MultiScience - XXX. microCAD International Multidisciplinary Scientific Conference
University of Miskolc, Hungary, 21-22 April 2016, ISBN 978-963-358-113-1

GENERAL THEOREMS ON EXPONENTIAL AND ROSENTHAL'S INEQUALITIES

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1 INTRODUCTION

It is known that in the proofs of asymptotic results for independent random variables (r.v.'s) exponential relations play fundamental role. The following definition gives the general form of these relations. The r.v.'s $X_1, X_2, \ldots, X_n$ are said to be acceptable if

$$\mathbb{E}e^{\sum_{i=1}^{n} \lambda X_i} \leq \prod_{i=1}^{n} \mathbb{E}e^{\lambda X_i} \tag{1.1}$$

for any real number λ, see [1]. Exponential inequalities were obtained for acceptable sequences, furthermore some versions of the notion of acceptability were introduced, see e.g. [7] and [10].

In this paper we shall present that an appropriate version of inequality (1.1) implies an exponential inequality. Then the exponential inequality implies a Rosenthal type inequality. To obtain the above results no additional dependence conditions are needed. Our general theorems will be applied to weakly orthant dependent (WOD) sequences of r.v.'s.

2 EXPONENTIAL INEQUALITIES

Let $d > 0$ be a real number, let X be a random variable (r.v.). Throughout the paper

$$X^{(d)} = \min\{X, d\}$$

will denote the r.v. truncated from above.

For the sequence of r.v.'s $\eta_1, \eta_2, \ldots, \eta_n$ we shall consider certain versions of the condition

$$\mathbb{E}e^{\sum_{i=1}^{n} \lambda \eta_i} \leq g(n) \prod_{i=1}^{n} \mathbb{E}e^{\lambda \eta_i}. \tag{2.1}$$

Here $g(n)$ is finite and non-negative. If (2.1) is satisfied for all $\lambda \in \mathbb{R}$ then we obtain the notion of acceptable r.v.'s ([1], [7]). If (2.1) is true for $\eta_1, \eta_2, \ldots, \eta_n$, then it remains true for $\eta_1 - a_1, \eta_2 - a_2, \ldots, \eta_n - a_n$ for any real numbers $a_1, \ldots, a_n$, in particular for $\eta_1 - \mathbb{E}\eta_1, \eta_2 - \mathbb{E}\eta_2, \ldots, \eta_n - \mathbb{E}\eta_n$.

DOI: 10.26649/musci.2016.157

If we assume condition (2.1) for positive values of λ and for the appropriately truncated r.v's, then we shall obtain a one-sided exponential inequality. If we assume that condition (2.1) is true both for positive and negative values of λ, then we can obtain a two-sided exponential inequality.

The following exponential inequality was presented for negatively orthant dependent r.v.'s in Lemma 3 of [3] and for extended negatively dependent r.v.'s in Lemma 1.2 of [11].

Theorem 2.1. *Let $X_1, X_2, \dots, X_n$ be a sequence of zero mean r.v.'s, $d > 0$. Let $S_n = \sum_{i=1}^n X_i$ be the sum and $B_n = \sum_{i=1}^n \mathbb{E} X_i^2$ be the sum of variances.*

Assume that (2.1) is satisfied for $\eta_i = X_i^{(d)}$, $i = 1, 2, \dots, n$ and $0 < \lambda \le \lambda_0$. Then for any x with $0 < x \le (B_n e^{d\lambda_0} - B_n)/d$, we have

$$\mathbb{P}(S_n > x) \le \mathbb{P}\left(\max_{1 \le i \le n} X_i > d\right) + g(n) \exp\left(\frac{x}{d} - \frac{x}{d} \ln\left(1 + \frac{xd}{B_n}\right)\right). \quad (2.2)$$

If (2.1) is satisfied both for $\eta_i = X_i^{(d)}$ $i = 1, 2, \dots, n$, and $\eta_i = (-X_i)^{(d)}$, $i = 1, 2, \dots, n$ and $0 < \lambda \le \lambda_0$, then for any x with $0 < x \le (B_n e^{d\lambda_0} - B_n)/d$ we have

$$\mathbb{P}(|S_n| > x) \le \mathbb{P}\left(\max_{1 \le i \le n} |X_i| > d\right) + 2g(n) \exp\left(\frac{x}{d} - \frac{x}{d} \ln\left(1 + \frac{xd}{B_n}\right)\right). \quad (2.3)$$

Proof. The proof applies the classical ideas of Fuk and Nagaev [2] (see also [6]). The same method was used in the proofs of Lemma 3 in [3] and Lemma 1.2 in [11]. □

Now we shall study Hoeffding's inequality. It was found for independent random variables in [4]. Then it was extended to dependent sequences. Our next theorem is a version of Theorem 2.3 in [7], where acceptable r.v.'s were studied.

Theorem 2.2. *Let $X_1, X_2, \dots, X_n$ be a sequence of r.v.'s. Let $S_n = \sum_{i=1}^n X_i$ be the sum. Let the random variables be bounded, i.e. $a_i \le X_i \le b_i$ for $i = 1, 2, \dots, n$, where a_i and b_i are real numbers. Assume that (2.1) is satisfied with $\eta_i = X_i$, $i = 1, 2, \dots, n$ and $0 < \lambda \le \lambda_0$. Then for any ε with $0 < \varepsilon \le \frac{\lambda_0}{4} \sum_{i=1}^n (b_i - a_i)^2$, we have*

$$\mathbb{P}(S_n - \mathbb{E} S_n \ge \varepsilon) \le g(n) \exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \quad (2.4)$$

Assume that (2.1) is satisfied for $\eta_i = X_i$, $i = 1, 2, \dots, n$ and $|\lambda| \le \lambda_0$. Then for any ε with $0 < \varepsilon \le \frac{\lambda_0}{4} \sum_{i=1}^n (b_i - a_i)^2$, we have

$$\mathbb{P}(|S_n - \mathbb{E} S_n| \ge \varepsilon) \le 2g(n) \exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \quad (2.5)$$

Proof. One can follow the proof given for independent r.v.'s (see the original paper by Hoeffding [4], Theorem 2). For acceptable r.v.'s an appropriate version of the original proof is described in [7] (see the proof of Theorem 2.3 in [7]). □

We shall find the maximal version of Hoeffding's inequality. In [7] the maximal Hoeffding's inequality for acceptable r.v.'s was not considered.

Corollary 2.1. Let $X_1, X_2, \dots, X_n$ be a sequence of r.v.'s. Assume that the random variables are bounded, i.e. $a_i \le X_i \le b_i$ for $i = 1, 2, \dots, n$, where a_i and b_i are real numbers. For k, l with $1 \le k \le l \le n$ denote by $M_{k,l}$ the following maximum

$$M_{k,l} = \max_{k \le j \le l} \left| \sum_{t=k}^{j} (X_t - \mathbb{E}X_t) \right|. \tag{2.6}$$

Assume that

$$\mathbb{E}e^{\sum_{i=k}^{l} \lambda X_i} \le C \prod_{i=k}^{l} \mathbb{E}e^{\lambda X_i} \tag{2.7}$$

for any $1 \le k < l \le n$ and any $\lambda \in \mathbb{R}$. Let $\varepsilon > 0$. Then for any $0 < \delta < 1$ there exists a $C_1 = C_1(\delta)$ such that

$$\mathbb{P}\left(M_{k,l} \ge \varepsilon\right) \le 2CC_1 \exp\left(-\frac{2\varepsilon^2(1-\delta)}{\sum_{i=k}^{l}(b_i - a_i)^2}\right) \tag{2.8}$$

for any $1 \le k \le l \le n$.

Proof. By (2.5) we have

$$\mathbb{P}\left(\left|\sum_{t=k}^{l}(X_t - \mathbb{E}X_t)\right| \ge \varepsilon\right) \le 2C \exp\left(-\frac{2\varepsilon^2}{\sum_{i=k}^{l}(b_i - a_i)^2}\right) \tag{2.9}$$

for any $1 \le k < l \le n$. We see that the function $g(k,l) = \sum_{i=k}^{l}(b_i - a_i)^2$ is superadditive. So Theorem 1 of Móricz [5] implies the result. □

3 ROSENTHAL'S INEQUALITY

We show that a general exponential inequality implies a Rosenthal type inequality.

Theorem 3.1. *Let $X_1, X_2, \dots, X_n$ be a sequence of zero mean r.v.'s, let $S_n = \sum_{i=1}^{n} X_i$ be their sum and B_n be a sequence of positive numbers. Assume that*

$$\mathbb{P}\left(|S_n| > x\right) \le l(n)\mathbb{P}\left(\max_{1 \le i \le n} |X_i| > d\right) + h(n) \exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd}{B_n}\right)\right) \tag{3.1}$$

is satisfied for any $x > 0$ and $d > 0$, where $h(n)$ and $l(n)$ are finite and non-negative. Then, for $p > 0$ we have

$$\mathbb{E}|S_n|^p \leq C_1 l(n) \mathbb{E} \max_{1\leq i\leq n} |X_i|^p + C_2 h(n) B_n^{p/2}, \tag{3.2}$$

where $C_1 = p^p$, $C_2 = \frac{1}{2} p^{1+p/2} e^p B\left(\frac{p}{2}, \frac{p}{2}\right)$ are absolute constants with $B(u, v)$ being the beta function.

Proof. We can use the classical method, see [6]. □

Remark 3.1. Let $X_1, X_2, \ldots, X_n$ be a sequence of zero mean r.v.'s, let $S_n = \sum_{i=1}^n X_i$ be their sum and $B_n = \sum_{i=1}^n \mathbb{E}X_i^2$ be the sum of variances. Assume that (2.1) is satisfied both for $\eta_i = X_i^{(d)}$, $i = 1, 2, \ldots, n$ and for $\eta_i = (-X_i)^{(d)}$, $i = 1, 2, \ldots, n$ for any $\lambda > 0$ and $d > 0$. Then Theorem 2.1 and inequality (3.2) imply

$$\mathbb{E}|S_n|^p \leq C_1 \mathbb{E} \max_{1\leq i\leq n} |X_i|^p + 2C_2 g(n) B_n^{p/2}, \tag{3.3}$$

where $p > 0$.

4 WIDELY ORTHANT DEPENDENT SEQUENCES OF R.V.'S

The sequence of r.v.'s $X_1, X_2, \ldots$ is said to be widely orthant dependent (WOD) if for any positive integer n there exists a finite $g(n)$ so that for any real numbers $x_1, \ldots, x_n$ we have

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq g(n) \prod_{i=1}^n \mathbb{P}(X_i > x_i) \tag{4.1}$$

and

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq g(n) \prod_{i=1}^n \mathbb{P}(X_i \leq x_i), \tag{4.2}$$

see [8]. Extended negatively orthant dependent sequences, negatively orthant dependent sequences, negatively superadditive dependent sequences, negatively associated and independent sequences are WOD, see [9]. In [9] exponential inequalities were proved for WOD sequences.

We shall see that the results of the previous sections are true for WOD sequences. To see this we list known facts on WOD sequences.

If $X_1, X_2, \ldots$ is a WOD sequence and the real functions $f_1, f_2, \ldots$ are either all non-decreasing of all non-increasing, then the sequence

$f_1(X_1), f_2(X_2), \ldots$ is WOD. In particular, the truncated sequence $X_1^{(t)}, X_2^{(t)}, \ldots$ is WOD. Moreover

$$\mathbb{E}e^{\sum_{i=1}^n \lambda X_i} \leq g(n) \prod_{i=1}^n \mathbb{E}e^{\lambda X_i} \tag{4.3}$$

for any real number λ.

Theorem 4.1. *Let $X_1, X_2, \ldots, X_n$ be a sequence of zero mean WOD r.v.'s. Let $S_n = \sum_{i=1}^n X_i$ be the sum and $B_n = \sum_{i=1}^n \mathbb{E}X_i^2$ be the sum of variances. Then for $d > 0$ and $x > 0$ we have*

$$\mathbb{P}(S_n > x) \leq \mathbb{P}\left(\max_{1\leq i\leq n} X_i > d\right) + g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd}{B_n}\right)\right) \tag{4.4}$$

and

$$\mathbb{P}(|S_n| > x) \leq \mathbb{P}\left(\max_{1\leq i\leq n} |X_i| > d\right) + 2g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd}{B_n}\right)\right). \tag{4.5}$$

Proof. It is a consequence of Theorem 2.1 and inequality (4.3). We remark that (4.5) is a special case of Lemma 2.2 in [9], where no detailed proof was presented. □

Now we shall find Hoeffding's inequality for WOD sequences.

Theorem 4.2. *Let $X_1, X_2, \ldots, X_n$ be a WOD sequence of r.v.'s. Let $S_n = \sum_{i=1}^n X_i$ be the sum. Let the random variables be bounded, i.e. $a_i \leq X_i \leq b_i$ for $i = 1, 2, \ldots, n$, where a_i and b_i are real numbers. Let $\varepsilon > 0$. Then we have*

$$\mathbb{P}(S_n - \mathbb{E}S_n \geq \varepsilon) \leq g(n)\exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right), \tag{4.6}$$

$$\mathbb{P}(|S_n - \mathbb{E}S_n| \geq \varepsilon) \leq 2g(n)\exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \tag{4.7}$$

Let $M_{k,l}$ the maximum defined by (2.6). Assume that (4.1) and (4.2) are satisfied with $g(n) = C$. Then for any $0 < \delta < 1$ there exists a $C_1 = C_1(\delta)$ such that

$$\mathbb{P}(M_{k,l} \geq \varepsilon) \leq 2CC_1\exp\left(-\frac{2\varepsilon^2(1-\delta)}{\sum_{i=k}^l (b_i - a_i)^2}\right) \tag{4.8}$$

for any $1 \leq k \leq l \leq n$.

Proof. It is a consequence of Theorem 2.2, Corollary 2.1 and inequality (4.3). □

Now we consider the Rosenthal type inequality for WOD sequences.

Theorem 4.3. *Let $X_1, X_2, \dots, X_n$ be a WOD sequence of zero mean r.v.'s, let $S_n = \sum_{i=1}^n X_i$ be their sum and $B_n = \sum_{i=1}^n \mathbb{E}X_i^2$. Then, for $p > 0$ we have*

$$\mathbb{E}|S_n|^p \leq C_1 \mathbb{E} \max_{1\leq i\leq n} |X_i|^p + C_2 g(n) B_n^{p/2}, \tag{4.9}$$

where C_1 and C_2 are absolute constants.

Proof. It is a consequence of Remark 3.1 and inequality (4.3). □

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