to π, then the resulting amplitude is very small, which we call destructive interference. In order that the resulting wave pattern be constant in time, the interfering waves must have the same phase difference at a given point of space. Such waves are called coherent waves.

Thus when traveling waves, originating from different wave centers, meet they interfere each other. The interference produces stationary wave patterns if the combining waves are coherent. For instance, in a ripple tank, two vibrating prongs create two patterns of circular ripples. If the prongs vibrate simultaneously, then the phase difference at any fixed point on the surface will be a constant in time, and depend only on the difference in path that the two rays need to travel from the two centers to reach that point. Thus the combining spherical waves produce a pattern of maxima and minima in the waves. The positions of these are along hyperboles (the position of points in the plane for that the difference in distance from two fixed points is a constant). If we view the wave pattern at a fixed distance from the line of the two centers, we observe alternating sequence of maxima and minima. This kind of interference pattern is also a signal of the presence of (coherent) waves.

5.9.2 Diffraction

Let us first produce plane waves in a ripple tank. The direction of the wave motion is at right angle to the wavefront. A line normal to the wavefronts, indicating the direction of motion of the waves, is called a ray. In the case of plane waves the rays are parallel lines (provided the medium is of uniform density, which is fulfilled in a ripple tank). If the wave center is a single point than we produce circular waves, the rays are straight lines again, but they are along the radii of a circle. If we put a wall with a slit on it into the tank such that it is perpendicular to the rays, than we expect that the rays falling on the wall get stopped, while the rays falling on the slit propagate without any change. However, what we observe is that the rays near the edge of the slit get bended towards the wall and enter into regions that are obstructed by the wall. In fact if the slit is narrow as compared to the wavelength, the observed wave pattern after the slit is a spherical wave as if the wave center were pointlike at the position of the slit. This phenomenon, the spreading of waves as it passes through a slit, is called diffraction.

5.9.3 Reflection, refraction

Reflection is the change in direction of a wavefront at an interface between two different media so that the wavefront returns into the medium from which it originated. The law of reflection says that for specular reflection the angle α at which the wave is incident on the surface equals the angle α' at which it is reflected, thus $\alpha = \alpha'$.

Refraction is the change in direction of a wave passing from one medium to another or from a gradual change in the medium. Refraction follows Snell's law, which states that, for a given pair of media, the ratio of the sines of the angle of incidence α_1 and angle of refraction α_2 is equal to the ratio of phase velocities (v_1/v_2) in the two media, or equivalently, to the indices of refraction (n_2/n_1) of the two media,

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1} = n_{21} \tag{427}$$

5.9.4 Huygens and Huygens-Fresnel principle

Christian Huygens Dutch physicist constructed a simple principle for understanding diffraction. According to this principle we can tell where a given wavefront will be at any time in the future if we know its present position:

> All points on a wavefront can be considered as point sources for the production of spherical secondary wavelets. After a time t the new position of a wavefront is the surface tangent to these secondary wavelets.

Huygens' principle is not useful for explaining interference of waves, but it can be altered easily as done by Augustin Fresnel:

> All points on a wavefront can be considered as point sources for the production of spherical secondary wavelets. After a time t the new wave pattern is an interference of these secondary wavelets.

5.10 Doppler effect

The Doppler effect (or the Doppler shift) is the change in frequency of a wave in relation to an observer who is moving relative to the wave source. It is named after the Austrian physicist Christian Doppler, who described the phenomenon in 1842. A common example of Doppler shift is the change of pitch heard when a vehicle sounding a horn approaches and recedes from an observer. Compared to the emitted frequency, the received frequency is higher during the approach, identical at the instant of passing by, and lower during the recession. For waves that propagate in a medium, such as sound waves, the velocity of the observer and of the source are relative to the medium in which the waves are transmitted. The total Doppler effect may therefore result from motion of the source, motion of the observer, or motion of the medium. The latter case, i.e., the motion of the medium is not considered here.

Let as assume a moving observer with the velocity v and a stationary source emitting waves with an actual frequency ν. In this case, the wavelength keeps constant, but due to the motion, the rate at which the observer receives waves and hence the transmission velocity of the wave (with respect to the observer) is changed. This yields change in the observed frequency ν',

$$\nu' = \frac{c'}{\lambda} = \frac{c \pm v}{\lambda} = \frac{c}{\lambda}\left(1 \pm \frac{v}{c}\right) = \nu\left(1 \pm \frac{v}{c}\right) \tag{428}$$

where c is the velocity of the wave.

Let us assume that the observer is stationary relative to the medium and a moving source is emitting waves with an actual frequency ν. In this case, the wavelength is changed, the transmission velocity of the wave keeps constant (note that the transmission velocity of the wave does not depend on the velocity of the source), then the observer detects waves with a frequency ν' given by

$$\nu' = \frac{c}{\lambda'} = \frac{c}{\lambda \pm vT} = \frac{c}{\lambda}\frac{1}{1 \pm \frac{v}{c}} = \nu\frac{1}{1 \pm \frac{v}{c}} \tag{429}$$

where T is the time period of the wave.

5.11 Physical characterisation of perception of sound

A most important type of waves are the three dimensional longitudinal waves of frequency between 16 and 20 000 Hz, which are called sound waves. Let us consider sound waves in a narrow tube of cross sectional area A, aligned with the x axis. A harmonic sound wave in this tube are described by a wave function $\xi(x,y,z,t) = \xi_0 \sin(\omega t - kx)$. We know that the change in pressure in a compressed volume of gas is proportional to the relative change in volume,

$$\Delta p = -\kappa \frac{\Delta V}{V}\,, \tag{430}$$

Imagine now a small part of the air of length Δx at position x in the tube. The relative change of the volume of the air belonging to Δx is

$$\frac{A\Delta\xi(x,t)}{A\Delta x}\,. \tag{431}$$

Taking the limit $\Delta x \to 0$, we find that the change of pressure due to the sound wave at position x is

$$\Delta p(x,t) = -\kappa \frac{\partial \xi}{\partial x}(x,t) = \kappa k \xi_0 \cos(\omega t - kx)\,. \tag{432}$$

Hence the pressure variation at each position is also harmonic, but its phase is shifted by 90° with respect to the oscillation of the particles. The speed of the longitudinal wave is $c = \sqrt{\kappa/\rho}$, therefore,

$$\Delta p(x,t) = \rho c^2 k \xi_0 \cos(\omega t - kx)\,. \tag{433}$$

The maximum change in pressure is called the pressure amplitude and is given by $\Delta p_{\mathrm{m}} = \rho c^2 k \xi_0$. We see that sound waves can be described by either displacement waves or pressure waves. However, multidimensional displacement waves add like vectors, while pressure waves add like scalars. It is usually preferable to describe sound waves as pressure waves. Moreover, our ear also detect pressure change not displacement.

The maximum pressure variation that the ear can tolerate is about 28 Pa, while the faintest sound that can be heard has a pressure amplitude of $2.8 \cdot 10^{-5}$ Pa (both values at 1000 Hz, because these depend on the frequency). We can compute the corresponding displacement amplitudes:

$$\xi_0^{\max} = \frac{\Delta p_m}{\rho c^2 k} \approx 10^{-5}\,\mathrm{m}\,, \qquad \xi_0^{\min} \approx 10^{-11}\,\mathrm{m}\,, \tag{434}$$

where we used $c^2 k = 2\pi\nu c \approx 2.1\,\mathrm{Mm/s^2}$. The value of $\xi_0^{\min}$ is especially interesting because it is about one tenth of the atomic diameter!

Sound waves are characterized by their (i) intensity, (ii) pitch and (iii) tone. The sources of sound are vibrating strings, air columns or membranes. If a vibrating system is capable of vibrating at frequencies $\nu_1 < \nu_2 < \nu_3 \ldots$, then ν_1 is called the fundamental frequency and the higher frequencies are the overtones. If the overtones are all integer multiples of the fundamental frequency, such as in the case of a vibrating string, then the overtones are called harmonics. When several frequencies are heard simultaneously, a pleasant sensation occurs when the ratio of the frequencies are ratios of small whole numbers, for instance 3:2, or 5:4. In designing musical instruments it is crucial to find the proper shape so that the overtones are harmonics.

We can express the intensity (related to loudness) as a function of the pressure amplitude,

$$I \equiv \langle j \rangle = \frac{1}{2} \frac{(\Delta p_m)^2}{\rho c}\,. \tag{435}$$

A natural question is how to characterize the loudness of a sound. In other words how to relate the actual change in a physical stimulus and the perceived change which can be done by using the Weber–Fechner laws. The Weber–Fechner laws are two related hypotheses in the field of psychophysics, known as Weber's law and Fechner's law. Both laws relate to human perception, more specifically the relation between the actual change in a physical stimulus and the perceived change. This includes stimuli to all senses: vision, hearing, taste, touch, and smell. According to

Weber–Fechner laws, the perceived loudness/brightness is proportional to logarithm of the actual intensity measured with an accurate nonhuman instrument, i.e., the relationship between stimulus and perception is logarithmic.

Thus, in order to characterize the loudness of a sound it is more useful to introduce the sound level via

$$\mathrm{SL} = 10\,\mathrm{dB}\log_{10}\frac{I}{I_0}\,, \tag{436}$$

where $I_0 = 10^{-12}\,\mathrm{W/m^2}$, which is the threshold of human hearing. The corresponding sound level is $0\,\mathrm{dB}$. At the threshold of pain $I = 1\,\mathrm{W/m^2}$, or $\mathrm{SL} = 120\,\mathrm{dB}$

5.12 Relativistic kinematics

The light travels with finite velocity. This statement can be and should be clarified or falsified by experimental tests. The first considerable attempt for such experimental test was done by Olaf Rhomer and it is based on astrophysical observation. Ole Christensen Romer was a Danish astronomer who, in 1676, made the first quantitative measurements of the speed of light. Romer compared the duration of Io's orbits as Earth moved towards Jupiter and as Earth moved away from Jupiter. Romer's view that the velocity of light was finite was not fully accepted until measurements of the so-called aberration of light were made by James Bradley in 1727. The first terrestrial measurement of the speed of light was done by Hippolyte Fizeau in 1849 when he projected a pulsed beam of light onto a distant mirror. Fizeau determined the speed of light between an intense light source and a mirror about 8 km distant. The light source was interrupted by a rotating cogwheel with 720 notches that could be rotated at a variable speed of up to hundreds of times a second. Fizeau adjusted the rotation speed of the cogwheel until light passing through one notch of the cogwheel would be completely eclipsed by the adjacent tooth. Spinning the cogwheel at 3, 5 and 7 times this basic rotation rate also resulted in eclipsing of the reflected light by the cogwheel teeth next in line. Given the rotational speed of the wheel and the distance between the wheel and the mirror, Fizeau was able to calculate a value of 315000 km/s for the speed of light.

5.12.1 Principle of relativity

Light is known to show wave properties and it was also known that it is a transverse wave, one may speculate or assume a supposed medium permeating space (aether) in which the transverse light wave propagates. The Michelson–Morley experiment was an attempt to detect the existence of the luminiferous aether, a supposed medium permeating space that was thought to be the carrier of light waves. The experiment was performed between April and July 1887 by American physicists Albert A. Michelson and Edward W. Morley. The experiment compared the speed of light in perpendicular directions in an attempt to detect the relative motion of matter through the stationary luminiferous aether ("aether wind"). The result was negative, in that Michelson and Morley found no significant difference between the speed of light in the direction of movement through the presumed aether, and the speed at right angles. This result is generally considered to be the first strong evidence against the then-prevalent aether theory, and initiated a line of research that eventually led to special relativity, which rules out a stationary aether.

The principle of relativity is the requirement that the equations describing the laws of physics have the same form in all admissible frames of reference. Special principle of relativity tells us that if a system of coordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the same laws hold good in relation to any other system of coordinates K' moving in uniform translation relatively to K. In order words, Einstein determined that the laws of physics are the same for all non-accelerating observers, and that the speed of light in a vacuum was independent of the motion of all observers. This requires the modification of the Galilean transformation which is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics.

5.12.2 Lorentz transformation

Let us consider two events: one is the emission of a light pulse and the second one is its detection at some distance Δx, measured in an inertial reference frame V. Both events are assumed to be along the x axis. The time needed for light to travel this distance is $\Delta t = \Delta x/c$. The coordinate difference of the same pair of events in another reference frame V' traveling with speed v in the x direction is $\Delta x = c\Delta t'$, where we used the second postulate. According to the Galilean transformation, these coordinate differences are related,

$$\Delta x = \Delta x' + v\Delta t' \,, \qquad \Delta x' = \Delta x - v\Delta t \,.$$

Assuming that time elapses at the same rate in the two frames, $\Delta t = \Delta t'$, which coincides with our common sense, dividing the equations with Δt, we obtain

$$c = c + v \,, \qquad c = c - v \,,$$

which are contradicting equations for $v \neq 0$. So the Galilean transformation of coordinates contradicts to our second postulate. This is not surprising because we have already discussed that the Galilean addition of velocities is also in contradiction with the constancy of speed of light. We would like to modifiy the Galilean transformations such that the following four requirements are kept:

1. The new transformation rules should also be symmetric in interchanging the reference frames.
2. The speed of light should be the same in both frames.
3. In the limit of small v ($\beta_v \equiv v/c \to 0$) the new transformation rules assume the form of the Galilean transformations, which were found valid at small v.
4. Event pairs that occur at the same position and time in V must happen at the same position and time in V', too.

A simple extension of the transformation rules is

$$\Delta x = \gamma(v)(\Delta x' + v\Delta t') \,, \qquad \Delta x' = \gamma(v)(\Delta x - v\Delta t) \tag{437}$$

where $\gamma(v)$ is a dimensionless function of v. This form satisfies the first condition. Using the second postulate (hence the second conition is automatically satsified), we have $\Delta t' = \Delta x'/c$, $\Delta t = \Delta x/c$, and the product of the two equations gives

$$\Delta x\, \Delta x' = \gamma(v)^2(\Delta x' + \beta_v\Delta x')(\Delta x - \beta_v\Delta x) = \gamma(v)^2\Delta x\, \Delta x'(1 - \beta_v^2) \,.$$

Therefore, $\gamma(v) \equiv \gamma_v = (1 - \beta_v^2)^{-1/2}$ where $\beta_v = v/c$. Knowing γ_v, we can deduce the transformaition properties of time. For instance, from the first equation of (437) we find

$$\Delta t' = \tfrac{1}{v}\left(\tfrac{\Delta x}{\gamma_v} - \Delta x'\right) = \tfrac{1}{v}\left(\tfrac{\Delta x}{\gamma_v} - \gamma_v\Delta x + \gamma_v v\Delta t)\right) \tag{438}$$

$$= \gamma_v\left[\Delta t - \tfrac{1}{v}\left(1 - \tfrac{1}{\gamma_v^2}\Delta x)\right)\right] \tag{439}$$

$$= \gamma_v\left(\Delta t - \beta_v\tfrac{\Delta x}{c}\right) \,. \tag{440}$$

Starting from the second equation of (437) and following the same steps, we can derive the inverse relation,

$$\Delta t = \gamma_v\left(\Delta t' + \beta_v\frac{\Delta x'}{c}\right) \,,$$

which clearly obeys the first condition. These transformation rules are called Lorentz transformations, and can be summarized in an even more symmetric form:

$$\Delta x = \gamma_v(\Delta x' + \beta_v c\Delta t') \,, \qquad \Delta x' = \gamma_v(\Delta x - \beta_v c\Delta t) \,, \tag{441}$$

$$c\Delta t = \gamma_v(c\Delta t' + \beta_v \Delta x')\,, \quad c\Delta t' = \gamma_v(c\Delta t - \beta_v \Delta x)\,. \tag{442}$$

In the limit of small v, $\beta_v \to 0$ and $\gamma_v \to 1$ and we recover the Galilean transformations which shows that the third condition is satisfied. The first and the fourth conditions are trivially satisfied (because the transformation is linear, homogeneous and symmetric) and the second condition can be verified as

$$c' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma_v(\Delta x - v\Delta t)}{\gamma_v\left(\Delta t - v\frac{\Delta x}{c^2}\right)} = \frac{\left(\frac{\Delta x}{\Delta t} - v\right)}{\left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right)} = \frac{c - v}{1 - \frac{v}{c}} = c\frac{c - v}{c - v} = c. \tag{443}$$

We note that under this transformation (called Lorentz boost in the x direction) the other space-components remain unchanged, $\Delta y' = \Delta y$, $\Delta z' = \Delta z$.