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COMPLEMENTARY ENERGY METHOD FOR CURVED COMPOSITE BEAMS

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Abstract

In this paper the complementary energy method is formulated for curved two-layer composite beam with interlayer slip. It is assumed that each curved layer separately follows the Euler-Bernoulli hypothesis and the load-slip relation for the flexible shear connection is a linear relationship. An example illustrates the application of the developed formulation.

1. INTRODUCTION

The considered curved two-layer composite beam with uniform cross section is shown in Fig. 1. In the cylindrical coordinate system $Or\varphi z$ the curved layer i $(i=1,2)$ occupies the space domain B_i $(i=1,2)$

$$B_i = \left\{ \left(r,\varphi,z \right) | \left(r,z \right) \in A_i,\ 0 \le \varphi \le \alpha \le 2\pi \right\},\quad \left(i=1,2 \right), \tag{1}$$

where A_i is the cross section of beam component B_i $(i=1,2)$ and the common boundary of B_1 and B_2 is denoted by ∂B_{12}

$$\partial B_{12} = \left\{ \left(r,\varphi,z \right) | \, r=c,\ 0<\varphi<\alpha,\ \left| z \right| \le 0.5t \right\}, \tag{2}$$

where t is the thickness of the cross section. The plane $z=0$ is the plane of symmetry for the whole curved beam. The connection between B_1 and B_2 is perfect in radial direction, but in the displacements it may have jump in circumferential direction which is called the interlayer slip. The derivation of the governing equations is based on the next displacement field [1,2,3]

$$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{e}_z,\quad w=0,\quad u=u\left(\varphi\right),\quad \left(r,\varphi,z \right) \in B = B_1 \cup B_2, \tag{3}$$

$$v = r\phi_i\left(\varphi\right) + \frac{\mathrm{d}u}{\mathrm{d}\varphi},\quad \left(r,\varphi,z \right) \in B_i,\quad \left(i=1,2 \right). \tag{4}$$

From the definition of the interlayer slip it follows that (Fig. 1)

$$s\left(\varphi\right) = c\left(\phi_1\left(\varphi\right) - \phi_2\left(\varphi\right) \right). \tag{5}$$

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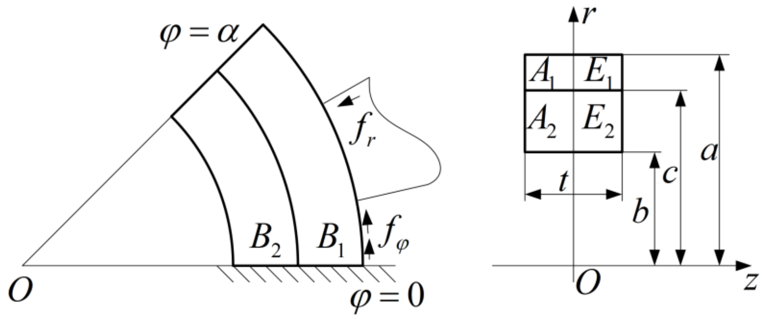

Figure 1. Two-layer composite beam and its cross section.

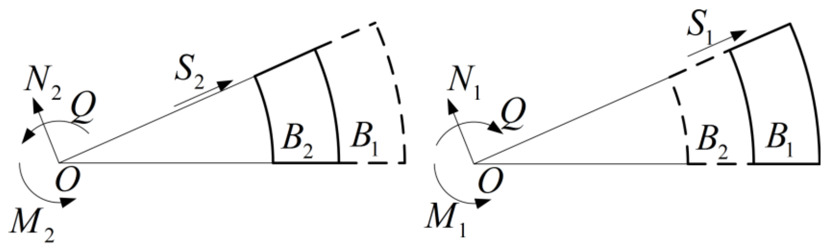

Figure 2. Illustration of internal forces and couples.

According to paper [2], the expression of the interlayer shear force is

$$T(\varphi) = kc^2 t\left(\phi_1(\varphi) - \phi_2(\varphi)\right), \tag{6}$$

where k is the slip modulus. Formulae of internal forces and couples in terms of $u = u(\varphi)$ and $\phi_i = \phi_i(\varphi)$ $(i = 1,2)$ are as follows (Fig. 2)

$$N_i = \int_{A_i} \sigma_\varphi \mathrm{d}A = \frac{AE_0}{R} W + A_i E_i \frac{\mathrm{d}\phi_i}{\mathrm{d}\varphi}, \quad (i = 1,2), \tag{7}$$

$$M_i = \int_{A_i} r\sigma_\varphi \mathrm{d}A = AE_0 W + r_i A_i E_i \frac{\mathrm{d}\phi_i}{\mathrm{d}\varphi}, \quad (i = 1,2), \tag{8}$$

$$N = \int_{A} \sigma_\varphi \mathrm{d}A = N_1 + N_2, \quad M = \int_{A} r\sigma_\varphi \mathrm{d}A = M_1 + M_2, \quad A = A_1 \cup A_2, \tag{9}$$

$$S = \int_{A} \tau_{r\varphi} \mathrm{d}A = -\frac{\mathrm{d}N}{\mathrm{d}\varphi} - f_\varphi, \tag{10}$$

$$Q = K\left(\phi_1 - \phi_2\right), \quad K = kc^3 t. \tag{11}$$

In Eqs. (7-11) N is the normal force, M is the „bending” moment, S is the shear force, Q is the moment of interlayer shear force about axis z, f_φ is the applied external force in circumferential direction [1,2] as shown in Fig. 1, furthermore

$$E_0 = \frac{A_1 E_1 + A_2 E_2}{A_1 + A_2}, \quad r_i = \frac{1}{A_i} \int_{A_i} r \mathrm{d}A, \tag{12}$$

$$\frac{AE_0}{R} = E_1 \int_{A_1} \frac{\mathrm{d}A}{r} + E_2 \int_{A_2} \frac{\mathrm{d}A}{r}, \quad W = \frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u. \tag{13}$$

Here E_i $(i = 1, 2)$ is the modulus of elasticity of beam component B_i $(i = 1, 2)$. The expression of the strain energy of the two-layer composite beam with imperfect shear connection in terms of $W = W(\varphi)$, $\phi_i = \phi_i(\varphi)$ is as follows [3]

$$\begin{aligned} U = \frac{1}{2}\int_0^{\alpha} \Bigg[& \frac{AE_0}{R} W^2 + 2A_1E_1W \frac{\mathrm{d}\phi_1}{\mathrm{d}\varphi} + 2A_2E_2W \frac{\mathrm{d}\phi_2}{\mathrm{d}\varphi} + \\ & + r_1A_1E_1 \left(\frac{\mathrm{d}\phi_1}{\mathrm{d}\varphi}\right)^2 + r_2A_2E_2 \left(\frac{\mathrm{d}\phi_2}{\mathrm{d}\varphi}\right)^2 + K\left(\phi_1 - \phi_2\right)^2 \Bigg] \mathrm{d}\varphi. \end{aligned} \tag{14}$$

From the system of equations

$$\frac{AE_0}{R} W + A_1E_1 \frac{\mathrm{d}\phi_1}{\mathrm{d}\varphi} + A_2E_2 \frac{\mathrm{d}\phi_2}{\mathrm{d}\varphi} = N, \tag{15$_1$}$$

$$A_1E_1W + r_1A_1E_1 \frac{\mathrm{d}\phi_1}{\mathrm{d}\varphi} = M_1, \tag{15$_2$}$$

$$A_2E_2W + r_2A_2E_2 \frac{\mathrm{d}\phi_2}{\mathrm{d}\varphi} = M_2 \tag{15$_3$}$$

we obtain

$$W = \frac{Nr_1r_2 - M_1r_2 - M_2r_1}{H}, \tag{16}$$

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}\varphi} = \frac{-Nr_2 + M_1\left(\frac{AE_0}{A_1E_1}\frac{r_2}{R} - \frac{A_2E_2}{A_1E_1}\right) + M_2}{H}, \tag{17}$$

$$\frac{\mathrm{d}\phi_2}{\mathrm{d}\varphi} = \frac{-Nr_1 + M_2\left(\frac{AE_0}{A_2E_2}\frac{r_1}{R} - \frac{A_1E_1}{A_2E_2}\right) + M_1}{H}, \tag{18}$$

$$H = \frac{r_1r_2}{R} AE_0 - r_2A_1E_1 - r_1A_2E_2. \tag{19}$$

The expression of strain energy in terms of N_1, M_1, M_2 and Q can be obtained from Eq. (14) and Eqs. (16-19).

$$U\left(N,M_1,M_2,Q\right)=\frac{1}{2H^2}\int_0^{\alpha}\left\{\frac{AE_0}{R}\left[Nr_1r_2-M_1r_2-M_2r_1\right]^2+\right.$$

$$+2A_1E_1\left[Nr_1r_2-M_1r_2-M_2r_1\right]\left[-Nr_2+M_1\left(\frac{AE_0}{A_1E_1}\frac{r_2}{R}-\frac{A_2E_2}{A_1E_1}\right)+M_2\right]+$$

$$+2A_2E_2\left[Nr_1r_2-M_1r_2-M_2r_1\right]\left[-Nr_1+M_2\left(\frac{AE_0}{A_2E_2}\frac{r_1}{R}-\frac{A_1E_1}{A_2E_2}\right)+M_1\right]+ \tag{20}$$

$$+r_1A_1E_1\left[-Nr_2+M_1\left(\frac{AE_0}{A_1E_1}\frac{r_2}{R}-\frac{A_2E_2}{A_1E_1}\right)+M_2\right]^2+$$

$$\left.+r_2A_2E_2\left[-Nr_1+M_2\left(\frac{AE_0}{A_2E_2}\frac{r_1}{R}-\frac{A_1E_1}{A_2E_2}\right)+M_1\right]^2+\frac{H^2}{K}Q^2\right\}\mathrm{d}\varphi.$$

2. PRINCIPLE OF THE MINIMUM OF STRAIN ENERGY

Equations of equilibrium for the two-layer composite beam with weak shear connection can be written in the form [2,3]

$$\frac{\mathrm{d}^2N}{\mathrm{d}\varphi^2}+N-f_r+\frac{\mathrm{d}f_\varphi}{\mathrm{d}\varphi}=0, \tag{21}$$

$$\frac{\mathrm{d}M_1}{\mathrm{d}\varphi}-Q=0,\quad \frac{\mathrm{d}M_2}{\mathrm{d}\varphi}+Q=0. \tag{22}$$

Next, we give some typical boundary conditions [2]

- Fixed end: $u=0,\quad \frac{\mathrm{d}u}{\mathrm{d}\varphi}=0,\quad \phi_1=0,\quad \phi_2=0.$ (23)
- Simply supported end in radial direction: $N=0,\quad M_1=0,\quad M_2=0,\quad u=0.$ (24)
- Free end: $N=0,\quad S=0,\quad M_1=0,\quad M_2=0.$ (25)

Internal forces which satisfy equations of equilibrium which are Eq. (10), Eqs. (21-22) and the force boundary conditions which refer to N, S and M_1, M_2 are called statically admissible internal forces. Next, we formulate the principle of minimum of complementary energy for homogeneous geometrical boundary conditions such as (23), $(24)_4$. In this case the principle of minimum of complementary energy says that

$$U\left[N,S,M_1,M_2,Q\right]\le U\left[\tilde{N},\tilde{S},\tilde{M}_1,\tilde{M}_2,\tilde{Q}\right], \tag{26}$$

where $N=N(\varphi)$, $S=S(\varphi)$, $M_1=M_1(\varphi)$, $M_2=M_2(\varphi)$, $Q=Q(\varphi)$ are the exact solutions of the considered boundary value problem of statical equilibrium and $\tilde{N}=\tilde{N}(\varphi)$, $\tilde{S}=\tilde{S}(\varphi)$, $\tilde{M}_1=\tilde{M}_1(\varphi)$, $\tilde{M}_2=\tilde{M}_2(\varphi)$, $\tilde{Q}=\tilde{Q}(\varphi)$ are only a statically admissible fields of the system of internal forces for the considered equilibrium problem. Equality in (26) is reached only if

$$N=\tilde{N},\ S=\tilde{S},\ M_1=\tilde{M}_1,\ M_2=\tilde{M}_2,\ Q=\tilde{Q}. \tag{27}$$

3. DETERMINATION OF STRESSES

From Eqs. (3), (4) and the strain-displacement relationship and Hooke's law the following formula can be derived for the circumferential normal stress [2,3]

$$\sigma_\varphi = E_i\left(\frac{W}{r}+\frac{\mathrm{d}\phi_i}{\mathrm{d}\varphi}\right),\quad (r,\varphi,z)\in B_i,\quad (i=1,2). \tag{28}$$

Assuming that $\tau_{\varphi z}=0$ and starting from the equilibrium equation

$$\frac{\partial \tau_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\varphi}{\partial \varphi}+\frac{\partial \tau_{\varphi z}}{\partial z}+2\frac{\tau_{r\varphi}}{r}=0, \tag{29}$$

we get for the shearing stress $\tau_{r\varphi}$

$$\tau_{r\varphi}(r,\varphi)=-\frac{1}{r^2}\int_b^r \rho\frac{\partial \sigma_\varphi(\rho,\varphi)}{\partial \varphi}\mathrm{d}\rho,\quad b\le r\le a. \tag{30}$$

4. NUMERICAL EXAMPLE

The curved beam and its load is shown in Fig. 3. The next data are used: $a=0.04$ [m], $b=0.02$ [m], $c=0.03$ [m], $t=0.03$ [m], $E_1=10\times10^{11}$ [Pa], $E_2=8\times10^{10}$ [Pa], $f=1000$ [N], $\alpha=\frac{2\pi}{3}$. The statically admissible internal forces and couples are as follows

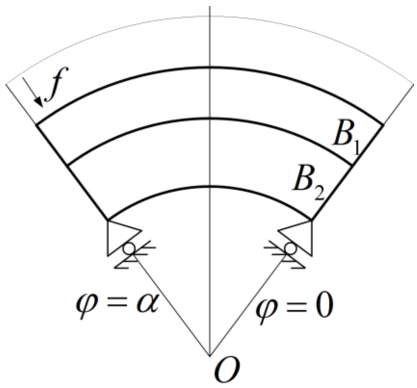

Figure 3. Simply supported curved beam loaded by uniform radial load.

$$N(\varphi) = f\left[\cos\varphi - 1 + \sin\varphi \tan\frac{\alpha}{2}\right], \tag{31}$$

$$M_1(\varphi) = -M_2(\varphi) = \sum_{p=1}^{q} m_p \sin\frac{p\pi\varphi}{\alpha}, \tag{32}$$

$$Q(\varphi) = \sum_{p=1}^{q} \frac{p m_p \pi}{\alpha} \cos\frac{p\pi\varphi}{\alpha}. \tag{33}$$

These internal forces and couples satisfy the equations of equilibrium (21), (22) and the forced boundary conditions, that is

$$N(0) = N(\alpha) = M_1(0) = M_1(\alpha) = 0. \tag{34}$$

In the present case, we have $U = U\left(m_p,\ p = 1,2,...,q\right)$. The unknown constant m_p $\left(p = 1,2,...,q\right)$ is obtained from the minimum condition of the complementary energy which is expressed as

$$\frac{\partial U}{\partial m_p} = 0, \quad \left(p = 1,2,...,q\right). \tag{35}$$

Figure 4 shows the bending moment diagrams M_1 and M_2 for two different values of the slip modulus $\left(k = 5\times 10^{11}\left[\mathrm{N/m^3}\right] \text{ and } k = 5\times 10^{8}\left[\mathrm{N/m^3}\right]\right)$. The slip function is obtained as

$$s(\varphi) = \frac{c}{K} Q(\varphi) \tag{36}$$

and its figures for $k = 5\times 10^{11}\left[\mathrm{N/m^3}\right]$ and $k = 5\times 10^{8}\left[\mathrm{N/m^3}\right]$ are shown in Fig. 5. The normal stress distribution in some cross sections are illustrated in Fig. 6. The positions of the cross sections are given by $\varphi = 0,\ \varphi = \frac{\alpha}{4},\ \varphi = \frac{\alpha}{2},\ \varphi = \frac{3\alpha}{4},\ \varphi = \alpha.$

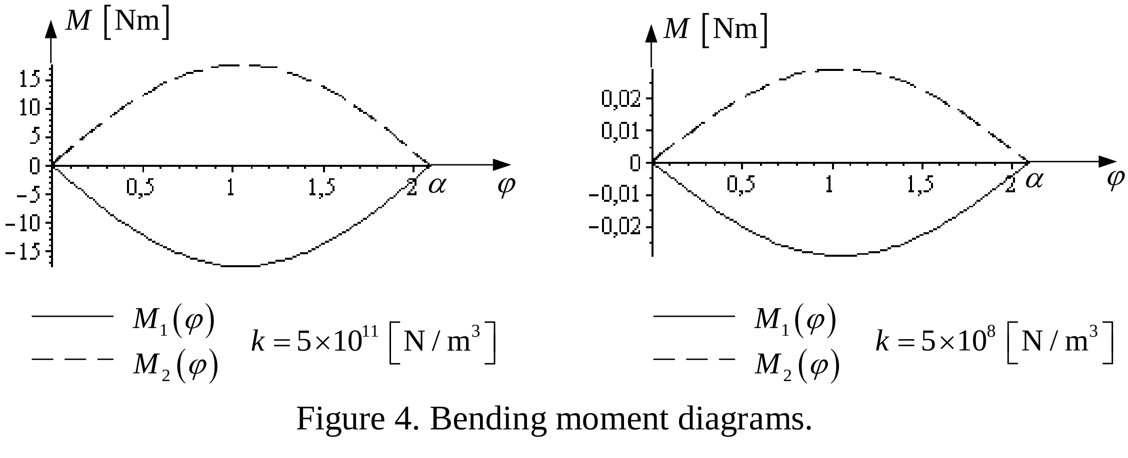

Figure 4. Bending moment diagrams.

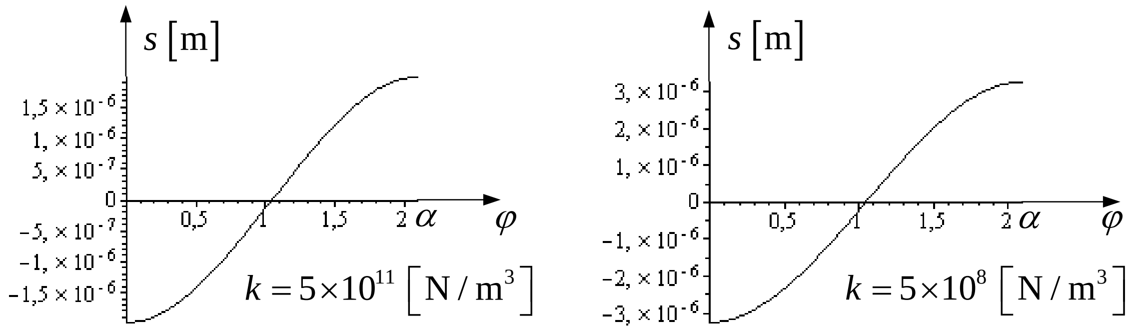

Figure 5. Plots of slip functions.

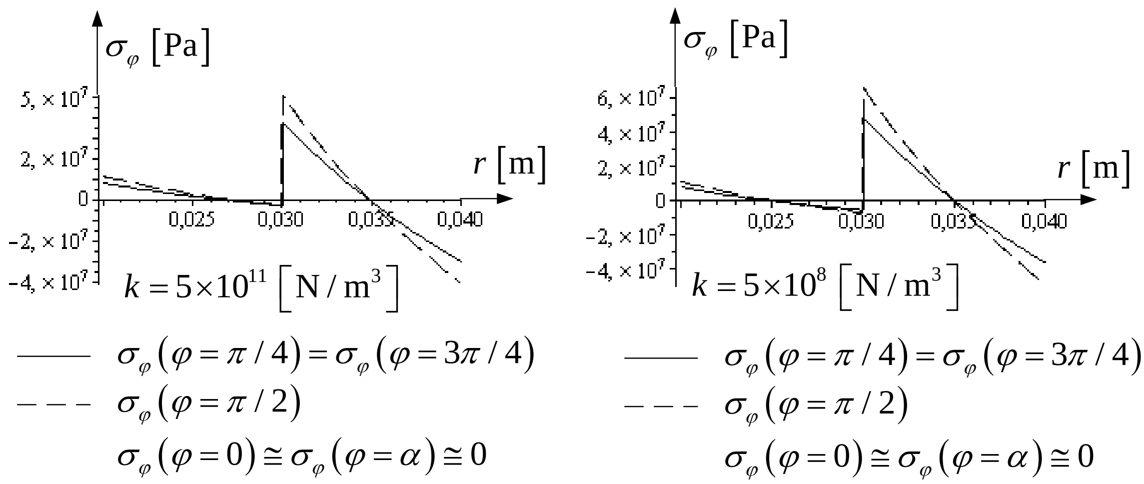

Figure 6. Plots of σ_φ.

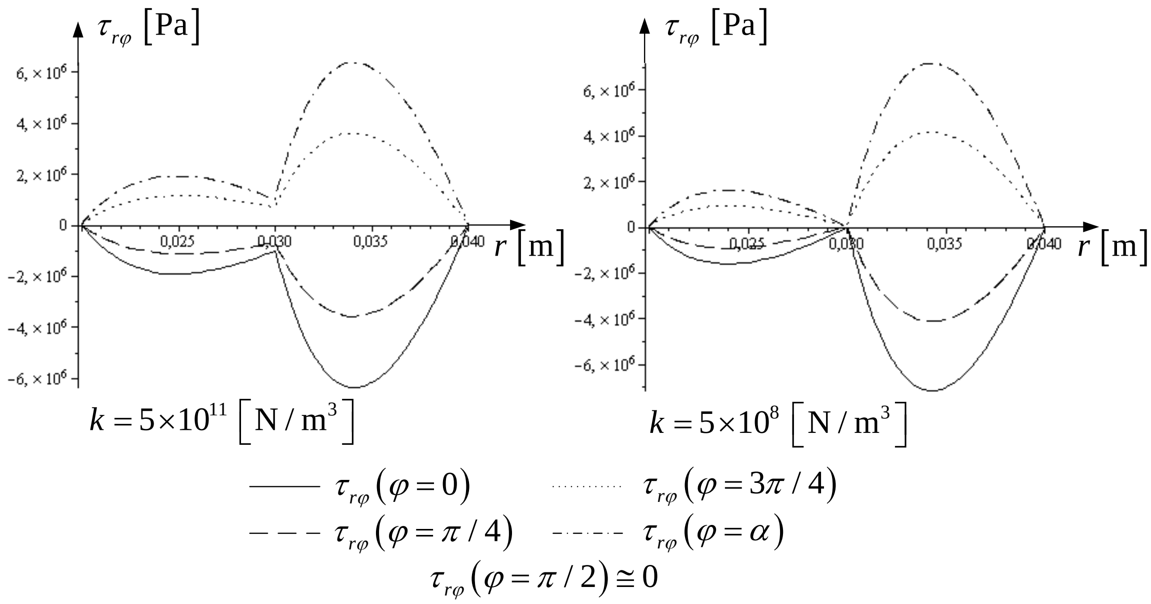

Figure 7. Graphs of $\tau_{r\varphi}$.

In the same cross section as in above, the shearing stress distributions are presented in Fig. 7. The graphs of the von Mises stresses $\sigma_M = \sqrt{\sigma_\varphi^2 + 3\tau_{r\varphi}^2}$ are shown in Fig. 8 for

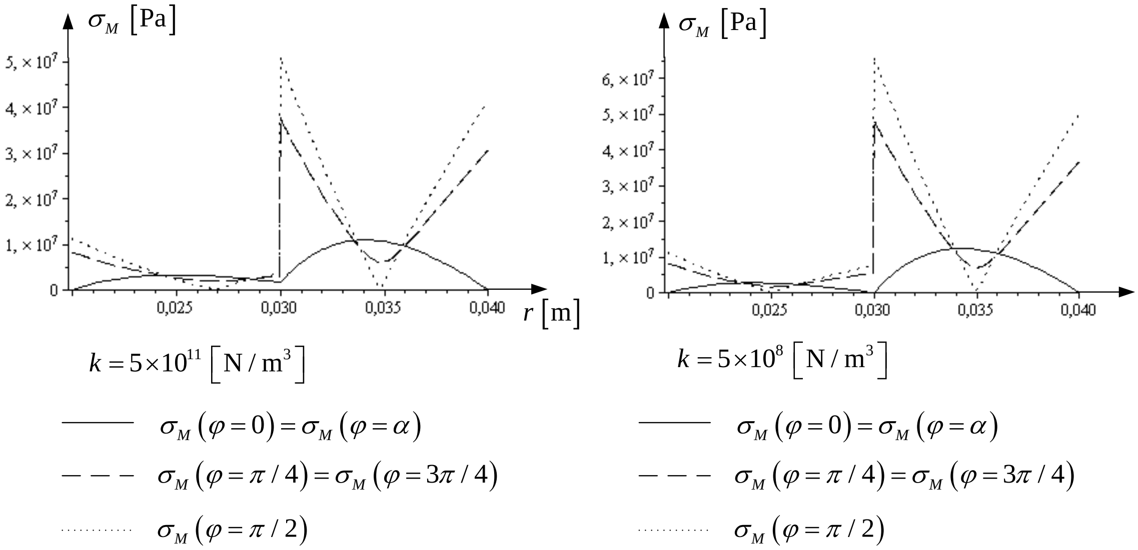

Figure 8. The plots of Von Mises stresses.

some typical cross section as considered in Figs. 6 and 7. All numerical solutions are obtained with $q = 25$.

4. CONCLUSIONS

The paper presents the application of the minimum theorem of complementary energy for two-layer composite curved beam with weak shear connection. By the combination of formulated minimum theorem with the Ritz method the solution of two-layer composite beam with simply supported end conditions loaded by uniform radial load is given. The obtained numerical results can be used as benchmark solution to check the accuracy of solutions obtained by different approximate methods such as FEM, finite differences, etc.

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