

THERMAL STRESSES IN FUNCTIONALLY GRADED CIRCULAR CYLINDERS

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Abstract: The object of this paper is the application of the theory of thermoelasticity for thermal stress analysis of solid circular cylinder made of functionally graded material. The considered steady-state problem is solved by the elementary beam theory.

1. INTRODUCTION

The solid circular cylinder made of functionally graded material is shown in Fig.1. The material properties of circular cylinder are smooth functions of the radial coordinate. The considered problem is a steady-state thermoelastic boundary value problem in the framework of beam theory. The temperature difference is denoted by

$$T=t-t_0, \quad (1)$$

where t is the absolute temperature and t_0 is the reference temperature at which the stresses are zero if the cylinder is undeformed. The formulation of the governing equations are given in cylindrical coordinate system $Or\varphi z$ (Fig. 1). The space domain occupied by the circular cylinder is B and

$$B = \{(r, \varphi, z) | 0 \leq r \leq a, 0 \leq \varphi \leq 2\pi, 0 \leq z \leq L\}. \quad (2)$$

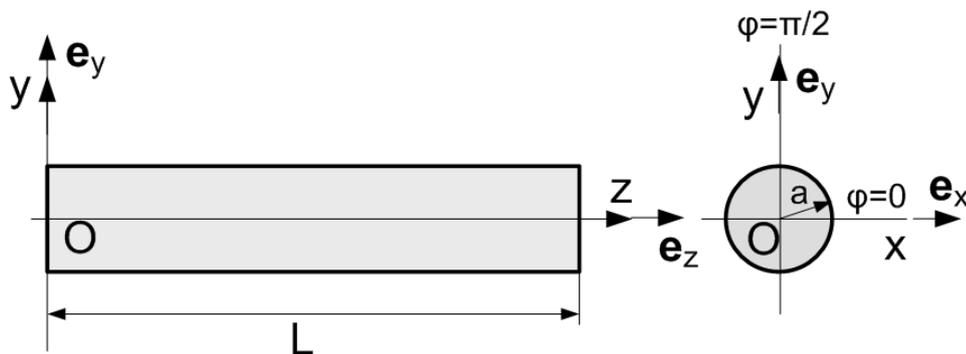


Fig. 1. The solid circular cylinder.

We consider an axisymmetric steady-state thermal stress problem. This means that $T=T(r,z)$. It is assumed that the following thermal boundary conditions are prescribed:

$$\frac{\partial T}{\partial z} = 0, \quad 0 \leq r \leq a, \quad z = 0, \quad (3)$$

$$\frac{\partial T}{\partial z} = 0, \quad 0 \leq r \leq a, \quad z = L, \quad (4)$$

$$T(r, z) = \theta(z), \quad r = a, \quad 0 \leq z \leq L. \quad (5)$$

The steady-state temperature distribution satisfies the heat condition equation [1,2]

$$\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{\lambda} \frac{d\lambda}{dr} + \frac{1}{r} \right) \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \quad 0 < r < a, \quad 0 < z < L, \quad (6)$$

where $\lambda = \lambda(r)$ is the thermal conduction coefficient. The solution of boundary value problem given by Eqs. (3-6) leads to the temperature difference function $T(r, z)$. All material properties of the nonhomogeneous solid circular cylinder are power functions of the radial coordinate r . We have

$$\lambda(r) = \lambda_0 \left(\frac{r}{a} \right)^{k_1}, \quad E(r) = E_0 \left(\frac{r}{a} \right)^{k_2}, \quad \alpha(r) = \alpha_0 \left(\frac{r}{a} \right)^{k_3}. \quad (7)$$

Here E is the modulus of elasticity, α is the coefficient of linear thermal expansion and k_1, k_2, k_3 are given constants (power indexes). We assume that all of the power indexes are non-negative, $k_j \geq 0$ ($j = 1, 2, 3$).

2. DETERMINATION OF THE TEMPERATURE DISTRIBUTION

Substitution of Eq. (7)₁ into Eq. (6) gives

$$\frac{\partial^2 T}{\partial r^2} + (k_1 + 1) \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \quad 0 < r < a, \quad 0 < z < L. \quad (8)$$

We look for the solution of Eq. (8) in the next form

$$T(r, z) = \sum_{i=0}^{\infty} T_i(r) \cos \frac{i\pi}{L} z. \quad (9)$$

It is obvious that the above defined temperature field satisfies the thermal boundary conditions formulated by Eqs. (3) and (4) with arbitrary $T_i = T_i(r)$ ($i=1, 2, \dots$). From boundary condition Eq. (5) it follows that

$$\theta(z) = \sum_{i=0}^{\infty} T_i(a) \cos \frac{i\pi}{L} z. \quad (10)$$

We have for $i^2 + j^2 \neq 0$

$$\int_0^L \cos\left(\frac{i\pi}{L}z\right)\cos\left(\frac{j\pi}{L}z\right)dz = \begin{cases} 0 & \text{if } i \neq j \\ \frac{L}{2} & \text{if } i = j \end{cases} \quad (11)$$

Combination of Eq. (10) and Eq. (11) gives

$$T_0(a) = \frac{1}{L} \int_0^L \theta(z)dz, \quad T_i(a) = \frac{2}{L} \int_0^L \cos\left(\frac{i\pi}{L}z\right)\theta(z)dz, \quad (i=1,2,\dots). \quad (12)$$

Substitution of assumed form of the temperature difference field into Eq. (9) yields the next differential equation for $T_i(r)$ ($i=0,1,2,\dots$)

$$\frac{d^2T_i}{dr^2} + (k_1 + 1)\frac{1}{r}\frac{dT_i}{dr} - \left(\frac{i\pi}{L}\right)^2 T_i = 0, \quad (i=0,1,2,\dots). \quad (13)$$

The general solution of Eq. (13) are

$$T_0(r) = C_{01} + C_{02}r^{-k_1}, \quad (14)$$

$$T_i(r) = C_{i1}r^{-\frac{k_1}{2}}I\left(\frac{k_1}{2}, \frac{i\pi r}{L}\right) + C_{i2}r^{\frac{k_1}{2}}K\left(\frac{k_1}{2}, \frac{i\pi r}{L}\right), \quad (i=1,2,\dots), \quad (15)$$

where C_{i1} and C_{i2} ($i=0,1,2,\dots$) are arbitrary constants and $I\left(\frac{k_1}{2}, \frac{i\pi r}{L}\right)$, $K\left(\frac{k_1}{2}, \frac{i\pi r}{L}\right)$ are the modified Bessel functions of first and second kinds of order $\frac{k_1}{2}$. For $k_1 > 0$ the bounded solution at $r=0$ is obtained from Eq. (14) and Eq. (15) with

$$C_{i2} = 0, \quad (i=0,1,2,\dots). \quad (16)$$

The combination of Eq. (12) with Eqs. (14) and (15) gives

$$C_{01} = \frac{1}{L} \int_0^L \theta(z)dz = T_0(a), \quad (17)$$

$$C_{i1} = \frac{\frac{2}{L} \int_0^L \cos\left(\frac{i\pi}{L}z\right)\theta(z)dz}{a^{-\frac{k_1}{2}}I\left(\frac{k_1}{2}, \frac{i\pi a}{L}\right)L} = \frac{T_i(a)}{a^{-\frac{k_1}{2}}I\left(\frac{k_1}{2}, \frac{i\pi a}{L}\right)}, \quad (i=1,2,\dots). \quad (18)$$

The final form of the temperature field can be obtained from Eq. (9) and Eqs. (17), (18) as

$$T(r, z) = \sum_{i=0}^{\infty} T_i(a) \frac{I\left(\frac{k_1}{2}, \frac{i\pi r}{L}\right)}{a^{-\frac{k_1}{2}} I\left(\frac{k_1}{2}, \frac{i\pi a}{L}\right)} \cos\left(\frac{i\pi}{L} z\right). \quad (19)$$

3. THERMAL STRESSES

To obtain the thermal stresses we define the following thermal loads which are used in the beam theory [2,3]

$$N_T(z) = \int_A E\alpha T dA = 2\pi E_0 \alpha_0 \int_0^a \left(\frac{r}{a}\right)^{k_3+k_2} r T(r, z) dr, \quad (20)$$

$$M_{T_x}(z) = \int_A E\alpha T r \sin \varphi dA = 2\pi E_0 \alpha_0 \int_0^a \left(\frac{r}{a}\right)^{k_3+k_2} r^2 T(r, z) dr \int_A \sin \varphi dA, \quad (21)$$

$$M_{T_y}(z) = \int_A E\alpha T r \cos \varphi dA = 2\pi E_0 \alpha_0 \int_0^a \left(\frac{r}{a}\right)^{k_3+k_2} r^2 T(r, z) dr \int_A \cos \varphi dA. \quad (22)$$

It is evident that $M_{T_x}=M_{T_y}=0$. According to Nowinski [2], Boley and Weiner [3] for the normal stress $\sigma_z=\sigma_z(r, z)$ the next formula can be derived

$$\sigma_z(r, z) = \frac{N_T(z)}{a^2 \pi} - E_0 \alpha_0 \left(\frac{r}{a}\right)^{k_2+k_3} T(r, z). \quad (23)$$

In the beam theory the normal stresses σ_r , σ_φ are neglected and $\tau_{r\varphi}$, $\tau_{z\varphi}$ vanish since the problem is axially symmetric. The shear stress is τ_{rz} obtained from the next equilibrium equation

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\varphi z}}{\partial \varphi} + \frac{\tau_{rz}}{r} + \frac{\partial \sigma_z}{\partial z} = 0. \quad (24)$$

In the present problem Eq. (24) can be written in the next form

$$\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial}{\partial z}(r\sigma_z) = 0. \quad (25)$$

A simple computation based on Eq. (25) gives for $\tau_{rz}=\tau_{rz}(r, z)$

$$\tau_{rz}(r, z) = E_0 \alpha_0 \left[\frac{1}{r} \int_0^r \rho \left(\frac{\rho}{a}\right)^{k_2+k_3} \frac{\partial T(\rho, z)}{\partial z} d\rho - \frac{r}{a^2} \int_0^a r \left(\frac{r}{a}\right)^{k_2+k_3} \frac{\partial T(r, z)}{\partial z} dr \right]. \quad (26)$$

By the application of the L'Hospital theorem it can be proven

$$\lim_{r \rightarrow 0} \tau_{rz}(r, z) = 0, \quad (27)$$

and from Eq. (26) it follows that $\tau_{rz}(a, z) = 0$ according to the stress free cylindrical boundary surface of the solid circular cylinder.

4. NUMERICAL EXAMPLE

For the numerical example the following data are used:

$$a = 0.05 \text{ m}; L = 5 \text{ m}; \lambda_0 = 58 \frac{\text{W}}{\text{mK}}; \alpha_0 = 1.2 \cdot 10^{-6} \frac{1}{\text{K}}, E_0 = 100 \text{ GPa}.$$

The prescribed surface temperature on the cylindrical boundary surface is

$$\theta(z) = \frac{\theta_0}{L} z, \quad r = a, \quad \theta_0 = 300 \text{ C}^\circ, \quad 0 \leq z \leq L.$$

Figure 2 shows the graphs of $N_T = N_T(z)$ for different values of power indexes which are $k_1 = k_2 = k_3 = k = (0, 0.2, 1, 2)$.

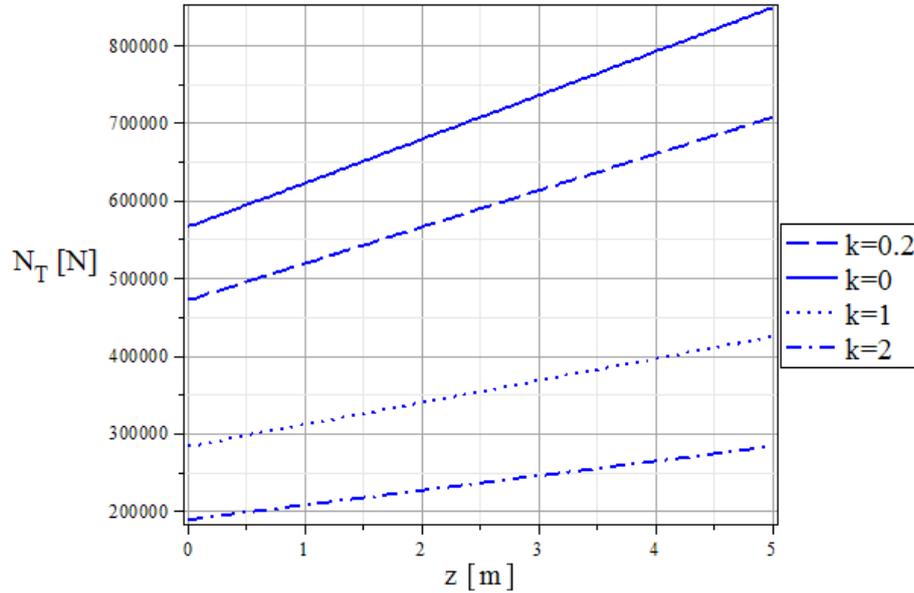


Fig. 2. Plots of N_T by 4 power indexes.

Figure 3 illustrates the plots of σ_z in the cross section for four different values of $k = (0, 0.2, 1, 2)$.

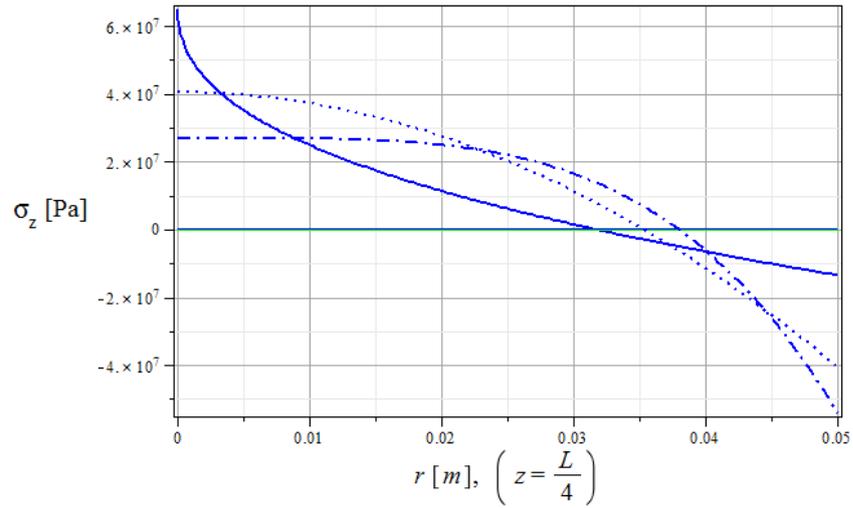


Fig. 3. The plots of the axial normal stresses.

In Fig. 4 the plots of shearing stresses are shown in the cross section $z=0.25L$ as a function of r for $k=(0, 0.2, 1, 2)$.

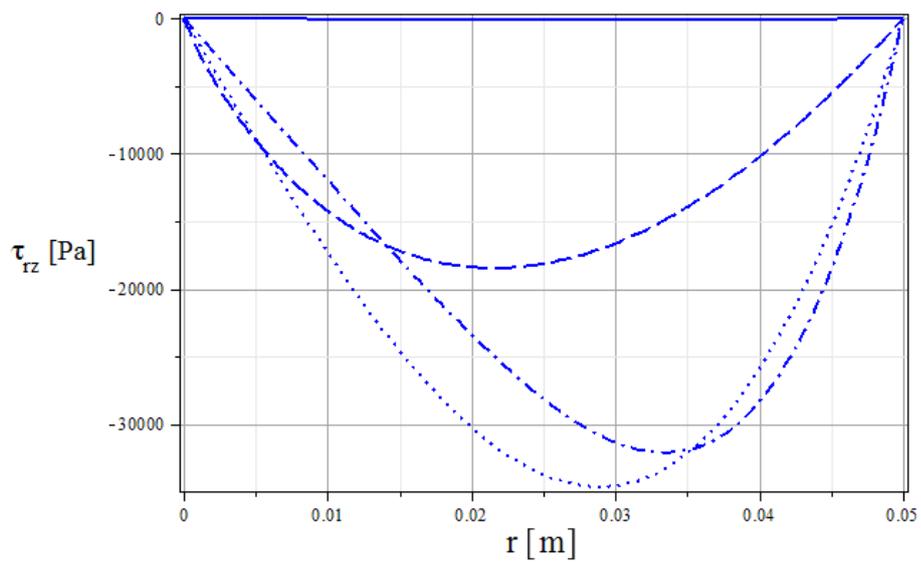


Fig. 4. The plots of the shearing stresses.

For $k=0.2$, the results obtained by the presented method are compared with FEM solution (Fig. 5).

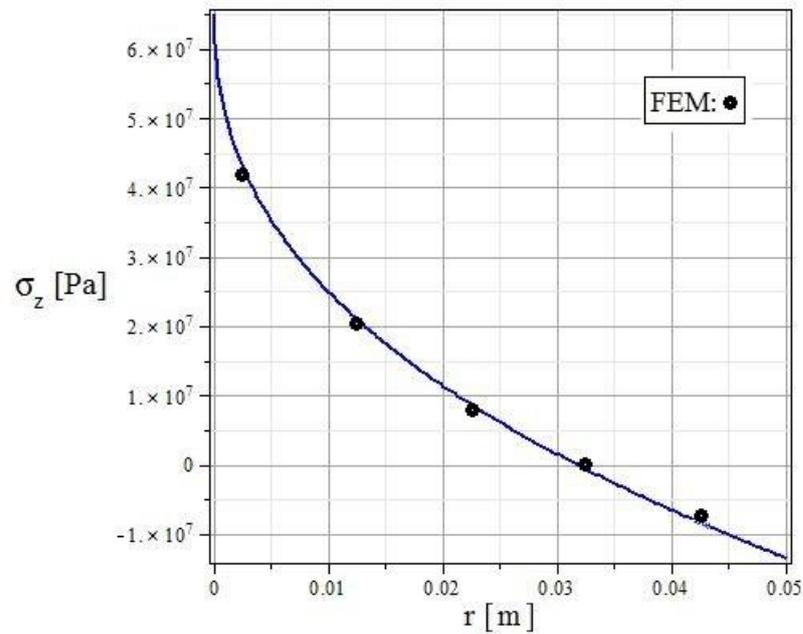


Fig. 5. The comparison of the axial normal stress function of the developed method with the FE simulation at $z=L/4$.

5. CONCLUSIONS

In this paper the thermal stresses are determined by the application of equations of thermoelasticity and assumptions of beam theory. Obtained results for normal stresses are checked by FEM solution which shows good agreement with the solution based on beam theory.

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