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RICHARDSON EXTRAPOLATION FOR SIMULATION OF LAMINAR FLOW PAST A CIRCULAR CYLINDER

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ABSTRACT

In the present study incompressible, constant property, laminar Newtonian unsteady fluid flow around a stationary circular cylinder is computed at different Reynolds numbers using a two-dimensional finite difference method. The time history of lift and drag coefficients are computed to produce root-mean-square (rms) and time-mean values. The objective of this paper is to improve the accuracy of the time discretization and to estimate discretization errors using generalized Richardson extrapolation.

INTRODUCTION

Flow past a stationary circular cylinder is widely investigated in fluid mechanics [1-5]. When the vortex-shedding frequency is near the natural frequency of the body and the damping is low, resonance can occur which can induce high-amplitude oscillation. This phenomenon played a role in the collapse of the Tacoma Narrows Bridge in 1940. Tall slender structures can be found in everyday life, for example chimney stacks, silos, telephone poles or underwater structures. The noisy operation and vibration of heat exchangers are often caused by this phenomenon.

In this study an in-house code based on the two-dimensional finite difference method (FDM) is used for the computation of low Reynolds number (Re=90–200) unsteady, incompressible, Newtonian constant property fluid flow past a stationary circular cylinder. For temporal discretization first order Eulerian method is applied. The objective of this paper is to improve the accuracy of the results using Richardson extrapolation. Richardson extrapolation is widely used for grid refinement studies with the aim of increasing the accuracy of the solution using computational results obtained from different grid sizes. These grids can be either spatial (computational mesh) or temporal (time step) [6–8].

The main objective of the present study is to carry out grid refinement investigations. Using an FDM code computations are carried out for three pairs of time steps and the rms and time-mean values of lift and drag coefficients are evaluated in order to examine the effect of Reynolds number and of time step. The two solutions for each time step pair can be combined to yield a solution with higher accuracy. The relative difference and normalized relative difference between the solutions are investigated as the first steps in estimating computational error.

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COMPUTATIONAL METHOD

The non-dimensional governing equations for the incompressible, constant property, laminar Newtonian fluid flow around a stationary circular cylinder are the two components of the equations of motion, the continuity equation and the Poisson equation for pressure. Figure 1 shows the physical and computational domains, where R_1 is the dimensionless radius of the cylinder and R_2 is that of the far field. No-slip boundary conditions are used for velocity and a Neumann-type boundary condition is applied for pressure both on the surface of the cylinder and on the outer surface.

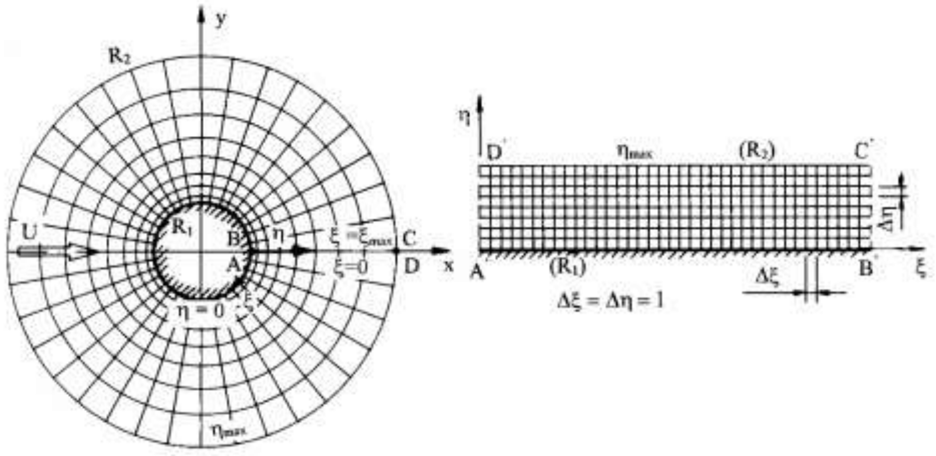


Figure 1
The physical and computational domains

In order to impose boundary conditions accurately and avoid inaccuracies, boundary-fitted coordinates are used. The physical domain is transformed into the rectangular computational domain applying linear mapping functions [2,3]. Due the properties of the mapping functions, the grid on the physical plane is very fine in the vicinity of the cylinder surface and coarse in the far field, but the grid is equidistant on the computational domain. The transformed governing equations with the boundary conditions are solved applying finite difference method [2]. The space derivatives are discretized using fourth order schemes except for the convective terms which are approximated by third order upwind difference schemes. The Poisson equation is solved using successive over-relaxation (SOR), the equation of motion is integrated explicitly and continuity equation is satisfied at every time step.

During the computations lift and drag coefficients (C_L and C_D) are obtained. Both lift and drag coefficients can be divided into two basic parts: one is due to the shear stress ($C_{L\tau}$ and $C_{D\tau}$), the other is due to pressure (C_{Lp} and C_{Dp}). From the time histories of these signals the time-mean (denoted by an overbar) and root-mean-square (rms) values are computed as

$$\bar{h} = \frac{1}{mT}\int_{t_1}^{t_1+mT} h(t)dt\,; \qquad h_{rms} = \sqrt{\frac{1}{mT}\int_{t}^{t+mT} \left[h(t)-\bar{h}\right]^2 dt} \quad . \qquad (1)$$

The dimensionless vortex shedding frequency or Strouhal number (St) can be obtained by taking Fast Fourier Transform (FFT) of the time history of lift [2].

During the computations the radius ratio R_2/R_1=160 and the computational grid is characterized by grid points 360×292 (peripheral × radial). The dimensionless time step (Δt) is varied between 0.001 and 0.00025.

GENERALIZED RICHARDSON EXTRAPOLATION

The main drawback of our code is that it uses first order Euler method for time discretization, which is accurate only if small Δt is used. The main objective of this paper is to increase the computational accuracy and to approximate the discretization error. Richardson extrapolation or its generalized version is widely used for these purposes [6-8]. Two solutions in series representations belonging to two different time steps Δt_1 and Δt_2 can be written as

$$f_1 = f_{exact} + g_p(\Delta t_1)^p + \mathcal{O}[(\Delta t_1)^{p+1}], \tag{2}$$

$$f_2 = f_{exact} + g_p(\Delta t_2)^p + \mathcal{O}[(\Delta t_2)^{p+1}], \tag{3}$$

where f_1 and f_2 are the numerical approximations of the exact solution f_{exact}. In equations (2) and (3) g_p is the pth order error term coefficient and $\mathcal{O}[(\Delta t)^{p+1}]$ represents the higher-order terms that will be neglected. The time step reference is denoted by Δt_1 and the ratio of the two time steps $r = \Delta t_1/\Delta t_2$ is called the refinement ratio. In the case of the original Richardson extrapolation, r=2 and the method is also generalized for cases r≠2. The feasible limits of the non-dimensional time step Δt are between 0.00025 and 0.001. Below this excessive computational time would be needed, and above this the SOR method diverges. We decided to use pairs of time steps within these limits, giving us a practical choice for the refinement ratio of $r = \Delta t_1/\Delta t_2 = 1.4$. Three reference time steps Δt_1 were chosen in order to investigate the effect of time step on the solution (Table 1).

Table 1
The applied time step values

Notation	$10^4 \times \Delta t_1$ (reference)	$10^4 \times \Delta t_2$
I	10.0	7.14286
II	5.0	3.57143
III	3.5	2.5

The most difficult task in numerical studies is to estimate the discretization error. If the computational grid is refined and the spacing tends to zero, the relative difference

$$\varepsilon[\%] = \frac{f_2 - f_1}{f_1} 100 \tag{4}$$

between two “neighboring” solutions decreases. If the relative difference approaches zero while the spacing tends to zero, the solution is said to be asymptotic [8].

Dropping the higher-order terms in equations (2) and (3) and solving them for the exact solution f_{exact} yields

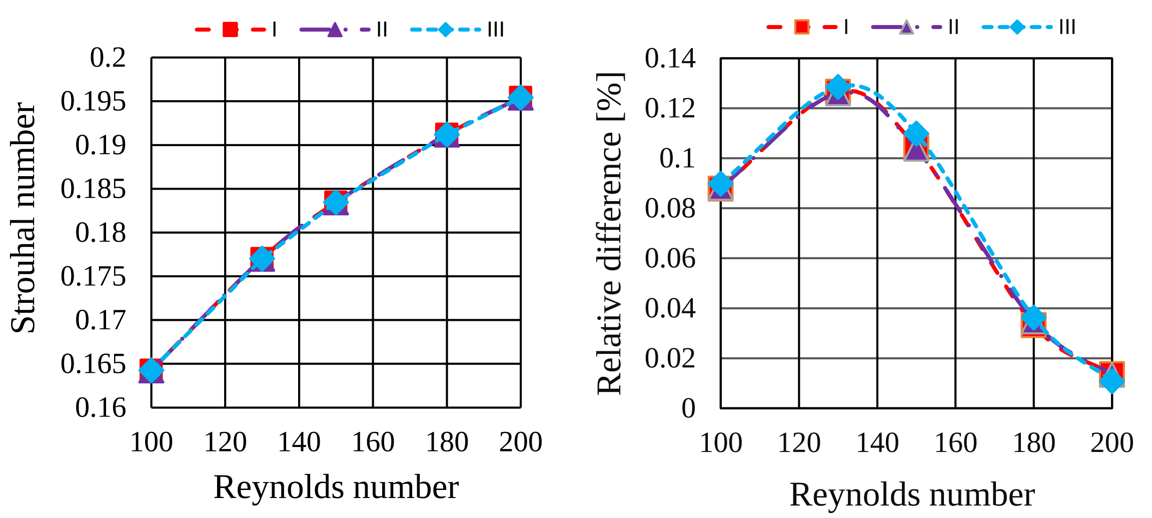
$$f_{exact} \cong f_2 + \frac{f_2 - f_1}{r^p - 1}, \tag{5}$$

the solution of the generalized Richardson extrapolation. Temporal discretization is first-order accurate, therefore p=1. Substituting p=1 and r=1.4 into equation (5),

$$f_{exact} \cong 3.5 f_2 - 2.5 f_1. \tag{6}$$

COMPUTATIONAL RESULTS

We carried out computations for the three reference time steps (see Δt_1 values in Table 1) at five Re values. Applying (6) for each pair of results, higher-order solutions can be reached (see Figure 2a) which practically collapse into a single curve. The relative differences ε between the extrapolated St values and those reported in [5] are shown in Figure 2b). As can be seen, the agreement is excellent, with the maximum relative difference of 0.128% indicating that our results are reliable. It can also be observed that the relative difference depends on Reynolds number: it increases until Re reaches 130, then decreases.



a) Extrapolated Strouhal numbers

b) Relative differences in St (our data and [5])

Figure 2
The results of the extrapolation

For further increasing the accuracy, three time steps are used and Richardson extrapolation is repeated using the extrapolated values. However, in our case this led to only marginal improvement, meaning the extra computational effort was not justified. This was probably because the SOR method allows only small time steps.

Next, the relative difference between each pair of time steps (the reference and its finer version are shown in Table 1) was calculated at different Reynolds numbers and plotted in Figure 3. This allows us to investigate how the relative difference changes with Reynolds number for the rms values of lift and drag (C_L and C_D) and the portion due to shear stress ($C_{L\tau}$ and $C_{D\tau}$).

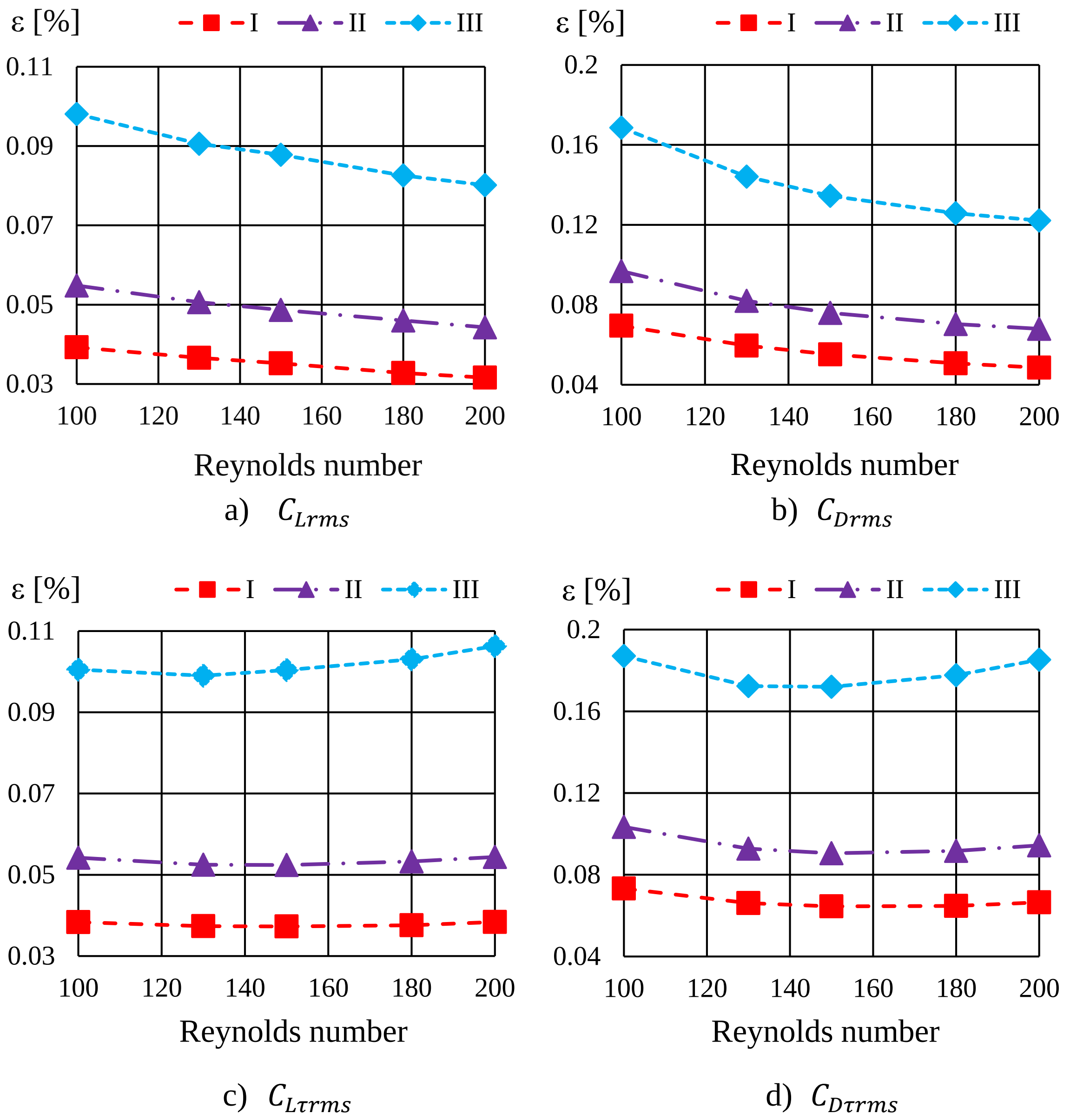


Figure 3
The computed relative differences (ε)

It can be seen in Figure 3 that the discretization error is lower for lower Δt_1, as expected. In addition, the relative difference ε for $C_{L\tau rms}$ and $C_{D\tau rms}$ (Figures 3c) and 3d)) decreases up to Re=130, and then increase with Re. The slope of the curves differ. The relative differences for C_{Drms} and C_{Lrms} (Figures 3a and 3b) decrease monotonously, and no extreme values can be observed. One possible reason could be that the role of shear stresses increases with increasing Reynolds number. This requires further investigation.

As can be seen in Figure 3, the scale of ε differs somewhat for the rms values investigated, making comparisons difficult. To be able to compare relative differences for different flow properties it seems reasonable to introduce the normalized relative difference (ε_{norm}), defined as the absolute value of the ratio of relative differences for the same quantity belonging to time step references Δt_1 and $\Delta t_1 = 0.001$ (the largest time step reference) at the same Re

$$\varepsilon_{norm}(Re_i, \Delta t_1) = \left|\frac{\varepsilon(Re_i, \Delta t_1)}{\varepsilon(Re_i, \Delta t_1 = 0.001)}\right|. \tag{7}$$

The values of ε_{norm} are in the range of [0,1] if the extrapolated solution is more accurate for the Δt_1 value than for $\Delta t_1 = 0.001$. If $\varepsilon_{norm} > 1$, Richardson extrapolation does not lead to improved accuracy.

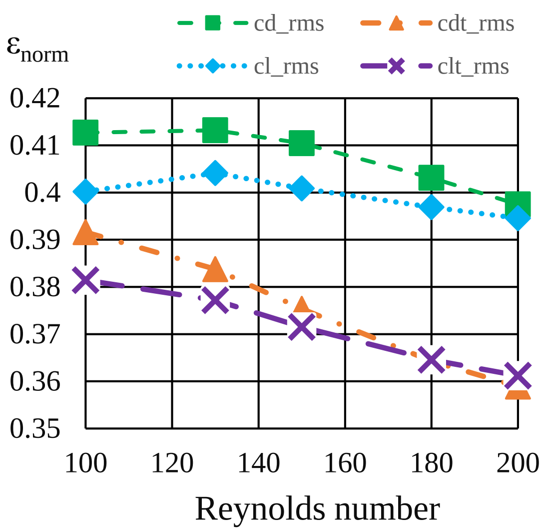
Figure 4 shows the ε_{norm} of the *rms* values of lift and drag coefficients against Re for two distinct time step reference values of $\Delta t_1 = 0.00035$ (see Fig. 4a) and $\Delta t_1 = 0.0005$ (See Fig. 4b). From the figure we can see that

(1) the ε_{norm} values are smaller for the smaller Δt_1 value (finer time step), as expected;

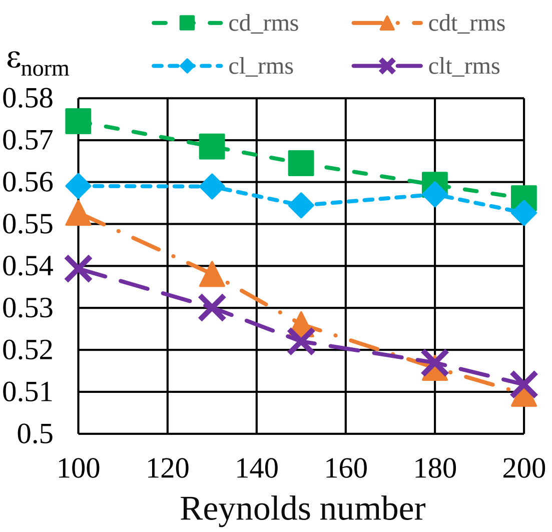
(2) the total rms of lift and drag coefficients approach each other more closely as Re increases, and this is also true for the portion due to shear stress ($C_{L\tau}$ and $C_{D\tau}$);

(3) the difference in ε_{norm} for C_L and $C_{L\tau}$ and that for C_D and $C_{D\tau}$ increases with Re, indicating that pressure (the other portion of the total rms) increases with Re; and

(4) all of the ε_{norm} values calculated were well below 1, meaning that the Richardson extrapolation improved the accuracy of the solution.



a) $\Delta t_1 = 0.00035$



b) $\Delta t_1 = 0.0005$

Figure 4
Comparison of ε_{norm} for rms of lift and drag

CONCLUSIONS

This paper deals with the numerical simulation of a two-dimensional incompressible, constant property, laminar Newtonian fluid flow around a stationary circular cylinder, where the main objective is to increase the accuracy of time discretization and the estimation of the discretization errors using generalized Richardson extrapolation. Relative differences and normalized relative differences ε_{norm} are used for approximating the discretization errors. Computational results are in good agreement with those in the literature. For the normalized relative difference, it was found that

- the ε_{norm} values are smallest for the smallest reference time step value Δt_1;
- as Re increases, the total rms of lift and drag coefficients approach each other more closely, and the same is true for the portion of rms due to shear stress ($C_{L\tau}$ and $C_{D\tau}$);
- as Re increases, the difference in ε_{norm} for C_L and $C_{L\tau}$ and that for C_D and $C_{D\tau}$ also increases. From this, it seems that the role of pressure (the other portion of the total rms) increases with Re;
- The discretization error depends on the Reynolds number.

Using generalized Richardson extrapolation, we succeeded in increasing the accuracy of the solution (all ε_{norm} values were well below 1). Further investigation is needed to determine whether the increased accuracy compensates for the increased computational cost of carrying out computations with two different time steps.

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