Differenciálegyenletek
Differential Equations

2

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Chapter 1

Kalkulus előismeretek

1.1 Newton-Leibnitz tétel

$$\int f(t)\,dt = F(t) \quad \Longleftrightarrow \quad F'(t) = f(t)$$
$$\int_a^b f(t)\,dt = F(t)|_a^b = F(b) - F(a)$$
$$\frac{d}{db}\int_a^b f(t)\,dt = \frac{d}{db}\left(F(b) - F(a)\right) = F'(b) - 0 = f(b)$$

1.2 Lineáris approximació

Lineáris kozelítés:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + hiba(\Delta x) \approx f(x) + f'(x)\Delta x, \tag{1.1}$$
$$\lim_{\Delta x \to 0}\left|\frac{hiba(\Delta x)}{\Delta x}\right| = 0.$$

Hibabecslés:

Problema 1. *Adott egy $f(t)$ függéeny. Négy, egyirányba haladó auto versenyez: F, Lin, Lassú, Gyors. $t = 0$-kor mindegyik pozíciója es sebessége $f(0)$, illetve $f'(0)$. Legyen $A = max_{s\in[0,\Delta t]}|f''(s)|$. Legyenek az autok pozíciói:*

$$\begin{aligned} F:&\quad f(t),\\ Lin:&\quad f(0) + f'(0)t,\\ Gyors:&\quad f(0) + f'(0)t + \frac{A}{2!}t^2,\\ Lassu:&\quad f(0) + f'(0)t - \frac{A}{2!}t^2. \end{aligned}$$

Becsüld meg felülről $|F(t) - Lin(t)|$-t!

Megoldas 1. *Lin állandó sebességel halad (lineáris approximació), míg F es Lin mindketten Gyors es Lassú között helyezkednek el, mivel Gyors es Lassú* *kihasználja a maximálisan elérhető gyorsulást (vagy Lassú esetében lassúlást). Tehát*

$$|F(t) - Lin(t)| \leq |Gyors(t) - Lin(t)| = \frac{1}{2}t^2 max_{s\in[0,\Delta t]}|f''(s)|$$

Tehát a lineáris közelítés (2.1) hibájára igaz, hogy

$$|hiba(\Delta x)| \leq \frac{1}{2}\Delta x^2 max_{s\in[x,x+\Delta x]}|f''(s)| \tag{1.2}$$

Problema 2.

$$a)\ f(x) = \sin x,\ \ x_0 = \pi/3; \qquad b)\ f(x) = \sqrt{x},\ \ x_0 = 9;$$
$$c)\ f(x) = 1/x,\ \ x_0 = 2;$$

Írd fel f-nek a lineáris $f(x_0+\Delta x) \approx T_1(x_0+\Delta x) = f(x_0)+f'(x_0)\Delta x$ közelítését, ha $\Delta x = 0.1$! Mennyi $max_{z\in[x_0,x_0+\Delta x]}|f''(z)|$? Adj felső korlátot a közelítés $|hiba(\Delta x)| = |f(x_0 + \Delta x) - T_1(x_0 + \Delta x)|$ hibájárá!

Megoldas 2.

$$a) \qquad \sin(\pi/3 + 0.1) = \sin(\pi/3) + \cos(\pi/3) \cdot 0.1 + hiba(0.1)$$
$$|hiba(0.1)| \leq \frac{1}{2}\Delta x^2 max_{z\in[x_0,x_0+\Delta x]}|f''(z)|$$
$$= \frac{1}{2}0.1^2 max_{z\in[\pi/3,\pi/3+0.1]}|\cos(z)| \leq \frac{1}{2}0.1^2 \cdot 1$$

$$c) \qquad 1/(2 + 0.1) = 1/2 - 1/2^2 \cdot 0.1 + hiba(0.1)$$
$$|hiba(0.1)| \leq \frac{1}{2}\Delta x^2 max_{z\in[x_0,x_0+\Delta x]}|f''(z)|$$
$$= \frac{1}{2}0.1^2 max_{z\in[2,2+0.1]}|2/z^3| = \frac{1}{2}0.1^2 \cdot 2/8$$

ÖkÖlszabály:

$$|f(x + \Delta x) - f(x)| = |hiba(\Delta x)| \approx \frac{1}{2}\Delta x^2 f''(x)$$

Egy példa, amikor ez az eljárás nem alkalmazható: Mennyi az $f(x) = |x|^{1.5}$ függvény lineáris approximaciója $x = 0$ körűl? Mennyi a közelítés hibája?

Egy kissé matematikusabb bizonyítása (2.2)-nak ezen alapulhatna:

$$\begin{aligned} f(t) = f(0) + \int_0^t f'(s)\,ds &= f(0) + \int_0^t \left(f'(0) + \int_0^s f''(u)\,du\right) ds \\ &= f(0) + f'(0)t + \int_0^t \left(\int_0^s f''(u)\,du\right) ds \\ &\leq f(0) + f'(0)t + max_{u\in[0,t]}|f''(u)| \cdot \int_0^t \left(\int_0^s 1\,du\right) ds \\ &= f(0) + f'(0)t + \frac{1}{2}t^2 max_{u\in[0,t]}|f''(u)| \end{aligned}$$

1.3 Többváltozós lineáris approximáció

Lineáris közelítés:

$$f(x_1+\Delta x_1, x_2+\Delta x_2) \approx f(x_1,x_2) + f'_{x_1}(x_1,x_2)\Delta x_1 + f'_{x_2}(x_1,x_2)\Delta x_2,$$

áltaáanosabban:

$$f(\overline{x+\Delta x}) \approx f(\overline{x}) + \sum_i f'_{x_i}(\overline{x})\Delta x_i$$

1.4 Taylor sor

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots, \\ f(x+\Delta x) &= f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \frac{f^{(3)}(x)}{3!}\Delta x^3 + \ldots, \end{aligned}$$

ami formálisan felírható mint:

$$f(x+\Delta x) = \left[\left(1 + \Delta x\partial_x + \frac{1}{2!}\Delta x^2\partial_x^2 + \ldots\right) f\right](x) = \left[e^{\Delta x\partial_x} f\right](x),$$

ahol e^A "defiíiciója" valamilyen A operátor esetében:

$$e^A = 1 + A + \frac{1}{2!}A^2 + \cdots.$$

Ha az A operátor nem "korlátos" (sajnos ∂_x ilyen), akkor a konvergencia kérdése a sorfejtésnél eléggé komplikált.

Problema 3. *Írd fel a következő függvények Taylor sorát az $x = x_0$ pont körűl!*

$$a)\ e^{3x},\ x_0 = 0; \qquad b)\ \sin(3x),\ x_0 = 0; \qquad c)\ \log(x),\ x_0 = 1;$$

$$d)\ \frac{1}{1-x},\ x_0 = 0; \qquad e)\ \frac{1}{x^2+1},\ x_0 = 0.$$

Megoldas 3.

$$a)\ 1 + 3x + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + O\left(x^4\right) = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + O\left(x^4\right)$$

$$b)\ 3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \frac{3^7x^7}{7!} + O\left(x^9\right) = 3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \frac{243x^7}{560} + O\left(x^9\right)$$

$$c)\ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + O\left((x-1)^5\right)$$

$$d)\ 1 + x + x^2 + x^3 + x^4 + O\left(x^5\right)$$

$$e)\ 1 - x^2 + x^4 - x^6 + O\left(x^7\right)$$

Chapter 2

Calculus prerequisites

2.1 Newton-Leibnitz theorem

$$\int f(t)\,dt = F(t) \quad \Longleftrightarrow \quad F'(t) = f(t)$$
$$\int_a^b f(t)\,dt = F(t)|_a^b = F(b) - F(a)$$
$$\frac{d}{db}\int_a^b f(t)\,dt = \frac{d}{db}\left(F(b) - F(a)\right) = F'(b) - 0 = f(b)$$

2.2 Linear approximation

Linear approximation:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + err(\Delta x) \approx f(x) + f'(x)\Delta x, \qquad (2.1)$$
$$\lim_{\Delta x\to 0}\left|\frac{err(\Delta x)}{\Delta x}\right| = 0.$$

Error estimate:

Problema 4. *Take $f(t)$. Take the car race of four cars moving in the same diretion: F, Lin, Slow, Fast. At $t = 0$ they have the same position and speed: $f(0)$ and $f'(0)$. Let $A = max_{s\in[0,\Delta t]}|f''(s)|$. The positions of the cars are:*

$$\begin{aligned} F:&\quad f(t),\\ Lin:&\quad f(0) + f'(0)t,\\ Fast:&\quad f(0) + f'(0)t + \frac{A}{2!}t^2,\\ Slow:&\quad f(0) + f'(0)t - \frac{A}{2!}t^2. \end{aligned}$$

Estimate $|F(t) - Lin(t)|$-t!

Megoldas 4. *Lin's speed is constant (linear approximation), while F and Lin are somewhere between Fast and Slow, as Fast and Slow are moving with the* *maximal available acceleration or deceleration. Consequently*

$$|F(t) - Lin(t)| \leq |Fast(t) - Lin(t)| = \frac{1}{2}t^2 max_{s\in[0,\Delta t]}|f''(s)|$$

So the error of the linear approximation (2.1) satisfies

$$|err(\Delta x)| \leq \frac{1}{2}\Delta x^2 max_{s\in[x,x+\Delta x]}|f''(s)| \tag{2.2}$$

Problema 5.

$$a)\ \ f(x) = \sin x,\ \ x_0 = \pi/3; \qquad b)\ \ f(x) = \sqrt{x},\ \ x_0 = 9;$$
$$c)\ \ f(x) = 1/x,\ \ x_0 = 2;$$

Find the linear approximation of f:

$$f(x_0 + \Delta x) \approx T_1(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x,$$

if $\Delta x = 0.1$! How much is $max_{z\in[x_0,x_0+\Delta x]}|f''(z)|$? Find an upper bound for the $|err(\Delta x)| = |f(x_0 + \Delta x) - T_1(x_0 + \Delta x)|$ error!

Megoldas 5.

$$\begin{aligned} a) \qquad \sin(\pi/3 + 0.1) &= \sin(\pi/3) + \cos(\pi/3)\cdot 0.1 + err(0.1) \\ |err(0.1)| &\leq \frac{1}{2}\Delta x^2 max_{z\in[x_0,x_0+\Delta x]}|f''(z)| \\ &= \frac{1}{2}0.1^2 max_{z\in[\pi/3,\pi/3+0.1]}|\cos(z)| \leq \frac{1}{2}0.1^2\cdot 1 \end{aligned}$$

$$\begin{aligned} c) \qquad 1/(2 + 0.1) &= 1/2 - 1/2^2\cdot 0.1 + err(0.1) \\ |err(0.1)| &\leq \frac{1}{2}\Delta x^2 max_{z\in[x_0,x_0+\Delta x]}|f''(z)| \\ &= \frac{1}{2}0.1^2 max_{z\in[2,2+0.1]}|2/z^3| = \frac{1}{2}0.1^2\cdot 2/8 \end{aligned}$$

Rule of thumb:

$$|f(x + \Delta x) - f(x)| = |err(\Delta x)| \approx \frac{1}{2}\Delta x^2 f''(x)$$

This approximation does not work all the time: Compute the lin.app. of $f(x) = |x|^{1.5}$ around $x = 0$! What can you say about the error?

A somewhat more rigorous proof of (2.2):

$$\begin{aligned} f(t) = f(0) + \int_0^t f'(s)\,ds &= f(0) + \int_0^t \left(f'(0) + \int_0^s f''(u)\,du\right) ds \\ &= f(0) + f'(0)t + \int_0^t \left(\int_0^s f''(u)\,du\right) ds \\ &\leq f(0) + f'(0)t + max_{u\in[0,t]}|f''(u)| \cdot \int_0^t \left(\int_0^s 1\,du\right) ds \\ &= f(0) + f'(0)t + \frac{1}{2}t^2 max_{u\in[0,t]}|f''(u)| \end{aligned}$$

2.3 Several variables lin.app.

Linear approximation:

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2) \approx f(x_1, x_2) + f'_{x_1}(x_1, x_2)\Delta x_1 + f'_{x_2}(x_1, x_2)\Delta x_2,$$

more generally:

$$f(\overline{x + \Delta x}) \approx f(\overline{x}) + \sum_i f'_{x_i}(\overline{x})\Delta x_i$$

2.4 Taylor series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots,$$
$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \frac{f^{(3)}(x)}{3!}\Delta x^3 + \ldots,$$

which can be written as:

$$f(x + \Delta x) = \left[\left(1 + \Delta x \partial_x + \frac{1}{2!}\Delta x^2 \partial_x^2 + \ldots\right) f\right](x) = \left[e^{\Delta x \partial_x} f\right](x),$$

where e^A is "defined" as:

$$e^A = 1 + A + \frac{1}{2!}A^2 + \cdots .$$

If the operator A is "unbounded" (∂_x is such one), then the convergence of the series is a difficult question.

Problema 6. *Find the Taylor series of the following functions around $x = x_0$!*

$$a)\ e^{3x},\ x_0 = 0; \qquad b)\ \sin(3x),\ x_0 = 0; \qquad c)\ \log(x),\ x_0 = 1;$$
$$d)\ \frac{1}{1-x},\ x_0 = 0; \qquad e)\ \frac{1}{x^2+1},\ x_0 = 0.$$

Megoldas 6.

$$a)\ 1 + 3x + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + O\left(x^4\right) = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + O\left(x^4\right)$$
$$b)\ 3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \frac{3^7x^7}{7!} + O\left(x^9\right) = 3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \frac{243x^7}{560} + O\left(x^9\right)$$
$$c)\ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + O\left((x-1)^5\right)$$
$$d)\ 1 + x + x^2 + x^3 + x^4 + O\left(x^5\right)$$
$$e)\ 1 - x^2 + x^4 - x^6 + O\left(x^7\right)$$

Chapter 3

Közönséges DE, numerikus megoldás, geometriai interpretáció

Evolúciós DE-k:

3.1 Geometria

$$y'(t) = f(y(t), t)), \quad \frac{d}{dt}y = f(y, t), \quad y' = f(y, t).$$

Pl.:

$$y'(t) = 2t + y(t)$$

Ha a megoldásgörbe $r(t) = (t, y(t))$ átmegy (pl.) a $(t, y) = (2, 3)$ ponton, akkor az y meredeksége ebben a pontban $y' = 2 \cdot 2 + 3 = 7$, vagyis az $r(t)$ görbe irányvektora a $(t, y) = (2, 3)$ pontban: $v = (1, 2 \cdot 2 + 3) = (1, 7)$.

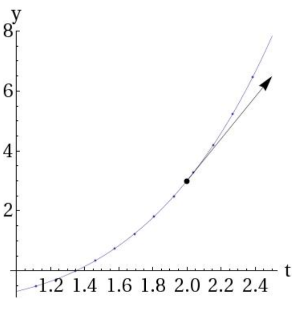


Figure 3.1: Az $y' = 2t + y(t)$ DE $y(2) = 3$ kezdeti feltételt kielégítő megoldásgörbéje. (Az érintővektor a $(2, 3)$ pontban 0.5 skálafaktorral van ábrázolva.)

Tehát egy adott ponton áthaladá megoldásgörbe érintővektora egy vektort rendel az adott ponthoz. Ezt "minden" pontban elvégezve egy vektormezőt kapunk. A DE sebességmezője és az általános megoldások görbeserege:

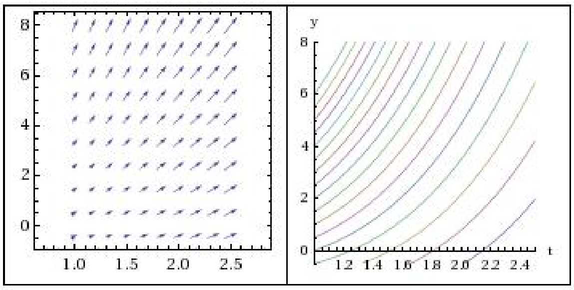


Figure 3.2: Az $y' = 2t + y(t)$ DE sebességmezője és az általános megoláasok görbeserege.

3.2 Magasabb rendű DE-k

Egy magasabb rendű deriváltakat tartalmazó DE átírható elsőrendű DE rendszerre. Legalábbis akkor, ha legmagasabb rendű derivált kifejezhető alacsonyabbrendűekkel (kvázilinearis DE):

Problema 7. *Írd át a következő egyenleteket elsőrendű DE rendszerre!*

$$a)\ y'' = -y' - 2y; \qquad b)\ y''' = y + x; \qquad c)\ \frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' - y_2 \\ y_2' y_1 \end{pmatrix}$$

Megoldas 7.

$$a)\ \frac{d}{dx}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -v - 2y \end{pmatrix}; \qquad b)\ \frac{d}{dx}\begin{pmatrix} y \\ v \\ a \end{pmatrix} = \begin{pmatrix} v \\ a \\ y + x \end{pmatrix};$$

$$c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 - y_2 \\ v_2 \\ v_2 y_1 \end{pmatrix}$$

Az explicit időfüggéstől is meg lehet szabadulni:

Problema 8. *Ird át a következő DE-ket időfüggetlen DE rendszerré!*

$$a)\ y' = xy^2 + x; \qquad b)\ y' = x - y; \qquad c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} xy_1 + y_2 \\ y_1 y_2 + x \end{pmatrix}$$

Megoldas 8.

$$a)\ \frac{d}{dx}\begin{pmatrix} y \\ s \end{pmatrix} = \begin{pmatrix} sy^2 + s \\ 1 \end{pmatrix}; \qquad b)\ \frac{d}{dx}\begin{pmatrix} y \\ s \end{pmatrix} = \begin{pmatrix} s - y \\ 1 \end{pmatrix}$$

$$c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ s \end{pmatrix} = \begin{pmatrix} sy_1 + y_2 \\ y_1 y_2 + s \\ 1 \end{pmatrix}$$

3.3 Numerikus megoldás

3.3.1 Euler (poligon) módszer

Adott $y(t_0) = y_0, \;\; y'(t) = f(y,t)$, ekkor

$$y(t_0 + \Delta t) \approx y_0 + f(y_0, t_0)\Delta t.$$

Pszeudokód :

$$\begin{aligned}
&input \;\; y_0, t_0, \Delta t, f, T \\
&\quad y_{old} \leftarrow y_0 \\
&\quad t \leftarrow t_0 \\
&repeat \\
&\quad y_{new} \leftarrow y_{old} + f(y_{old}, t)\Delta t \\
&\quad y_{old} \leftarrow y_{new} \\
&\quad t \leftarrow t + \Delta t \\
&\quad if(t > T) \; then \; break \\
&end \; loop \\
&output \quad "y(T) = " \;\; y_{old}
\end{aligned}$$

Pl.: $y' = y/2, \; y(0) = 1$. Mennyi $y(0.5)$?

t	y	y’	y-pontos	hiba
0.	1.	0.5	1.	0.
0.1	1.05	0.525	1.05127	−0.0012711
0.2	1.1025	0.55125	1.10517	−0.00267092
0.3	1.15763	0.578813	1.16183	−0.00420924
0.4	1.21551	0.607753	1.2214	−0.00589651
0.5	1.27628	0.638141	1.28403	−0.00774385

Itt (pl.) $y(0.3) = y(0.2) + 0.1 \cdot y(0.2)/2 = 1.1025 + 0.1 \cdot 1.1025/2 = 1.15763$.

Grafikusan ugyanez: $y' = y,\ y(0) = 1,\ \Delta t = 1/3$.

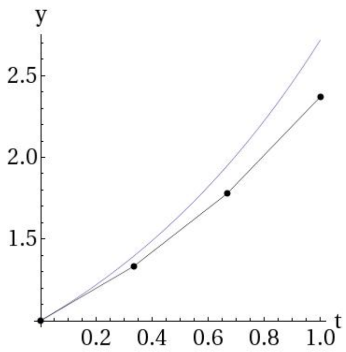


Figure 3.3: Az $y(t)' = y(t),\ y(0) = 1$ DE közelítő és pontos megoldása, ahol $\Delta t = 1/3$.

Problema 9. *Alkalmazd az Euler módszert az $y'(t) = 2y(t)$, $y(0) = 4$ DE megoldására!*

Megoldas 9. *Legyen $y_n = y(n\Delta t)$, $y_0 = y(0) = 4$,*

$$y_{n+1} = y(n\Delta t + \Delta t) = y(n\Delta t) + 2y(n\Delta t)\Delta t = (1 + 2\Delta t) \cdot y_n,$$

tehat $y_n = 4 \cdot (1 + 2\Delta t)^n y_0$.

Ha $\Delta t \to 0$, akkor

$$y(T) \approx (1 + 2\Delta t)^{T/\Delta t} y(0) \quad \to \quad e^{2T} \cdot 4,$$

tehát reproduáaltuk a pontos $y(t) = 4 \cdot e^{2t}$ megoldást.

Problema 10. *Alkalmazd az Euler módszert az $y'(t) = 3y(t) - 6$, $y(0) = 77$ DE megoldására!*

Megoldas 10. *Legyen $y_n = y(n\Delta t)$, $y_0 = y(0) = 77$,*

$$\begin{aligned} y_{n+1} &= y(n\Delta t + \Delta t) = y(n\Delta t) + (3y(n\Delta t) - 6)\Delta t \\ &= (1 + 3\Delta t)y_n - 6\Delta t, \end{aligned}$$

vagyis ha ki tudjuk számítani az

$$x_{n+1} = ax_n + b$$

rekuríiv sorozat áltaáanos tagját, akkor meg tudjuk oldani a probléáat.

3.3.2 Inhomogén geometriai sorozatok

1. Számtani sor:

$$x_{n+1} = x_n + d, \quad \Longrightarrow \quad x_n = x_0 + nd.$$

   Mértani sor:

$$x_{n+1} = qx_n, \quad \Longrightarrow \quad x_n = q^n x_0.$$

   A nekünk szükséges eset: $x_{n+1} = qx_n + d$. Ezt a következő alfejezetben tárgyaljuk.

3.4 Euler módszer hibája, inhomogén geometriai sorozatok

3.4.1 Inhomogén geometriai sorozatok

Legyen $x_{n+1} = ax_n + b = f(x_n)$. Ha adott x_0, mennyi x_n?

Linearizááas a fixpont körűl: Vegyük a következőo diszkrét idejű dinamikai rendszert amit az $f : x \to f(x)$ függvény generál:

$$x_0,\, x_1 = f(x_0),\, x_2 = f(x_1) = f(f(x_0)) = f^2(x_0),\, \ldots, x_{n+1} = f(x_n), \ldots$$

Fixpont, egyensúlyi állapot x_{fix}:

$$f(x_{fix}) = x_{fix}.$$

Esetünkben

$$f(x_{fix}) = ax_{fix} + b = x_{fix} \quad \Longrightarrow \quad x_{fix} = \frac{b}{1-a}.$$

Vezessünk be uj változót: $\Delta x = x - x_{fix}$. Ekkor

$$\Delta x_{n+1} = a\Delta x_n,$$

vagyis megszabadultunk az inhomogén tagtól. Tehát

$$x_n = a^n(x_0 - x_{fix}) + x_{fix}.$$

Problema 11. *Beteszek a bankba 777000 Forintot. Kapok 5% kamatot, de évente levonnak 300 Forint kezelései költséget. Mennyi pénzem lesz n év után?*

Megoldas 11. *Fixpont:* $(1 + 0.05) \cdot x_{fix} - 300 = x_{fix}, \quad \Rightarrow \quad x_{fix} = 6000.$ *Tehát 6000 forint nem kamatozik semmit, csak éppen fedezi a kezelési költséget. Viszont az ezen felüli összegre megkapom az 5% kamatot, tehát*

$$x_n = 1.05^n \cdot (777000 - 6000) + 6000.$$

Homogén koordináták: Egy inhomogén lineáris (más szóval affin) transzformáció elcserélhető egy homogén lineáris transzformációra eggyel öobb dimenzióban.

$$x \to ax + b; \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ 1 \end{pmatrix} \to \begin{pmatrix} ax+b \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = A \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Tehát az a kérdés, hogy mennyi

$$A^n \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Ez a kérdés megválaszloható A sajátvektorainak es sajátértékeinek ismeretében. Most csak az elfajult $a = 1$ esetet kezeljük:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^n = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right]^n$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^n + \binom{n}{1} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \binom{n}{2} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 + \cdots = \begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix},$$

mivel

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 = 0.$$

Itt a binomiális tétel

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \cdots + y^n$$

azért volt alkalmazható a

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right]^n$$

kifejezésre, mert a két tag a zárójelben kommutált (felcserélhetőek a mátrixszorzás szempontjából) egymással. Tehat

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_0 + nb \\ 1 \end{pmatrix}$$

3.4.2 Az Euler módszer hibája

Az Euler módszer hibája két forrásból származik. Legyen

$$y'(t) = f(t, y(t)),\ y(0) = y_0,$$
$$y_{n+1} = y_n + f(n \cdot \Delta t, y_n)\Delta t,$$
$$hiba_n = |y_n - y(n \cdot \Delta t)|$$

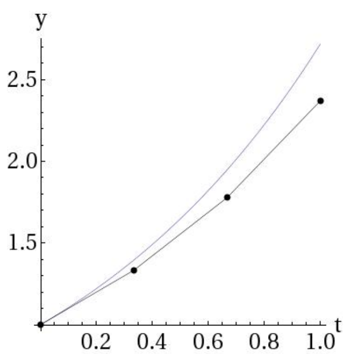


Figure 3.4: Az $y(t)' = y(t),\ y(0) = 1$ DE közelítő és pontos megoldása, ahol $\Delta t = 1/3$.

$t = 0..1/3$ között a hibát a lineáris közelítés hibája okozza. Viszont utána, pl. $t = 1/3$-kor már eleve rossz helyen számoljuk ki a sebességet, hiszen $f(1/3, y_1)$-ot használjuk a pontos $f(1/3, y(1/3))$ értek helyett.

Tételezzük fel, hogy

- $|y''(t)| \leq K$,
- $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ (Lipschitz-feltétel)

valamilyen K, L konstansokra. Ekkor

$$hiba_{n+1} \leq hiba_n + \frac{K}{2}\Delta t^2 + (L \cdot hiba_n)\,\Delta t, \tag{3.1}$$

hiszen a lineáris közelítés maximum $K/2 \cdot \Delta t^2$ hibát okozhat, míg az f sebesség értékében a hiba maximum $L \cdot hiba_n$ lehet a $t = n\Delta t$ pillanantban. A legrosszabb esetben egyenlőség áll fenn (4.1)-ben. Legyen

$$h_0 = 0, \quad h_{n+1} = h_n + \frac{K}{2}\Delta t^2 + (L \cdot hiba_n)\,\Delta t,$$

ekkor

$$\begin{aligned} h_n &= (1 + L\Delta t)^n \left(0 - \frac{K\Delta t^2/2}{1 - (1 + L\Delta t)}\right) + \frac{K\Delta t^2/2}{1 - (1 + L\Delta t)} \\ &= \left[(1 + L\Delta t)^n - 1\right] \frac{K\Delta t}{2L}. \end{aligned}$$

h_n felső korlátot ad az $|y_n - y(n\Delta t)|$ hibára.

Nézzük meg, hogy viselkedik $h_{T/\Delta t}$, ha $\Delta t \to 0$, vagyis mennyi a felső korlátunk $y(T)$ numerikus approximációjának a hibájárá.

$$\left[(1 + L\Delta t)^{T/\Delta t} - 1\right] \frac{K\Delta t}{2L} \approx \left(e^{LT} - 1\right) \frac{K\Delta t}{2L}, \quad \text{ha} \quad \Delta t \approx 0.$$

Ha T viszonylag kicsi, ez a kifejezés nagyjából $KT\Delta t/2$, vagyis a hiba a T idővel aáanyosan növekedhet, mivel a lineáris approximáció egyes lépésekben elkövetett hibája felhalmozodik. Ha T elég nagy, akkor az e^{LT} faktor dominál, vagyis a hiba exponenciálisan növekedhet. Itt már a hibát nem annyira a lin.approx. hibája okozza, mint az, hogy eleve rossz helyen számoljuk ki f értéket.

3.4.3 Heun modszer

Adott $y(t_0) = y_0, \;\; y'(t) = f(t, y)$, ekkor az Euler módszer jóslata:

$$y(t_0 + \Delta t) = y_0 + f(t_0, y_0)\Delta t.$$

Itt feltettük, hogy a sebesség a $[t_0, t_0 + \Delta t]$ intervallumon végig $f(t_0, y_0)$. Ennél jobb lenne, ha a sebességnek az intervallum két végponjaban mért értékeinek az átlagát használnánk. Sajnos nem tudjuk pontosan, hogy mennyi $y'(t_0 + \Delta t, y(t_0 + \Delta t))$, viszont használhatjuk az Euler módszer jóslatát $y(t_0 + \Delta t)$ kiszámítására:

$$\begin{aligned} k_1 &= f(y_0, t_0), \\ k_2 &= f(t_0 + \Delta t, y_0 + f(t_0, y_0)\Delta t), \\ y(t_0 + \Delta t) &= y_0 + \frac{1}{2}(k_1 + k_2)\Delta t. \end{aligned}$$

Problema 12. *Alkalmazd az Euler, illetve a Heun módszert a következő DE-ekre $\Delta x = 0.1$ lépésöozzel az $y(2) = 3$ kezdeti feltétel mellett!*

$$a)\;\; y' = f(x, y) = x - y; \qquad b)\;\; y' = x - y^2;$$

Végezd el ugyanezt az

$$c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}; \qquad d)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ y_1^2 + x \end{pmatrix}$$

egyenletekre az $\begin{pmatrix} y_1(2) \\ y_2(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *kezdeti feltétel mellett!*

Mit jósolnak ezek a módszerek y(2.1)-re?

Megoldas 12.

$$a) \quad Euler: \quad y(2.1) \approx y(2) + (2-3)\cdot 0.1 = 3 + (-1)\cdot 0.1 = 2.9,$$
$$Heun: \quad k_1 = f(2,3) = 2 - 3 = -1, \quad k_2 = f(2.1, 2.9) = 2.1 - 2.9 = -0.8,$$
$$y(2.1) \approx y(2) + \frac{1}{2}(k_1 + k_2)\cdot 0.1 = 3 + \frac{1}{2}(-1 - 0.8)\cdot 0.1.$$

Részletesebb diszkusszió: Kollár: Numerikus módszerek

3.4.4 A megoldás Taylor sora

Az Euler módszer az

$$y(t + \Delta t) = y(t) + y'(t)\Delta t = y(t) + f(t, y(t))\Delta t$$

becslést használja, ami nem más mint az elsőrendű Taylor közelítése y-nak t körűl. Magasabbrendű közelítések nyilván pontosabb eredményt adnának.

$$y(t + \Delta t) = y(0) + y'(0)\Delta t + \frac{y''(0)}{2!}\Delta t^2 + \dots$$

Egy kezdetiértek $y(0) = y_0$ feladatban:

$$y(0) = y_0, \qquad y'(0) = f(0, y_0),$$

viszont mennyi $y''(0)$? Általánossaban, a kétváltozós $G(t, y)$ függvenyre

$$\frac{d}{dt}G(t, y(t)) = \left[\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right) G\right](t, y(t)),$$

tehát

$$y'' = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right) f,$$
$$y''' = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right) y'' = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right)^2 f,$$
$$y^{(n)} = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right)^{n-1} f.$$

Problema 13.

$$a)\ y' = f(t, y) = t - y; \qquad b)\ y' = f(t, y) = y^2 + yt;$$

Mennyi y'' *es* y''' *? Írd fel y harmadrendű Taylor polinomját az* $t = 1$ *pont körűl, ha* $y(1) = 5$ *!*

Megoldas 13. *a)*

$$y(1) = 5, \;\; y'(1) = 1 - 5 = -4,$$
$$y''(t) = (t-y)'_t + (t-y)\cdot(t-y)'_y = 1 - t + y, \;\; y''(1) = 5,$$
$$y'''(t) = (1-t+y)'_t + (t-y)\cdot(1-t+y)'_y = -1 + t - y, \;\; y'''(1) = -5,$$
$$y(1+\Delta t) \approx 5 - 4\Delta t + \frac{5}{2!}\Delta t^2 + \frac{-5}{3!}\Delta t^3$$

b)

$$y(1) = 5, \;\; y'(1) = 30,$$
$$y''(t) = y(t+y)(t+2y) + y, \;\; y''(1) = 335,$$
$$y'''(t) = y\left(t^3 + 7t^2y + 3t\left(4y^2+1\right) + 6y^3 + 4y\right), \;\; y'''(1) = 5545,$$
$$y(1+\Delta t) \approx 5 + 30\Delta t + \frac{335}{2!}\Delta t^2 + \frac{5545}{3!}\Delta t^3$$

Sajnos az magasabbrendű $y^{(n)}$ deriváltak tipikúsan egyre bonyulultabbak lesznek, így sok esetben ez nem egy praktikus módszer numerikus számításokhoz.

3.4.5 Megoldas léteéese, egyértelműsége

Tetel 1. *Legyen adott az $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ diff.egy., ahol a kétvéltozós f függvény folytonos. Ekkor létezik olyan $\delta > 0$ szám, es olyan, a $(t_0 - \delta, t_0 + \delta)$ intevallumon értelmezett y függvény, ami megoldja a differencialegyenletünket.*

1. Nincs mindig globális, bármely t-re értelmezett megoldás:

   $$y'(t) = y^2(t), \; y(0) = 5, \quad \Longrightarrow \quad y(t) = -\frac{1}{t - 1/5},$$

   de a kapott függvény csak az $(-\infty, 1/5)$ intervallumon oldja meg a DE-t. Itt a Lipschitz-feléetel a $y(0) = 5$ kezdeti feltételnek csak egy véges környezetében teljesűl.

2. A megoláas nem feltetlenűl egyértelmű: Az $y' = 3\sqrt[3]{y^2}$ DE-t megoldáak az $y(t) = (t-C)^3$ függvények, de szintén megoldás pl. a következő $\tilde{y}$ függvény:

   $$\tilde{y} = \begin{cases} t^3 & \text{ha} \;\; t < 0 \\ 0 & \text{ha} \;\; 0 \le t < 2 \\ (t-2)^3 & \text{ha} \;\; 2 \le t \end{cases}$$

   (A 2 helyére itt bármilyen pozitív sáamot írhatnánk.) Itt a Lipschitz-feltétel a $y(t_0) = 0$ kezdeti feltételnek semmilyen véges környezetében sem teljesűl.

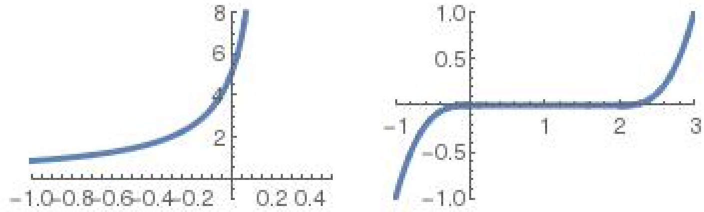


Figure 3.5: a) Az $y' = y^2$, $y(0) = 5$ DE megoldása csak $t = 0.2$-ig van értelmezve. b) A $y' = 3\sqrt[3]{y^2}$, $y(-1) = -1$ DE megoldása nem egyertelmű.

Problema 14. *Oldd meg a kovetkező DE-ket az $y(0) = 1$ kezdeti feltétel mellett, vizsgáld meg a megoldások egyertelműségét, illetve határozd meg, hogy milyen intervalumon vannak azok értelmezve! Magyarázd meg a felmerülő problémákat!*

$$a)\ \ y' = y, \qquad b)\ \ y' = y^4, \qquad c)\ \ y' = 2\sqrt{|y|},$$

Megoldas 14.

$$\begin{aligned}
&a: && y = e^t,\ t \in (-\infty, \infty),\\
&b: && y = \frac{1}{\sqrt[3]{1-3t}},\ t \in (-\infty, 1/3),\\
&c: && y = \begin{cases} (t+1)^2, & ha\ t > -1,\\ 0, & ha\ C \le t \le -1\\ -(t+C)^2, & ha\ t < C \end{cases}
\end{aligned}$$

Chapter 4

Ordinary DE, numerical solution, geometric interpretation

Evolutionary DE:

4.1 Geometry

$$y'(t) = f(y(t), t)), \quad \frac{d}{dt}y = f(y, t), \quad y' = f(y, t).$$

For example

$$y'(t) = 2t + y(t)$$

If the solution curve $\bar{r}(t) = (t, y(t))$ passes through $(t, y) = (2, 3)$ point, then the slope of y at that point $y' = 2 \cdot 2 + 3 = 7$, so the tangent vector of $\bar{r}(t)$ at $(t, y) = (2, 3)$ is $v = (1, 2 \cdot 2 + 3) = (1, 7)$.

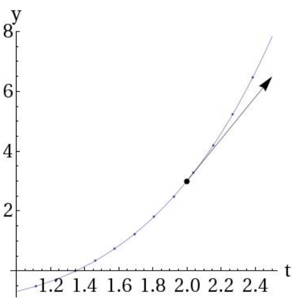


Figure 4.1: Solution of $y' = 2t + y(t)\, y(2) = 3$. (The tangent vector at $(2, 3)$ is scaled (halved) by 0.5.)

The tangent vectors of a of the curves $t \to \bar{r}(t)$ generate the vector field $(t, y) \to (1, f(t, y))$, the velocity field of the DE.

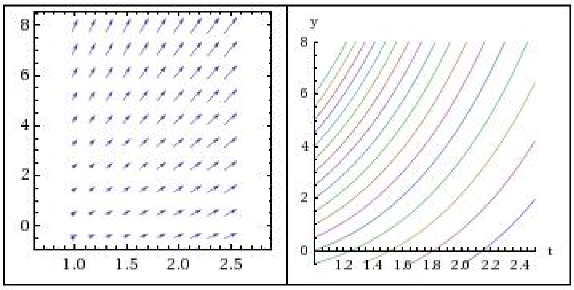


Figure 4.2: The velocity field and the solution curves of the DE $y' = 2t + y(t)$.

4.2 Higher order DE

A higher order DE can be transformed to a first order one, at least if the highest order derivative can be expressed with the lower order terms (quasilinear DE).

Problema 15. *Rewrite the following DE to first order systems!*

$$a)\ \ y'' = -y' - 2y; \qquad b)\ \ y''' = y + x; \qquad c)\ \ \frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' - y_2 \\ y_2' y_1 \end{pmatrix}$$

Megoldas 15.

$$a)\ \ \frac{d}{dx}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -v - 2y \end{pmatrix}; \qquad b)\ \ \frac{d}{dx}\begin{pmatrix} y \\ v \\ a \end{pmatrix} = \begin{pmatrix} v \\ a \\ y + x \end{pmatrix};$$

$$c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 - y_2 \\ v_2 \\ v_2 y_1 \end{pmatrix}$$

One can get rid of the explicit time dependence:

Problema 16. *Transform the following DE to their time-independent form!*

$$a)\ \ y' = xy^2 + x; \qquad b)\ \ y' = x - y; \qquad c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} xy_1 + y_2 \\ y_1 y_2 + x \end{pmatrix}$$

Megoldas 16.

$$a)\ \ \frac{d}{dx}\begin{pmatrix} y \\ s \end{pmatrix} = \begin{pmatrix} sy^2 + s \\ 1 \end{pmatrix}; \qquad b)\ \ \frac{d}{dx}\begin{pmatrix} y \\ s \end{pmatrix} = \begin{pmatrix} s - y \\ 1 \end{pmatrix}$$

$$c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ s \end{pmatrix} = \begin{pmatrix} sy_1 + y_2 \\ y_1 y_2 + s \\ 1 \end{pmatrix}$$

4.3 Numerical solution

4.3.1 Euler (polygon) method

Let $y(t_0) = y_0$, $y'(t) = f(y, t)$, then

$$y(t_0 + \Delta t) \approx y_0 + f(y_0, t_0)\Delta t.$$

Pseudocode:

$$
\begin{aligned}
&input\ \ y_0, t_0, \Delta t, f, T \\
&\quad y_{old} \leftarrow y_0 \\
&\quad t \leftarrow t_0 \\
&repeat \\
&\quad y_{new} \leftarrow y_{old} + f(y_{old}, t)\Delta t \\
&\quad y_{old} \leftarrow y_{new} \\
&\quad t \leftarrow t + \Delta t \\
&\quad if (t > T)\ then\ break \\
&end\ loop \\
&output\ \ \text{"}y(T) = \text{"}\ \ y_{old}
\end{aligned}
$$

For example $y' = y/2,\ y(0) = 1$. How much is $y(0.5)$?

t	y	y'	y-pontos	hiba
0.	1.	0.5	1.	0.
0.1	1.05	0.525	1.05127	-0.0012711
0.2	1.1025	0.55125	1.10517	-0.00267092
0.3	1.15763	0.578813	1.16183	-0.00420924
0.4	1.21551	0.607753	1.2214	-0.00589651
0.5	1.27628	0.638141	1.28403	-0.00774385

Here $y(0.3) = y(0.2) + 0.1 \cdot y(0.2)/2 = 1.1025 + 0.1 \cdot 1.1025/2 = 1.15763$.

Graphically: $y' = y,\ y(0) = 1,\ \Delta t = 1/3$.

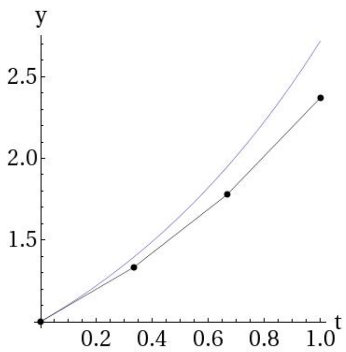


Figure 4.3: The exact and approximate solutions of $y(t)' = y(t),\ y(0) = 1$, where $\Delta t = 1/3$.

Problema 17. *Apply the Euler method to $y'(t) = 2y(t)$, $y(0) = 4$!*

Megoldas 17. *Let $y_n = y(n\Delta t)$, $y_0 = y(0) = 4$,*

$$y_{n+1} = y(n\Delta t + \Delta t) = y(n\Delta t) + 2y(n\Delta t)\Delta t = (1 + 2\Delta t) \cdot y_n,$$

so $y_n = 4 \cdot (1 + 2\Delta t)^n y_0$.

If $\Delta t \to 0$, then

$$y(T) \approx (1 + 2\Delta t)^{T/\Delta t} y(0) \quad \to \quad e^{2T} \cdot 4,$$

so we reproduced the exact $y(t) = 4 \cdot e^{2t}$ solution.

Problema 18. *Apply the Euler method for the $y'(t) = 3y(t) - 6$, $y(0) = 77$ DE!*

Megoldas 18. *Let $y_n = y(n\Delta t)$, $y_0 = y(0) = 77$,*

$$\begin{aligned} y_{n+1} &= y(n\Delta t + \Delta t) = y(n\Delta t) + (3y(n\Delta t) - 6)\Delta t \\ &= (1 + 3\Delta t)y_n - 6\Delta t, \end{aligned}$$

so if we can compute the general member of the

$$x_{n+1} = ax_n + b$$

recursive sequence, then we can solve the problem.

4.3.2 Inhomogeneous geometric sequences

1. Arithmetic sequence:

$$x_{n+1} = x_n + d, \quad \Longrightarrow \quad x_n = x_0 + nd.$$

   Geometeric sequence:

$$x_{n+1} = qx_n, \quad \Longrightarrow \quad x_n = q^n x_0.$$

   We are in the mixed case: $x_{n+1} = qx_n + d$. We will discuss it in the next subsection.

4.4 Error of the Euler method, inhomogeneous geometric sequences

4.4.1 Inhomogen geometriai sorozatok

Let $x_{n+1} = ax_n + b = f(x_n)$. If x_0 is given , how much is x_n?

Linearization around the fixed point: Take the following discrete time dynamical system:

$$x_0,\, x_1 = f(x_0),\, x_2 = f(x_1) = f(f(x_0)) = f^2(x_0),\, \dots, x_{n+1} = f(x_n), \dots$$

Fixed point, equilibrium or steady state x_{fix}:

$$f(x_{fix}) = x_{fix}.$$

In our case

$$f(x_{fix}) = ax_{fix} + b = x_{fix} \quad \Longrightarrow \quad x_{fix} = \frac{b}{1-a}.$$

Introduce the new variable: $\Delta x = x - x_{fix}$. then

$$\Delta x_{n+1} = a\Delta x_n,$$

so we managed to get rid of the inhomogeneous term. So

$$x_n = a^n(x_0 - x_{fix}) + x_{fix}.$$

Problema 19. *I deposit 777000 EUR to my bank account. The interest rate is 5%, but the yearly fee of the account is 300 EUR. What is my balance after n years?*

Megoldas 19. *Fixed point:* $(1 + 0.05) \cdot x_{fix} - 300 = x_{fix}, \quad \Rightarrow \quad x_{fix} = 6000.$ *So 6000 EUR yields nothing, it just covers the yearly fee. For the surplus over 6000 EUR I got the 5% interest:*

$$x_n = 1.05^n \cdot (777000 - 6000) + 6000.$$

Homogeneous coordinates: An inhomogeneous linear (affine) transformation can be trade for a linear one:

$$x \to ax + b; \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ 1 \end{pmatrix} \to \begin{pmatrix} ax+b \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = A \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

So how much is

$$A^n \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

This question can be answered if the eigenvalues and eigenvectors of A are known. Here we treat only the degenerate $a = 1$ case.

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^n = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right]^n$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^n + \binom{n}{1} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \binom{n}{2} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 + \dots = \begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix},$$

since

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 = 0.$$

Here the binomial theorem

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \cdots + y^n$$

can be applied to the expression

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right]^n,$$

since the two terms in the parentheses

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_0 + nb \\ 1 \end{pmatrix}$$

are commuting with each other.

4.4.2 Error of the Euler method

The error has two sources. Let

$$\begin{aligned} y'(t) &= f(t, y(t)),\ y(0) = y_0, \\ y_{n+1} &= y_n + f(n \cdot \Delta t, y_n)\Delta t, \\ hiba_n &= |y_n - y(n \cdot \Delta t)| \end{aligned}$$

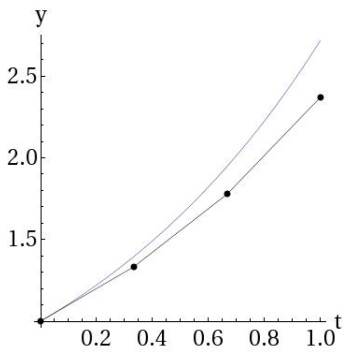


Figure 4.4: The exact and approximate solutions of $y(t)' = y(t),\ y(0) = 1$, where $\Delta t = 1/3$.

In the interval $t = 0..1/3$ the error is caused by the linear approximation. Then, say at $t = 1/3$ we compute the velocity at a wrong place, since we use $f(1/3, y_1)$ instead of $f(1/3, y(1/3))$.

Assume that

- $|y''(t)| \leq K$,
- $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ (Lipschitz-condition)

for some K, L constants. Then

$$hiba_{n+1} \leq hiba_n + \frac{K}{2}\Delta t^2 + (L \cdot hiba_n)\,\Delta t, \tag{4.1}$$

since the error of the linear approximation can not be more than $K/2 \cdot \Delta t^2$, while the maximal possible error of f is less or equal to $L \cdot hiba_n$ at $t = n\Delta t$. In the worst case there is an equality in (4.1). Let

$$h_0 = 0, \quad h_{n+1} = h_n + \frac{K}{2}\Delta t^2 + (L \cdot hiba_n)\,\Delta t,$$

then

$$\begin{aligned} h_n &= (1 + L\Delta t)^n \left(0 - \frac{K\Delta t^2/2}{1-(1+L\Delta t)}\right) + \frac{K\Delta t^2/2}{1-(1+L\Delta t)} \\ &= \left[(1+L\Delta t)^n - 1\right]\frac{K\Delta t}{2L}. \end{aligned}$$

h_n is an upper bound of the $|y_n - y(n\Delta t)|$ error.

Let us study the behaviour of $h_{T/\Delta t}$ when $\Delta t \to 0$.

$$\left[(1+L\Delta t)^{T/\Delta t} - 1\right]\frac{K\Delta t}{2L} \approx \left(e^{LT} - 1\right)\frac{K\Delta t}{2L}, \quad \text{ha} \quad \Delta t \approx 0.$$

If T is small, then this expression is about $KT\Delta t/2$, so the error can increase proportionally to T, due to the accumulation of the errors of the linear approximations. For large T the exponential e^{LT}

4.4.3 Heun method

Let $y(t_0) = y_0$, $y'(t) = f(t, y)$. Then the Euler method predicts:

$$y(t_0 + \Delta t) = y_0 + f(t_0, y_0)\Delta t.$$

Here we assumed that in the interval $[t_0, t_0+\Delta t]$ the velocity is $f(t_0, y_0)$. It would be better to use the average of the velocities at the endpoints, unfortunately we do not know exactly $y'(t_0 + \Delta t, y(t_0 + \Delta t))$, since $y(t_0 + \Delta t)$ is unknown. However we can use the prediction of the Euler method for $y(t_0 + \Delta t)$:

$$\begin{aligned} k_1 &= f(y_0, t_0), \\ k_2 &= f(t_0 + \Delta t, y_0 + f(t_0, y_0)\Delta t), \\ y(t_0 + \Delta t) &= y_0 + \frac{1}{2}(k_1 + k_2)\Delta t. \end{aligned}$$

Problema 20. *Apply the Euler and Heun methods with timestep $\Delta x = 0.1$ and initial condition $y(2) = 3$!*

$$a)\ y' = f(x, y) = x - y; \qquad b)\ y' = x - y^2;$$

Do the same for

$$c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}; \qquad d)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ y_1^2 + x \end{pmatrix}$$

with initial condition $\begin{pmatrix} y_1(2) \\ y_2(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *What are the predictions for $y(2.1)$-re?*

Megoldas 20.

$$a) \quad Euler: \quad y(2.1) \approx y(2) + (2-3)\cdot 0.1 = 3 + (-1)\cdot 0.1 = 2.9,$$
$$Heun: \quad k_1 = f(2,3) = 2-3 = -1, \quad k_2 = f(2.1, 2.9) = 2.1 - 2.9 = -0.8,$$
$$y(2.1) \approx y(2) + \frac{1}{2}(k_1 + k_2)\cdot 0.1 = 3 + \frac{1}{2}(-1-0.8)\cdot 0.1.$$

4.4.4 On the existence and uniqueness of the solution

Tetel 2. *Let $y'(t) = f(t, y(t))$, $y(t_0) = y_0$, where f is continuous . Then there exists a constant $\delta > 0$ and a function y defined on $(t_0 - \delta, t_0 + \delta)$, such that y solves the DE.*

1. It is possible that there is no global (i.e. defined for all t) solution:

$$y'(t) = y^2(t), \; y(0) = 5, \quad \Longrightarrow \quad y(t) = -\frac{1}{t - 1/5},$$

   here y solves the DE only on the interval $(-\infty, 1/5)$. The Lipschitz condition is satisfied only in finite neighbourhoods of the initial condition $y(0) = 5$.

2. The solution is not necessarily unique: The $y' = 3\sqrt[3]{y^2}$ DE is solved by the functions $y(t) = (t - C)^3$, but it is also solved by $\tilde{y}$:

$$\tilde{y} = \begin{cases} t^3 & \text{ha} \quad t < 0 \\ 0 & \text{ha} \quad 0 \le t < 33 \\ (t-33)^3 & \text{ha} \quad 33 \le t \end{cases}$$

   Here the Lipschitz condition is not satisfied in any neighborhood of $y(t_0) = 0$.

Problema 21. *Solve the following DE with initial condition $y(0) = 1$ and study the existence and uniqueness of the solutions!*

$$a) \; y' = y, \qquad b) \; y' = y^4, \qquad c) \; y' = 2\sqrt{|y|},$$

Megoldas 21.

$$a: \quad y = e^t, \; t \in (-\infty, \infty),$$
$$b: \quad y = \frac{1}{\sqrt[3]{1-3t}}, \; t \in (-\infty, 1/3),$$
$$c: \quad y = \begin{cases} (t+1)^2, & ha \; t > -1, \\ 0, & ha \; C \le t \le -1 \\ -(t+C)^2, & ha \; t < C \end{cases}$$

Chapter 5

Linearizáció

Példa: legyen adott az következő időfüggetlen DE:

$$\frac{d}{dt}y = f(y) = 3y(1-y) = 3y - 3y^2.$$

Ha a kezdeti feltétel $y(0) = 0$, akkor a megoldás $y(t) = 0 = konst.$, vagyis $y_{fix} = 0$ egyensúlyi állapot, fixpont. Ha $y \approx 0$, akkor $f(y) \approx 3y$ (mivel ekkor $|-3y^2| \ll |3y|$) , vagyis közelítőleg

$$\frac{d}{dt}y \approx 3 \cdot y,$$

ahol $3 = \frac{df(y)}{dy}_{|y=0}$. Tehát a fixpont körűl az eredeti nemlineáris DE-t elcserélhetjük egy homogén lineáris DE-re!

5.1 Egydimenzio, kvalitativ viselkedes

5.1.1 Linearizacio a fixpont korul

Legyen adott az $\frac{d}{dt}y = f(y)$ idofuggetlen DE. Legyen y_{fix} a DE-hez tartozo dinamikai rendszer fixpontja, egyensulyi allapota, vagyis legyen $f(y_{fix}) = 0$. Vezessuk be az uj $\Delta y = y - y_{fix}$ uj valtozot. Ekkor $\frac{dy}{dt} = \frac{d\Delta y}{dt}$, tehat

$$\frac{d}{dt}\Delta y = \frac{dy}{dt} = f(y_{fix} + \Delta y) \approx f(y_{fix}) + f'(y_{fix})\Delta y = f'(y_{fix})\Delta y.$$

A

$$\frac{d}{dt}\Delta y = f'(y_{fix})\Delta y$$

homogen linearis DE az eredeti nemlinearis DE linearizacioja az y_{fix} fixpont korul. (Itt $f' = \frac{df(y)}{dy}$.)

5.1.2 Kvalitativ viselkedes

Problema 22. *Rajzold le az $y' = f(y)$ DE iranymezojet es a megoldasgorbeit! Keresd meg a DE fixpontjait es ird fel a fixponttol valo elteresre vonatkozo DE*

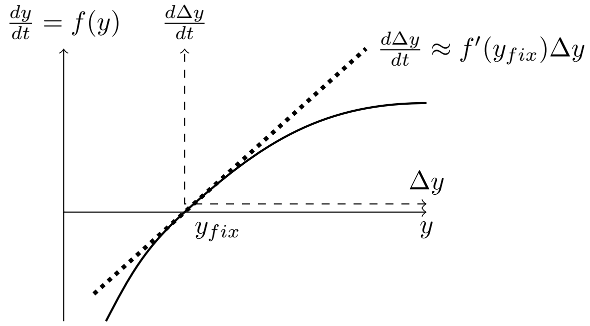


Figure 5.1: Az uj $\Delta y = y - y_{fix}$ koordinatarendszerben az eredeti $\frac{dy}{dt} = f(y)$ DE-t a linearizalt $\frac{d\Delta y}{dt} = f'(y_{fix})\Delta y$ verzioval kozelithetjuk.

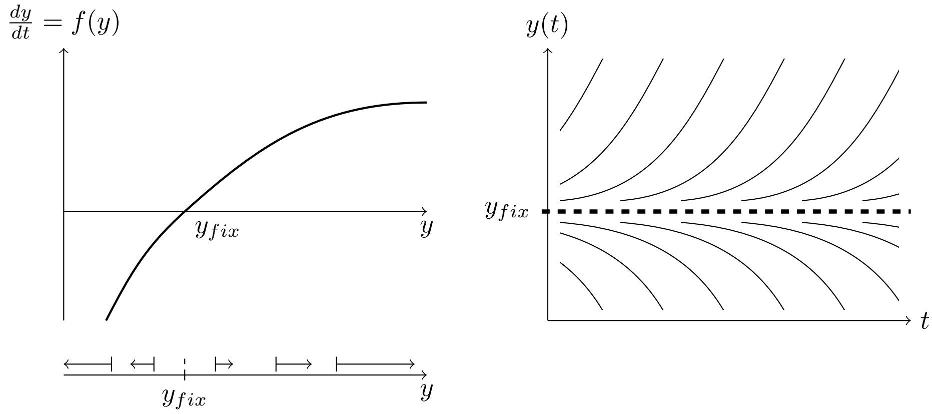


Figure 5.2: Az idofuggetlen $\frac{dy}{dt} = f(y)$ DE vektormezoje es megoldasgorbei.

linearizalt alakjat! Vizsgald meg a fixpontok stabilitasat!

$$a)\ y' = 1, \qquad b)\ y' = y, \qquad c)\ y' = -y, \qquad d)\ y' = y + 1,$$
$$e)\ y' = -1 + y^2, \qquad f)\ y' = y(1-y), \qquad g)\ y' = y(1-y)(1+y).$$

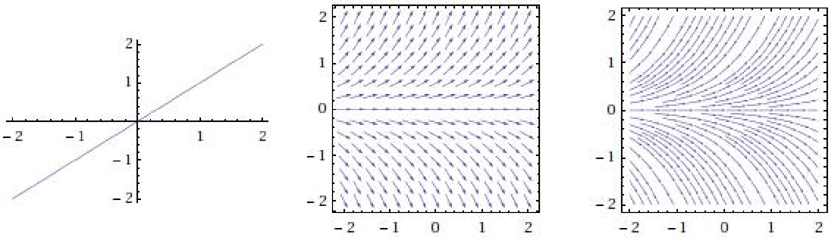


Figure 5.3: b) $y' = y$

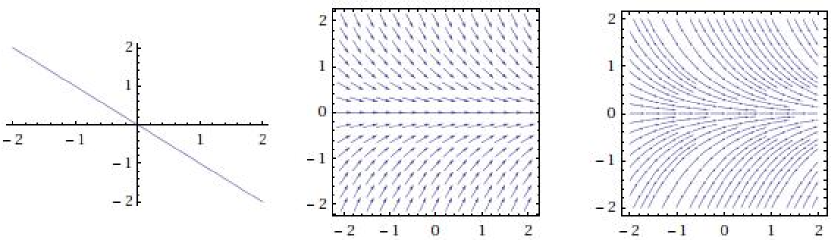


Figure 5.4: c) $y' = -y$

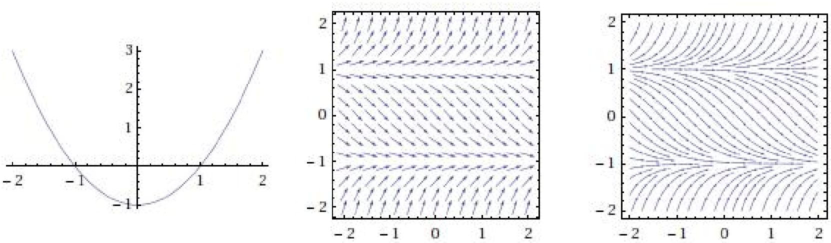


Figure 5.5: e) $y' = -1 + y^2$

Megoldas 22. .

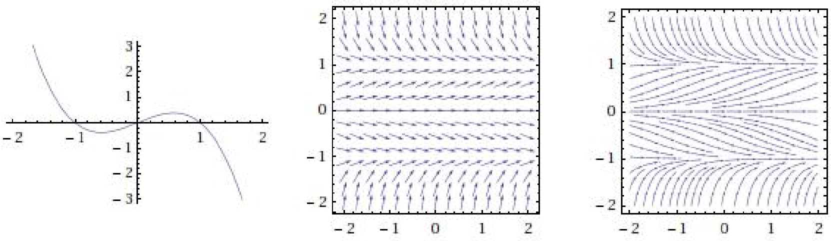


Figure 5.6: g) $y' = y(1-y)(1+y)$

g)

$$\frac{d}{dt}y = f(y) = y(1-y)(1+y) = +y - y^3.$$

$f'(y) = \frac{d}{dy}f(y) = 1 - 3y^2$. A fixpontok (vagyis $f(y)$ gyokei):

$$\begin{aligned} y_1 = 0, \quad & f'(y_1) &= f'(0) = 1 - 3 \cdot 0^2 = 1 > 0, \\ y_2 = 1, \quad & f'(1) &= -2 < 0, \\ y_3 = -1, \quad & f'(-1) &= -2 < 0. \end{aligned}$$

f' elojele alapjan az y_1, y_2, y_3 fixpontok stabilitasa: instabil, stabil, stabil.

A linearizalt egyenletek a fixpontok korul:

$$\begin{aligned} \frac{d}{dt}(y-0) =& \frac{d}{dx}\Delta y_1 = 1 \cdot \Delta y_1, \\ \frac{d}{dt}(y-1) =& \frac{d}{dx}\Delta y_2 = -2 \cdot \Delta y_2, \\ \frac{d}{dt}(y-(-1)) =& \frac{d}{dx}\Delta y_3 = -2 \cdot \Delta y_3, \end{aligned}$$

Legyen $y(t)$ az $y(0) = 0.7$, $\frac{d}{dt}y = f(y)$ kezdetiertek problema megoldasa. Ekkor

$$\lim_{t\to\infty} y(t) = 1, \qquad \lim_{t\to-\infty} y(t) = 0.$$

Ha a kezdeti feltetel $y(0) = -0.7$, akkor pedig

$$\lim_{t\to\infty} y(t) = -1, \qquad \lim_{t\to-\infty} y(t) = 0.$$

Reszletesebben errol a temarol: Karsai: DE modellek

5.2 Tobbdimenzio

5.2.1 Kvalitative viselkedes ket dimenzioban

Tegyuk fel, hogy egy sima idofuggetlen DE-nek a $P = \bar{y}(0)$ pontbol kiindulo megoldasa egy korlatos regioban marad. Ekkor az $t \to \bar{y}(t)$ trajektoria haromfelekeppen viselkedhet:

- Ha P fixpontja a DE-nek, akkor a trajektoria az egyetlen P pontbol all.

- A trajektoria egy mashol levo fixponthoz konvergal.

- A trajektoria egy hatarciklushoz konvergal. (Vagyis letezik olyan periodikus nem konstans $\bar{\gamma} : t \to \mathbb{R}^2$ fuggveny, hogy $\lim_{t\to\infty} |\bar{y}(t) - \bar{\gamma}(t)| = 0$.

Tobbe-kevesse ez a tartalma a Poincare-Bendixson tetelnek.

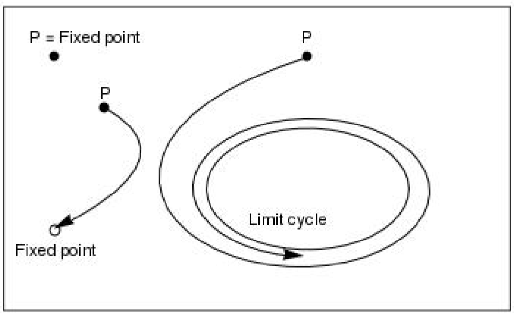


Figure 5.7: A Poincare-Bendixson tetel illusztracioja.

5.2.2 Kvalitative viselkedes magasabb dimenzioban

Kaosz: Itt mar nagyon bonyolult, kaotikus viselkedes is elofordulhat. Ennek talan legegyszerubb peldaja a haromdimenzios Lorenz-egyenlet.

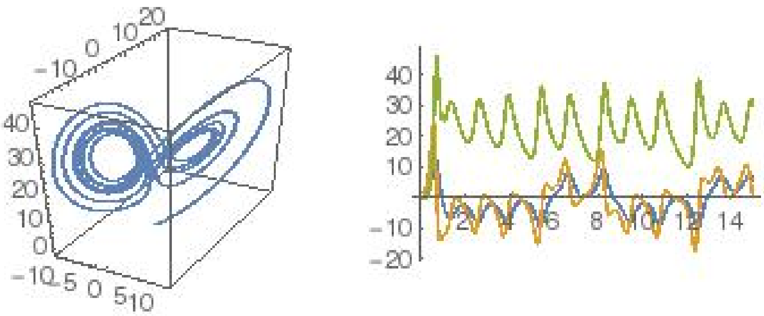


Figure 5.8: A kovetkezo kezdetiertek problema megoldasa: $x' = -3(x-y), y' = -xz + 26.5x - y, z' = xy - z, x(0) = z(0) = 0, y(0) = 1$. Az elso abra a $\bar{\gamma}(t) = (x(t), y(t), z(t))^T$ trajektoria abrazolasa, mig a masodik lerajzolja az x, y, z komponensek idofuggeset.

Hoszzutavon a megoldas kaotikusan oszcillal az elso abra "nyolcasa" ket fele kozott. Hogyan tudnank jellemezni a megoldas kaotikussagat? Ennek egyik meroszama a (maximalis) Lyapunov exponens.

Legyen a $\frac{d}{dt}\bar{y} = \bar{f}(\bar{y}),\ \bar{y}(0) = \bar{y}_0$ DE partikularis megoldasa $\bar{Y}(t, \bar{y}_0)$. Ekkor a maximalis Lyapunov exponens definicioja:

$$\lambda = \lim_{t\to\infty} \max \lim_{\overline{\delta y}(0)\to\bar{0}} \frac{1}{t} \ln \frac{|\bar{Y}(t,\bar{y}_0) - \bar{Y}(t,\bar{y}_0 + \overline{\delta y}(0))|}{|\overline{\delta y}(0)|}.$$

Ennek a kisse komplikalt kifejezes motivacioja az, hogy ha $f(y) = ay$, akkor ennek a kifejezesnek az erteke pontosan a. Vagyis a Lyapunov exponens felugyeli a kozeli trajektoriak exponencialis tavolodasat. Ha $y' = ay$, akkor

$$\begin{aligned}
y(0) = y_0 \quad &\Longrightarrow \quad y(t) = e^{at}y(0),\\
\tilde{y}(0) = y_0 + \delta y_0 \quad &\Longrightarrow \quad y(t) = e^{at}(y(0) + \delta y_0),\\
|\tilde{y}(t) - y(t)| &= e^{at}|\delta y_0|,\\
\lim_{t\to\infty} \lim_{\delta y(0)\to 0} \frac{1}{t} \ln \frac{|e^{at}\delta y_0|}{|\delta y_0|} &= a.
\end{aligned}$$

Oldjuk meg a Lorenz-egyenletet az eredeti $(x, y, z)^T(0) = (0, 1, 0)$, illetve a $(x, y, z)^T(0) = (0, 1, 10^{-6})$ kezdeti feltetelek mellett. Abrazoljuk a ket trajektoria tavolsagat es ennek a logaritmusat t fuggvenyeben!

Problema 23. *Tegyuk fel, hogy erre a Lorenz egyenletre a kezdeti erteket* 10^{-11} *pontossaggal ismerjuk. Korulbelul milyen* t *ertekekre tudunk megbizhato elortejelzest adni?*

Problema 24. *A Lyapunov exponens fugghet a kezdeti ertektol, nem csak a differencialegyenlettol. Van-e ilyen fugges esetunkben?*

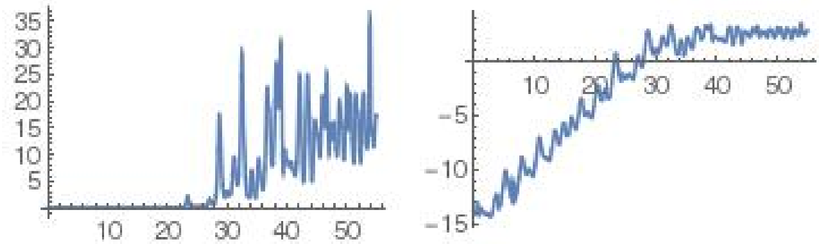


Figure 5.9: Mi tortenik a trajektoriakkal, ha egy kicsit megvaltoztatjuk a kezdeti feltetelt: $(z(0) = 0) \rightarrow (z(0) = 10^{-6})$. ? Az elso abra a ket trajektoria tavolsagat mutatja t fuggvenyeben, ez gyakorlatilag 0 az elso abran, ha $t \in [0, 20]$. Viszont $t = 30$ utan mar semmi korrelacio sincs (nagyon messze vannak egymastol) a ket trajektoria kozott. A Lyapunov exponens a kozeli trajektoriak exponencialis tavolodasat meri, ez a masodik abra atlagos meredekseget jelenti a (mondjuk) $t \in [0, 20]$ idointervallumban.

Problema 25. *Mennyi a Lyapunov exponens a Lotka-Volterra egyenlet eseteben?*

Problema 26. *Probald megmagyarazni, hogy mit jelent a max a Lyapunov exponens nem egeszen preciz definiciojaban!*

Gradiens aramlas: A

$$\frac{d}{dt}\bar{y} = -grad(V(\bar{y}))$$

DE megoldasgorbeinek a viselkedeset viszonylag konnyu megerteni, hiszen a trajektoriak a $V(\bar{y})$ fuggveny kritikus pontjaihoz konvergalnak. A legtobb trajektoria V lokalis minimumaihoz tart.

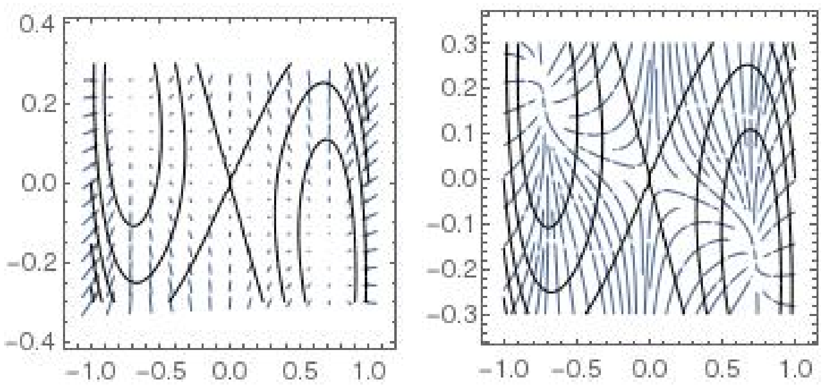


Figure 5.10: A $V(y_1, y_2) = y_1^4 - y_1^2 + y_2^2 + 0.5y_1y_2$ fuggveny gradiens aramlata. V a szintvonalaival lett abrazolva.

5.2.3 Stabilitas

Legyen $\frac{d}{dt}\bar{y} = \bar{f}(\bar{y})$, $\bar{f}(\bar{y}_{fix}) = 0$

1. Az $\bar{y}_{fix}$ egyensulyi allapotot Lyapunov stabilnak nevezzuk, ha barmely $\epsilon > 0$-hoz letezik olyan $\delta > 0$, hogy ha $|\bar{y}(0) - \bar{y}_{fix}| < \delta$, akkor $|\bar{y}(t) - \bar{y}_{fix}| < \epsilon$ barmely $t > 0$-ra.
2. Az $\bar{y}_{fix}$ egyensulyi allapotot aszimptotikusan stabilnak nevezzuk, ha letezik olyan $\delta > 0$, hogy ha $|\bar{y}(0) - \bar{y}_{fix}| < \delta$, akkor $\lim_{t\to\infty} |\bar{y}(t) - \bar{y}_{fix}| = 0$.

Pl. a csillapitatlan harmonikus oszcillator Lyapunov stabil de nem aszimptotikusan stabil, mig a csillapitott harmonikus oszcillator Lyapunov es aszimptotikusan stabil.

Tetel 3. *Tegyuk fel, hogy letezik olyan, az $\bar{y}_{fix}$ egyensulyi allapotot egy kornyezeteben definialt $V(\bar{y})$ fuggveny, hogy a kovetkezok teljesulnek:*

1. $V(\bar{y}_{fix}) = 0$,
2. $V(\bar{y}) > 0$ *ha* $\bar{y} \neq \bar{y}_{fix}$,
3. $\frac{d}{dt}V(\bar{y}(t)) < 0$.

Ekkor $\bar{y}_{fix}$ aszimptotikusan stabil. Ha az utolso feltetelben megengedjuk az egyenloseget is, akkor csak a gyengebb, Lyapunov fele stabilitas kovetkezik.

Problema 27. *Csillapitott harmonikus oszcillator:*

$$\frac{d}{dt}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & -\alpha \end{pmatrix}\begin{pmatrix} y \\ p \end{pmatrix},$$

ahol a rugoalland $k > 0$, mig a csillapitas $\alpha \geq 0$. Mutasd meg, hogy az energia fuggveny

$$H(y,p) = \frac{1}{2}p^2 + \frac{k}{2}y^2$$

kielegiti a tetel felteteleit! Vizsgald meg az $\alpha = 0$ esetet is! Mit modhatunk ez alapjan a fixpont $\bar{0}$ stabilitasarol?

Problema 28. *Anharmonikus oszcillator: Legyen*

$$H(y,p) = \frac{1}{2}p^2 - \frac{1}{2}y^2 + \frac{1}{2}y^4, \qquad \frac{d}{dt}p = -\frac{\partial H}{\partial y}, \quad \frac{d}{dt}y = \frac{\partial H}{\partial p}.$$

Mutasd meg, hogy $\frac{d}{dt}H = 0$ (ehhez nem kell kihasznalnod H aktualis formajat, eleg az utolso ket egyenlet)! Keresd meg a DE fixpontjait. Mely fixpontokban lehet H-t felhasznalni a fixpont stabilitasanak a bizonyitasara?

5.2.4 Linearizacio a fixpont korul

Legyen adott az $\frac{d}{dt}\bar{y} = \bar{f}(\bar{y})$ idofuggetlen DE. Legyen $\bar{z}$ a DE-hez tartozo dinamikai rendszer fixpontja, egyensulyi allapota, vagyis legyen $\bar{f}(\bar{z}) = 0$. Vezessuk be az uj $\overline{\Delta y} = \bar{y} - \bar{z}$ uj valtozot. Ekkor

$$\frac{d}{dt}\overline{\Delta y} = \frac{d}{dt}\bar{y}$$

tehat

$$\frac{d}{dt}\overline{\Delta y} = \frac{d}{dt}\bar{y} = \bar{f}(\bar{z} + \overline{\Delta y}) \approx \bar{f}(\bar{z}) + \sum_i \frac{\partial \bar{f}}{\partial y_i}_{|\bar{y}=\bar{z}} \Delta y_i = \sum_i \frac{\partial \bar{f}}{\partial y_i}_{|\bar{y}=\bar{z}} \Delta y_i.$$

A

$$\frac{d}{dt}\overline{\Delta y} = \sum_i \frac{\partial \bar{f}}{\partial y_i}_{|\bar{y}=\bar{z}} \Delta y_i$$

homogen linearis DE az eredeti nemlinearis DE linearizacioja a $\bar{z}$ fixpont korul. A $Jac = \frac{\partial \bar{f}}{\partial \bar{y}}$ matrixot az $\bar{f}$ fuggveny Jacobi matrixanak hivjuk.

5.2.5 Ugyanez ketdimenzioban

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}.$$

Fixpont:

$$\bar{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \text{ahol} \quad \begin{pmatrix} f_1(\bar{z}) \\ f_2(\bar{z}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Legyen $\overline{\Delta y} = \bar{y} - \bar{z}$. Ekkor

$$\frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} f_1(z_1 + \Delta y_1, z_2 + \Delta y_2) \\ f_2(z_1 + \Delta y_1, z_2 + \Delta y_2) \end{pmatrix}$$
$$\approx \begin{pmatrix} (f_1)'_{y_1}(\bar{z})\Delta y_1 + (f_1)'_{y_2}(\bar{z})\Delta y_2 \\ (f_2)'_{y_1}(\bar{z})\Delta y_1 + (f_2)'_{y_2}(\bar{z})\Delta y_2 \end{pmatrix} = \begin{pmatrix} (f_1)'_{y_1}(\bar{z}) & (f_1)'_{y_2}(\bar{z}) \\ (f_2)'_{y_1}(\bar{z}) & (f_2)'_{y_2}(\bar{z}) \end{pmatrix} \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}.$$

Tehat, ha

$$Jac = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix},$$

akkor a linearizalt egyenlet

$$\frac{d}{dt}\overline{\Delta y} = Jac(\bar{z})\ \overline{\Delta y}.$$

Problema 29. *Keresd meg a Lotka-Volterra (vagy ragadozo-zsakmany) DE fixpontjait es a linearizalt egyenleteket!*

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_1 - y_1 y_2 \\ -y_2 + y_1 y_2/2 \end{pmatrix}.$$

Megoldas 23. *1. Fixpontok:*

$$\begin{matrix} 2y_1 - y_1 y_2 = 0 \\ -y_2 + y_1 y_2/2 = 0 \end{matrix} \quad \Longrightarrow \quad \bar{z}_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{vagy} \quad \bar{z}_B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

2. Jacobi matrix:

$$Jac = \begin{pmatrix} \partial_{y_1}(2y_1 - y_1 y_2) & \partial_{y_2}(2y_1 - y_1 y_2) \\ \partial_{y_1}(-y_2 + y_1 y_2/2) & \partial_{y_2}(-y_2 + y_1 y_2/2) \end{pmatrix} = \begin{pmatrix} 2 - y_2 & -y_1 \\ y_2/2 & -1 + y_1/2 \end{pmatrix}$$

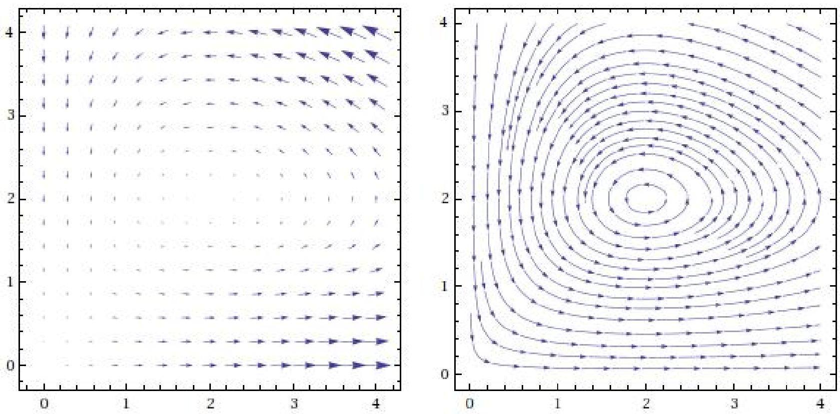


Figure 5.11: Lotka-Volterra DE. A fixpontok helyei: 1. $\bar{z}_A = (0,0)^T$, vagyis nulla ragadozo, nulla zsakmany, 2. $\bar{z}_A = (2,2)^T$, a nemtrivialis egyensulyi allapot.

3. *Tehat*

$$Jac(\bar{z}_A) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Jac(\bar{z}_B) = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

A linearizalt egyenletek:

$$\bar{z}_A : \quad \frac{d}{dt}\begin{pmatrix} y_1 - 0 \\ y_2 - 0 \end{pmatrix} = \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix},$$

$$\bar{z}_B : \quad \frac{d}{dt}\begin{pmatrix} y_1 - 0 \\ y_2 - 0 \end{pmatrix} = \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}.$$

A DE vektormezeje es megoldasgorbei:

Problema 30. *Keresd meg az inga mozgasat leiro $\phi''(t) = -\sin(\phi(t))$ DE elsorendo variansanak fixpontjait es a linearizalt egyenleteket!*

Megoldas 24.

$$\frac{d}{dt}\begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -\sin(\phi) \end{pmatrix}.$$

1. *Fixpontok:*

$$\begin{matrix} \omega = 0 \\ -\sin(\phi) = 0 \end{matrix} \qquad \Longrightarrow \qquad \bar{z}_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{vagy} \quad \bar{z}_B = \begin{pmatrix} \pi \\ 0 \end{pmatrix}.$$

2. *Jacobi matrix:*

$$Jac = \begin{pmatrix} \partial_\phi \omega & \partial_\omega \omega \\ \partial_\phi(-\sin(\phi)) & \partial_\omega(-\sin(\phi)) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos(\phi) & 0 \end{pmatrix}$$

3. *Tehat*

$$Jac(\bar{z}_A) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Jac(\bar{z}_B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

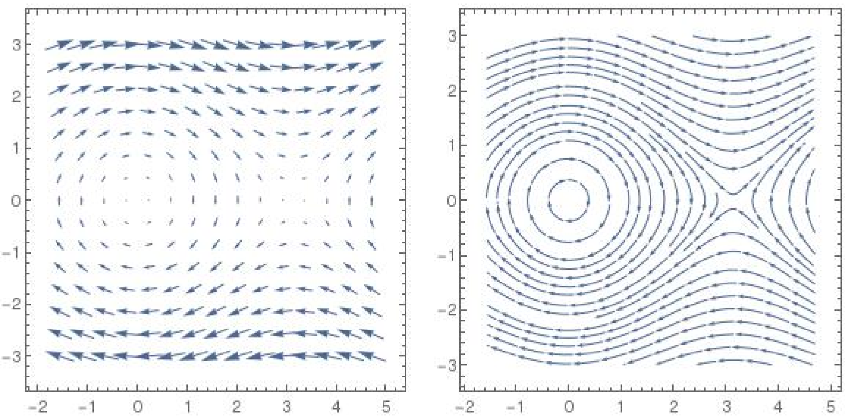


Figure 5.12: Egy inga mozgasanak a fazisportreja. $(\phi,\omega)^T = (0,0)^T$ a (Lyapunov) stabil egensulyi allapot, mig $(\phi,\omega)^T = (\pi,0)^T$ egy insabil fixpont.

A linearizalt egyenletek:

$$\bar{z}_A: \quad \frac{d}{dt}\begin{pmatrix}\phi-0\\ \omega-0\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix},$$

$$\bar{z}_B: \quad \frac{d}{dt}\begin{pmatrix}\phi-\pi\\ \omega-0\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix},$$

A DE vektormezeje es megoldasgorbei:

Problema 31. $y'' = y - y^3$. *Legyen* $p = y'$. *Mutasd meg, hogy a DE a kovetkezo (Hamilton fele) alakban is felirhato:*

$$y' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial y}.$$

Mennyi H ? Mutasd meg, hogy $H' = 0$ *!*

Ird at a DE-t egy elsorendu DE rendszerre, keresd meg annak a fixpontjait, ird fel a fixpontoktol valo elteresre vonatkozo liearizalt DE-ket! Vizsgald meg a fixpontok stabilitasat!

Megoldas 25.

$$y' = p = \frac{\partial H}{\partial p} \qquad \Longrightarrow \qquad H(y,p) = \frac{p^2}{2} + h(y),$$

$$p' = y'' = y - y^3 = -\frac{\partial H}{\partial y} \qquad \Longrightarrow \qquad H = \frac{p^2}{2} + \frac{y^4}{4} - \frac{y^2}{2} + konst.$$

H (mas neven az energia) megmarado mennyiseg:

$$H' = pp' + y^3y' - yy' = p(y-y^3) + (y-y^3)y' = p(y-y^3) + (y-y^3)p = 0.$$

Elsorendu DE-rendszer:

$$\frac{d}{dx}\begin{pmatrix}y\\ p\end{pmatrix} = \begin{pmatrix}p\\ y-y^3\end{pmatrix}$$

1. *Fixpontok:*

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}p\\y-y^3\end{pmatrix} \quad\Longrightarrow\quad \bar{z}_A = \begin{pmatrix}0\\0\end{pmatrix}, \qquad \bar{z}_B = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \bar{z}_C = \begin{pmatrix}-1\\0\end{pmatrix},$$

2. *Jacobi matrix:*

$$J = \begin{pmatrix}\frac{\partial}{\partial y}p & \frac{\partial}{\partial p}p\\ \frac{\partial}{\partial y}(y-y^3) & \frac{\partial}{\partial p}(y-y^3)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1-3y^2 & 0\end{pmatrix}$$

3. *Tehat*

$$J(\bar{z}_A) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad J(\bar{z}_B) = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}, \qquad J(\bar{z}_C) = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}$$

A linearizalt egyenletek: A linearizalt DE pl. az $\bar{z}_B$ fixpont korul:

$$\frac{d}{dx}\begin{pmatrix}y-1\\p-0\end{pmatrix} = \frac{d}{dx}\begin{pmatrix}\Delta y\\ \Delta p\end{pmatrix} = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}\begin{pmatrix}\Delta y\\ \Delta p\end{pmatrix}$$

A DE vektormezeje es megoldasgorbei:

$y'' = y - y^3 - y'$. Legyen $p = y'$ Ird at a DE-t egy elsorendu DE rendszerre, keresd meg annak a fixpontjait, ird fel a fixpontoktol valo elteresre vonatkozo liearizalt DE-ket! Vizsgald meg a fixpontok stabilitasat!

Legyen $H = p^2/2 - y^2/2 + y^4/4$. Mutasd meg, hogy $H' \leq 0$ a harom kozul ket fixpontban ! Bizonyitsd be ennek alapjan ezen fixpontok stabilitasat!

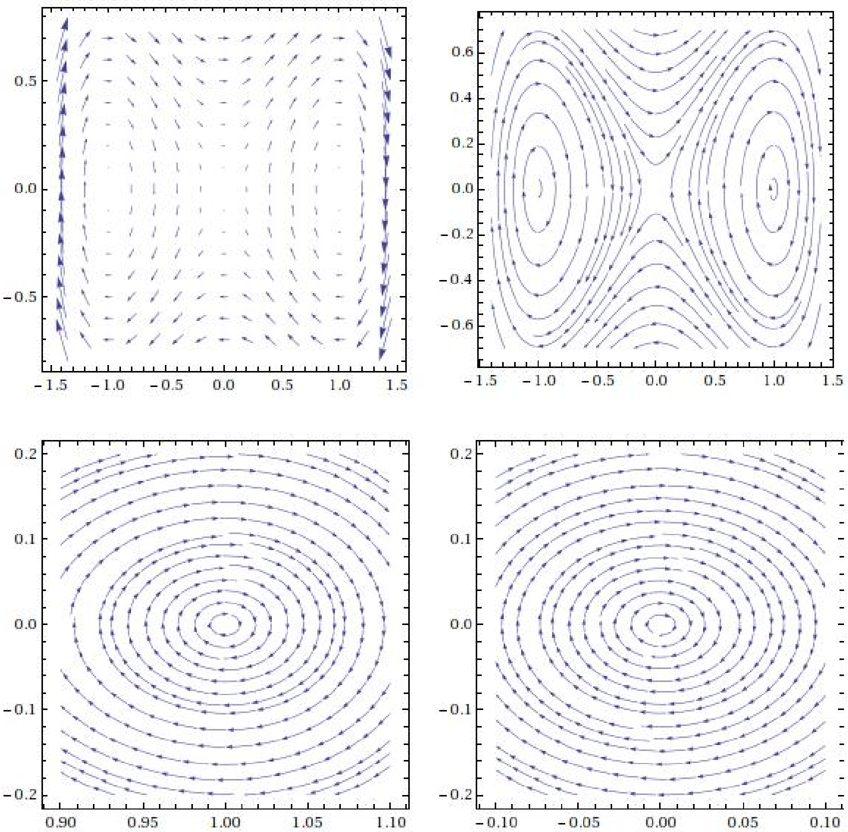


Figure 5.13: Az felso sor ket abraja a DE vektormezojet, illetve annak megoldasgorbeit mutatja. A masodik sor elso abraja a megoldasgorbeket abrazolja az $\bar{z}_B$ fixpont korul, mig a masodik abra a linearizalt, kozelito egyenlet megoldasgorbeit tartalmazza. Az also sor ket abraja kozotti kulonbseg szinte eszrevehetetlen.

Chapter 6

Linearization

Example: Let

$$\frac{d}{dt}y = f(y) = 3y(1-y) = 3y - 3y^2.$$

If $y(0) = 0$, then the solution is $y(t) = 0 = const.$, so $y_{fix} = 0$ is a fixed point (equilibrium position, steady state) of the dynamical system generated by the DE. If $y \approx 0$, then $f(y) \approx 3y$ (since $|-3y^2| \ll |3y|$), so approximately

$$\frac{d}{dt}y \approx 3 \cdot y,$$

where $3 = \frac{df(y)}{dy}_{|y=0}$. So we can trade the original nonlinear DE for an approximative homogeneous linear one.

6.1 One dimension, qualitative behaviour

6.1.1 Linearization around the fixed point

Let $\frac{d}{dt}y = f(y)$ be a time independent DE. Let y_{fix} be a fixed point, i.e. $f(y_{fix}) = 0$. introduce $\Delta y = y - y_{fix}$. Then $\frac{dy}{dt} = \frac{d\Delta y}{dt}$, so

$$\frac{d}{dt}\Delta y = \frac{dy}{dt} = f(y_{fix} + \Delta y) \approx f(y_{fix}) + f'(y_{fix})\Delta y = f'(y_{fix})\Delta y.$$

The

$$\frac{d}{dt}\Delta y = f'(y_{fix})\Delta y$$

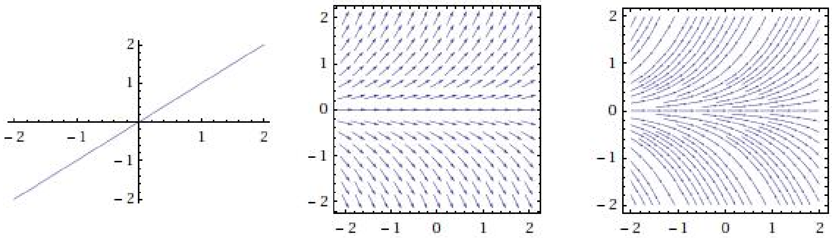
homogeneous linear DE is the linearization of the original DE around y_{fix}. (Here $f' = \frac{df(y)}{dy}$.)

6.1.2 Qualitative behaviour

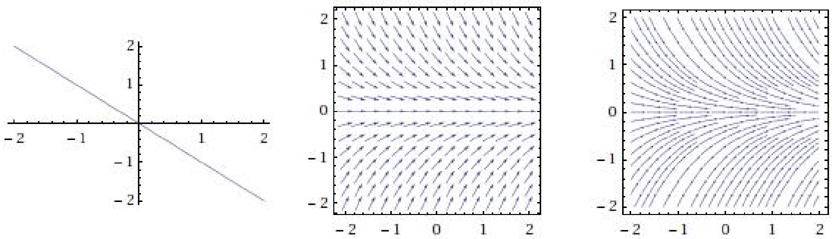
Problema 32. *Plot the velocity fields and solution curves of $y' = f(y)$! Find the fixed points, write down the linearized equations, and study the stability of the fixed points!*

$$a)\ y' = 1, \qquad b)\ y' = y, \qquad c)\ y' = -y, \qquad d)\ y' = y + 1,$$
$$e)\ y' = -1 + y^2, \qquad f)\ y' = y(1-y), \qquad g)\ y' = y(1-y)(1+y).$$

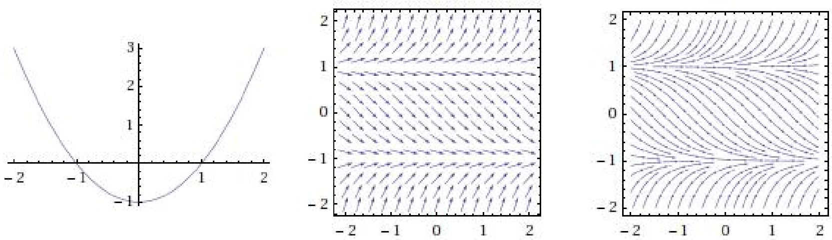
Megoldas 26. *b)* $y' = y$



c) $y' = -y$



e) $y' = y^2 - 1$



g)

$$\frac{d}{dt}y = f(y) = y(1-y)(1+y) = +y - y^3.$$

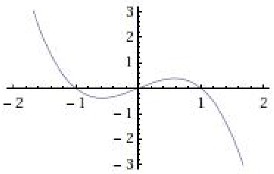
$f'(y) = \frac{d}{dy}f(y) = 1 - 3y^2$. *the fixed points (i.e. the roots of* $f(y)$*):*

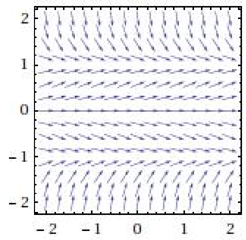
$$\begin{aligned}
y_1 &= 0, & f'(y_1) &= f'(0) = 1 - 3 \cdot 0^2 = 1 > 0,\\
y_2 &= 1, & f'(1) &= -2 < 0,\\
y_3 &= -1, & f'(-1) &= -2 < 0.
\end{aligned}$$

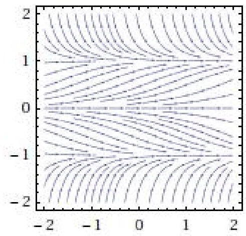
The sign off$'$ *determines the stability of the fixed points* $y_1,\ y_2,\ y_3$*: unstable, stable, stable.*

the linearized equations:

$$\begin{aligned}
\frac{d}{dt}(y-0) =& \frac{d}{dx}\Delta y_1 = 1 \cdot \Delta y_1,\\
\frac{d}{dt}(y-1) =& \frac{d}{dx}\Delta y_2 = -2 \cdot \Delta y_2,\\
\frac{d}{dt}(y-(-1)) =& \frac{d}{dx}\Delta y_3 = -2 \cdot \Delta y_3,
\end{aligned}$$







Assume that $y(t)$ solves the initial value problem $y(0) = 0.7$, $\frac{d}{dt}y = f(y)$. then

$$\lim_{t\to\infty} y(t) = 1, \qquad \lim_{t\to-\infty} y(t) = 0.$$

if the initial value is $y(0) = -0.7$, then

$$\lim_{t\to\infty} y(t) = -1, \qquad \lim_{t\to-\infty} y(t) = 0.$$

6.2 Several dimensions

6.2.1 Linearization around the fixed point

Let $\frac{d}{dt}\bar{y} = \bar{f}(\bar{y})$ be a time independent DE. Let $\bar{z}$ be a fixed point, so $\bar{f}(\bar{z}) = 0$. introduce $\overline{\Delta y} = \bar{y} - \bar{z}$. Then

$$\frac{d}{dt}\overline{\Delta y} = \frac{d}{dt}\bar{y},$$

so

$$\frac{d}{dt}\overline{\Delta y} = \frac{d}{dt}\bar{y} = \bar{f}(\bar{z} + \overline{\Delta y}) \approx \bar{f}(\bar{z}) + \sum_i \frac{\partial \bar{f}}{\partial y_i}_{|\bar{y}=\bar{z}} \Delta y_i = \sum_i \frac{\partial \bar{f}}{\partial y_i}_{|\bar{y}=\bar{z}} \Delta y_i.$$

The

$$\frac{d}{dt}\overline{\Delta y} = \sum_i \frac{\partial \bar{f}}{\partial y_i}_{|\bar{y}=\bar{z}} \Delta y_i$$

DE is the linearization around $\bar{z}$. The matrix $Jac = \frac{\partial \bar{f}}{\partial \bar{y}}$ is the Jacobian of $\bar{f}$.

6.2.2 The same in two dimensions

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}.$$

Fixed point:

$$\bar{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \text{ahol} \quad \begin{pmatrix} f_1(\bar{z}) \\ f_2(\bar{z}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\overline{\Delta y} = \bar{y} - \bar{z}$. then

$$\frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} f_1(z_1 + \Delta y_1, z_2 + \Delta y_2) \\ f_2(z_1 + \Delta y_1, z_2 + \Delta y_2) \end{pmatrix}$$

$$\approx \begin{pmatrix} (f_1)'_{y_1}(\bar{z})\Delta y_1 + (f_1)'_{y_2}(\bar{z})\Delta y_2 \\ (f_2)'_{y_1}(\bar{z})\Delta y_1 + (f_2)'_{y_2}(\bar{z})\Delta y_2 \end{pmatrix} = \begin{pmatrix} (f_1)'_{y_1}(\bar{z}) & (f_1)'_{y_2}(\bar{z}) \\ (f_2)'_{y_1}(\bar{z}) & (f_2)'_{y_2}(\bar{z}) \end{pmatrix} \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}.$$

So if

$$Jac = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix},$$

then the linearized equation is

$$\frac{d}{dt}\,\overline{\Delta y} = Jac(\bar{z})\;\overline{\Delta y}.$$

Problema 33. *Find the fixed points of the Lotka-Volterra (or predator-prey) DE and write down the linearized equations!*

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_1 - y_1y_2 \\ -y_2 + y_1y_2/2 \end{pmatrix}.$$

Megoldas 27. *1. Fixed points:*

$$\begin{aligned} 2y_1 - y_1y_2 &= 0 \\ -y_2 + y_1y_2/2 &= 0 \end{aligned} \qquad \Longrightarrow \qquad \bar{z}_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad or \quad \bar{z}_B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

2. Jacobian:

$$Jac = \begin{pmatrix} \partial_{y_1}(2y_1 - y_1y_2) & \partial_{y_2}(2y_1 - y_1y_2) \\ \partial_{y_1}(-y_2 + y_1y_2/2) & \partial_{y_2}(-y_2 + y_1y_2/2) \end{pmatrix} = \begin{pmatrix} 2 - y_2 & -y_1 \\ y_2/2 & -1 + y_1/2 \end{pmatrix}$$

3. So

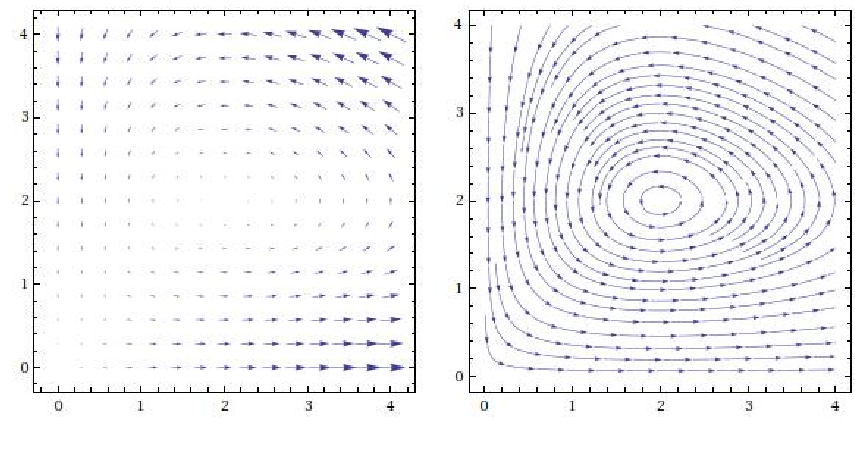
$$Jac(\bar{z}_A) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Jac(\bar{z}_B) = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

The linearized DE:

$$\bar{z}_A: \quad \frac{d}{dt}\begin{pmatrix} y_1 - 0 \\ y_2 - 0 \end{pmatrix} = \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix},$$

$$\bar{z}_B: \quad \frac{d}{dt}\begin{pmatrix} y_1 - 0 \\ y_2 - 0 \end{pmatrix} = \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}.$$

The velocity field and solution curves:



Problema 34. *The motion of a pendulum is described by the following DE: $\phi''(t) = -\sin(\phi(t))$. Rewrite it as a first order system, find the fixed points and write down the linearized equations!*

Megoldas 28.

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \omega\end{pmatrix} = \begin{pmatrix}\omega\\ -\sin(\phi)\end{pmatrix}.$$

1. *Fixed points:*

$$\begin{matrix}\omega = 0\\ -\sin(\phi) = 0\end{matrix} \qquad \Longrightarrow \qquad \bar{z}_A = \begin{pmatrix}0\\0\end{pmatrix} \quad or \quad \bar{z}_B = \begin{pmatrix}\pi\\0\end{pmatrix}.$$

2. *Jacobian:*

$$Jac = \begin{pmatrix}\partial_\phi\omega & \partial_\omega\omega\\ \partial_\phi(-\sin(\phi)) & \partial_\omega(-\sin(\phi))\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\cos(\phi) & 0\end{pmatrix}$$

3. *So*

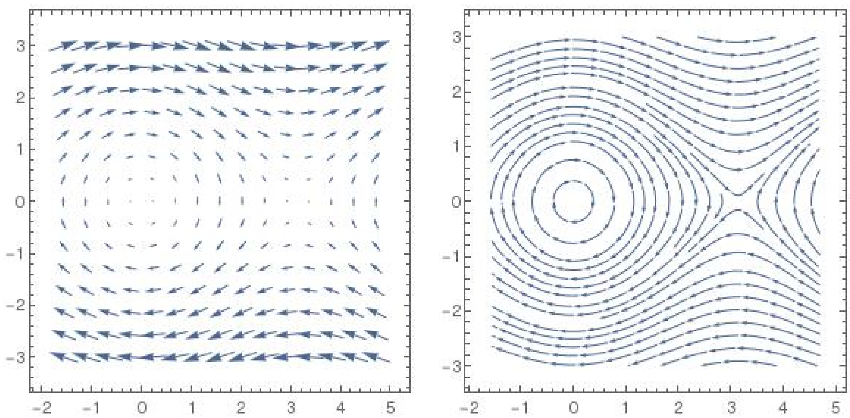
$$Jac(\bar{z}_A) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad Jac(\bar{z}_B) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

The linearized equations:

$$\bar{z}_A : \quad \frac{d}{dt}\begin{pmatrix}\phi - 0\\ \omega - 0\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix},$$

$$\bar{z}_B : \quad \frac{d}{dt}\begin{pmatrix}\phi - \pi\\ \omega - 0\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Delta\phi\\ \Delta\omega\end{pmatrix},$$

The velocity field and solution curves



Problema 35. *$y'' = y - y^3$. Let $p = y'$. Show that the DE can be written in the following Hamiltonian form*

$$y' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial y}.$$

How much is H ? Show that $H' = 0$!

Rewrite it as a first order system, find the fixed points and write down the linearized equations!

Megoldas 29.

$$y' = p = \frac{\partial H}{\partial p} \quad \Longrightarrow \quad H(y,p) = \frac{p^2}{2} + h(y),$$
$$p' = y'' = y - y^3 = -\frac{\partial H}{\partial y} \quad \Longrightarrow \quad H = \frac{p^2}{2} + \frac{y^4}{4} - \frac{y^2}{2} + konst.$$

H (the energy) is a conserved quantity:

$$H' = pp' + y^3y' - yy' = p(y - y^3) + (y - y^3)y' = p(y - y^3) + (y - y^3)p = 0.$$

The first order DE::

$$\frac{d}{dx}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} p \\ y - y^3 \end{pmatrix}$$

1. *Fix points:*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ y - y^3 \end{pmatrix} \quad \Longrightarrow \quad \bar{z}_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \bar{z}_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \bar{z}_C = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

2. *Jacobian:*

$$J = \begin{pmatrix} \frac{\partial}{\partial y} p & \frac{\partial}{\partial p} p \\ \frac{\partial}{\partial y}(y - y^3) & \frac{\partial}{\partial p}(y - y^3) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 3y^2 & 0 \end{pmatrix}$$

3. *so*

$$J(\bar{z}_A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J(\bar{z}_B) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \qquad J(\bar{z}_C) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

The linearized DE around $\bar{z}_B$:

$$\frac{d}{dx}\begin{pmatrix} y - 1 \\ p - 0 \end{pmatrix} = \frac{d}{dx}\begin{pmatrix} \Delta y \\ \Delta p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \Delta y \\ \Delta p \end{pmatrix}$$

Velocity field and solution curves:

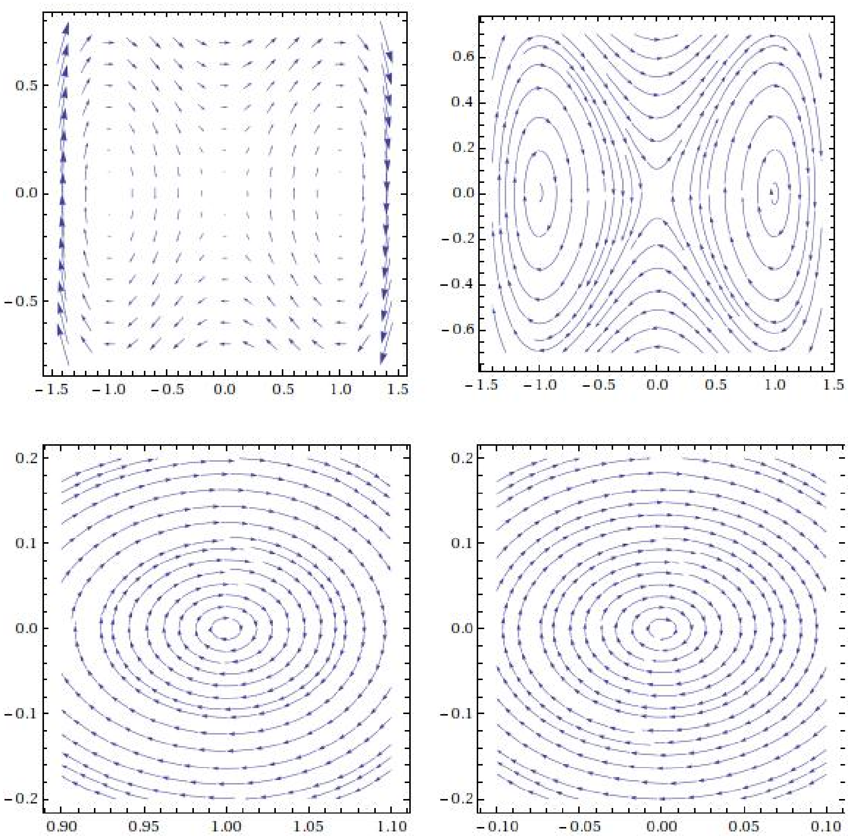


Figure 6.1: Upper row: velocity field and solution curves. Lower row: Solution curves around $\bar{z}_B$ for the nonlinear an the linearized equations. The shapes of the solution curves are very close to each other in these two plots.

Chapter 7

Homogen linearis rendszerek

Homogen, illetve inhomogen linearis egyenlet:

$$\text{hom:} \quad \frac{d}{dt}\bar{y} = A(t)\bar{y}, \qquad \text{inhom:} \quad \frac{d}{dt}\bar{y} = A(t)\bar{y} + \bar{f}(t),$$

($\bar{y}$ vektor, A matrix.)

1. Szuperpozicio elve: Ha

   $$\frac{d}{dt}\overline{y_1} = A(t)\overline{y_1} \quad \text{es} \quad \frac{d}{dt}\overline{y_2} = A(t)\overline{y_2}$$

   akkor

   $$\frac{d}{dt}\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right) = A(t)\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right),$$

   vagyis a homogen egyenletek megoldasainak linearis kombinacioja is megoldas.

2. Inhom. egyenlet altalanos megoldasa: Ha

   $$\frac{d}{dt}\overline{y}_{hom} = A(t)\overline{y}_{hom} \quad \text{es} \quad \frac{d}{dt}\overline{y_{in}} = A(t)\overline{y_{in}} + \bar{f}(t),$$

   akkor

   $$\frac{d}{dt}\left(\overline{y}_{hom} + \overline{y_{in}}\right) = A(t)\left(\overline{y}_{hom} + \overline{y_{in}}\right) + \bar{f}(t).$$

   Tehat az inhomogen egyenlet altalanos megoldasa felirhato a homogen egyenlet $\overline{y}_{hom}$ altalanos megoldasa es az inhom. egyenlet egy $\overline{y_i}$ partikularis megoldasanak az $\overline{y}_{hom} + \overline{y_i}$ osszegekent.

3. Linearis input-output relacio: Ha

   $$\frac{d}{dt}\overline{y}_1 = A(t)\overline{y}_1 + \bar{f}_1(t) \quad \text{es} \quad \frac{d}{dt}\overline{y}_2 = A(t)\overline{y}_2 + \bar{f}_2(t)$$

   akkor

   $$\frac{d}{dt}\left(\alpha_1\overline{y}_1 + \alpha_2\overline{y}_2\right) = A(t)\left(\alpha_1\overline{y}_1 + \alpha_2\overline{y}_2\right) + \left(\alpha_1\bar{f}_1(t) + \alpha_2\bar{f}_2(t)\right),$$

   tehat az $(\alpha_1\bar{f}_1(t) + \alpha_2\bar{f}_2(t))$ ”input” $(\alpha_1\overline{y}_1 + \alpha_2\overline{y}_2)$ ”output”-ot general.

7.1 Idofuggetlen homogen rendszerek

Problema 36. *Radioaktiv bomlas: $A \to B \to \emptyset$.*

$$\frac{d}{dt}\bar{y} = \frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2a \\ 2a - 3b \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Ennek a feladatnak nagyon konnyu a megoldasa, ha $a(0) = 0, b(0) = 1$ (ez a P pont). Magyarazd meg, hogy miert sokkal nehezebb a feladat, ha a kezdeti feltetel $a(0) = 1, b(0) = 0$ (ez pedig a Q pont) !

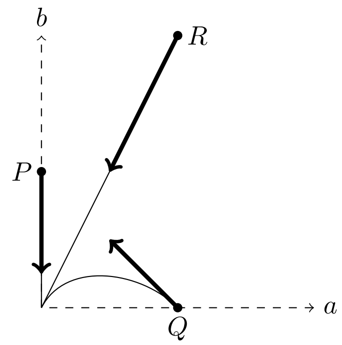


Figure 7.1: A sebessegvektorok kulonbozo $\bar{y} = (a, b)^T$ pontokban. A $P = (0, 1)^T$ es a $R = (1, 2)^T$ pontokban a sebessegvektor es a pozicio vektor egyiranyu. Ez nem igaz a $Q = (1, 0)^T$ pontra, igy az innen kiindulo trajektoria gorbevonalu.

Megoldas 30. *Ha*

$$\bar{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ akkor } \dot{\bar{y}}(0) = -3 \cdot \bar{y}(0),$$

vagyis a poziciovektor $\bar{y}$ es a sebessegvetor $\dot{\bar{y}}$ aranyos egymassal (vagyis egyiranyu), tehat a megoldas

$$\bar{y}(t) = e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Ez az aranyossag nem teljesul, ha $\bar{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, ebbol a kezdeti feltetelbol kiindulva a az $\bar{y}(t)$ trajektoria egy gorbe vonal lesz. Viszont legyen

$$\bar{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ ekkor } \dot{\bar{y}}(0) = \begin{pmatrix} -2 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2 \cdot \bar{y}(0),$$

vagyis a poziciovektor $\bar{y}$ es a sebessegvetor $\dot{\bar{y}}$ aranyos egymassal, igy az evolucio az $\bar{y}(0)$ vektor egyenesen tortenik, a megoldas tehat

$$\bar{y}(t) = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Mivel a DE hom.lin., igy az altalanos megoldas

$$\bar{y}(t) = C_1 e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Tetel 4. *Ha a $\bar{v}_1, \ldots, \bar{v}_n$ vektorok bazist alkotnak egy vektorterben es sajatvektorai A-nak, vagyis $A\bar{v}_i = \lambda_i \bar{v}_i$, akkor a*

$$\frac{d}{dt}\bar{y} = A\bar{y}$$

DE altalanos megoldasa

$$\bar{y}(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t} \bar{v}_i.$$

Valoban, ha kiszamitjuk a DE ket oldalat:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} C_i e^{\lambda_i t} \bar{v}_i\right) = \sum_{i=1}^{n} C_i \lambda_i e^{\lambda_i t} \bar{v}_i,$$
$$A \cdot \left(\sum_{i=1}^{n} C_i e^{\lambda_i t} \bar{v}_i\right) = \sum_{i=1}^{n} C_i e^{\lambda_i t} \lambda_i \bar{v}_i,$$

akkor ugyanazt kapjuk. Sajnos nincs arra garancia, hogy letezik sajatvektorokbol allo bazis. Azonban ez az eset is kezelheto, mivel

$$\frac{d}{dt}\bar{y} = A\bar{y}, \quad \bar{y}(0) = \bar{y}_0 \quad \Longrightarrow \quad \bar{y}(t) = e^{tA}\bar{y}_0.$$

Ehhez persze definialnunk kell egy matrix exponencialis fuggvenyet.

7.1.1 Exponencialis fuggveny

Valos szamok. $x \in \mathbb{R}$.

e^x mindenhol konvergens Taylor sora:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

$e^x e^y = e^{x+y}$, tehat

$$\begin{aligned}\left(1 + x + \frac{x^2}{2!} + \ldots\right)\left(1 + y + \frac{y^2}{2!} + \ldots\right) \\ = \left(1 + (x+y) + \frac{(x+y)^2}{2!} + \ldots\right).\end{aligned} \tag{7.1}$$

Ez azt jelenti, hogy ha peldaul kiszamitjuk xy egyutthatojat a ket oldalon, akkor ugyanazt, 1-et kapunk.

Komplex szamok. $x \in \mathbb{C}$.

Definialjuk az exponencialis foggveny komplex szamokra ugyanazzal a Taylor sorral:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Ekkor $e^x e^y = e^{x+y}$, mivel ugyanazok a muveleti szabalyok ervenyesek a komplex szamokra is, igy (7.1.1) itt is igaz.

Ha a valos, akkor

$$
\begin{aligned}
e^{ia} &= 1 + ia + \frac{(ia)^2}{2!} + \frac{(i)a^3}{3!} + \dots \\
= \left(1 - \frac{a^2}{2!} + \dots\right) &+ i\left(a - \frac{a^3}{3!} + \dots\right) = cos(a) + i\sin(a).
\end{aligned}
$$

Tehat pl.

$$e^{2+3i} = e^2\left(\cos(3) + i\sin(3)\right)$$

Matrix exponensek. $x \in Mat(n)$. Itt x, y $n \times n$ matrixok. e^x tovabbra is a Taylor sorral van definialva. (A tovabbiakban 1 az egysegmatrixot (is) jeloli.) Vajon igaz-e, hogy

$$
\begin{aligned}
e^x e^y &= \left(1 + x + \frac{x^2}{2!} + \dots\right)\left(1 + y + \frac{y^2}{2!} + \dots\right) \quad (7.2)\\
&= 1 + (x+y) + \left(\frac{x^2}{2!} + \frac{y^2}{2!} + xy\right) + \dots \\
??? = ??? \qquad e^{x+y} &= \left(1 + (x+y) + \frac{(x+y)^2}{2!} + \dots\right) \quad (7.3)\\
&= 1 + x + y + \frac{x^2}{2!} + \frac{y^2}{2!} + \frac{1}{2!}(xy + yx) + \dots
\end{aligned}
$$

Az masodik sor csak akkor egyenlo a negyedikkel, ha $xy = yx$, ami NEM igaz altalaban a matrixszorzasra. Tehat:

$$ha\ \ AB = BA,\ \ akkor\ \ e^A e^B = e^{A+B}$$

Kommutator: $[A, B] = AB - BA$. Vagyis $AB = BA$ ugyanazt jelenti, mint $[A, B] = 0$.

A skalaris $y' = ay$ megoldasa: $y = e^{at}y(0)$, mivel

$$
\begin{aligned}
\frac{d}{dt}\left[\left(1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots\right)y(0)\right] &= \left(0 + a + a^2t + \frac{a^3t^2}{2!} + \dots\right)y(0)\\
&= a\left(1 + at + \frac{(at)^2}{2!} + \dots\right)y(0) \quad (7.4)
\end{aligned}
$$

Ugyanez a levezetes igaz, ha a helyett egy A matrixot, y helyett egy $\bar{y}$ vektor t irunk:

$$\frac{d}{dt}\bar{y} = A\bar{y} \quad \Longrightarrow \quad \bar{y} = e^{tA}\bar{y}(0). \quad (7.5)$$

Megjegyzes 1: Persze joggal aggodhatunk a Taylor sor konvergenciaja miatt. Valos es komplex szamok eseteben a kulonbozo becsleseknel a szamok

$$|xy| = |x| \cdot |y|, \qquad |x + y| \leq |x| + |y|$$

tulajdonsagait hasznaljuk fel. Hasonlo oszzefuggesek matrixokra is igazak, ha egy eukledeszi (vagy a komplex esetben hermitikus) vektorterben hato linearis transzformacio normajat igy ertelmezzuk:

$$||A|| = \max_{||\bar{v}||=1} ||A\bar{v}||.$$

(Vagyis maximalisan hanyszorosara nyujthat meg a tr. egy egysegvektort.) Ekkor

$$||AB|| \leq ||A|| \cdot ||B||, \qquad ||A+B|| \leq ||A|| + ||B||.$$

Megjegyzes 2: Csabito lenne a (7.5) kepletet akkor is alkalmazni, ha a $\bar{y}$ egy vegtelen dimenzios vektorter eleme, pl. egy egyvaltozos sima fuggveny.

Pelda: Hoegyenlet.

$$\partial_t \phi(t,x) = \partial_x^2 \phi(t,x), \;\; \phi(0,x) = f(x),$$

ahol $f(x)$ a $t = 0$-kor adott homersekleteloszlas. Ennek a megoldasat formalisan a kovetkezo alakbn irhatjuk fel:

$$\phi(t,x) = \left(e^{t\partial_x^2} f \right)(x).$$

Viszont ez a kifejezes szinte mindig ertelmetlen, ha $t < 0$. (Ennek az a magyarazata, hogy a hoegyenlet altal leirt evolucio "kisimitja" a kezdeti homersekleteloszlast, tehat ha a kezdeti eloszlas nem extrem modon sima, akkor idoben visszafele nem lehet megoldani az PDE-t.) Ettol figgetlenul az ilyen formalis kifejezesek vegtelenul hsznosak lehetnek.

Hogyan tudjuk kiszamitani e^A-t? Ez nagyon konnyu, ha A diagonalis. A (peldaul ket dimenzios) diagonalis matrixok algebraja tulajdonkeppen ket fuggetlen kopiaja (direkt osszege) a kozonseges szamok algebrajanak:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{pmatrix},$$
$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{pmatrix}.$$

Emiatt

$$e^D = \exp \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}.$$

Ha A nem diagonalis, de diagonilzalhato (vagyis van olyan S invertalhato matrix, hogy $S^{-1}AS = D$ mar diagonalis (ekkor persze $SDS^{-1} = A$)), akkor

$$\begin{aligned} e^A &= SS^{-1} + SDS^{-1} + \frac{1}{2!} SDS^{-1}SDS^{-1} + \ldots \\ &= S\left(1 + D + \frac{1}{2!}D^2 + \ldots\right) S^{-1} = Se^D S^{-1}. \end{aligned}$$

7.1.2 Sajatertekek, sajatvektorok

Sajatertekegyenlet:

$$\begin{aligned} A\bar{v} &= \lambda \bar{v}, \qquad \bar{v} \neq \bar{0} \\ A\bar{v} &= \lambda E \bar{v}, \\ (A - \lambda E)\bar{v} &= \bar{0}. \\ \text{Viszont} \quad (A - \lambda E)\bar{0} &= \bar{0}, \end{aligned}$$

tehat a linearis transzformacio $A - \lambda E$ nem egy az egyhez tipusu, vagyis nem invertalhato, tehat az egyenletunknek csak akkor lesz nemtrivialis $\bar{v} \neq \bar{0}$ megoldasa, ha

$$\det(A - \lambda E) = 0.$$

Ebbol megkapjuk λ lehetseges ertekeit, ezutan $\bar{v}$ megkeresesehez mar csak egy linearis egyenlet megoldas szukseges.

Tetel 5. *Tegyuk fel, hogy a $\bar{v}_1, \ldots, \bar{v}_n$ vektorok bazist alkotnak egy vektorterben es sajatvektorai A-nak, vagyis $A\bar{v}_i = \lambda_i \bar{v}_i$. Legyen S a $\bar{v}_i$ oszlopvektorokbol alkotott matrix. Ekkor*

$$S^{-1}AS = diag(\lambda_1, \ldots, \lambda_n) = D.$$

Itt $diag(\lambda_1, \ldots, \lambda_n)$ azt a diagonalis matrixot jelenti, ahol nem nulla elemek csak a diagonalison vannak, az i-edik sorban ez az elem eppen λ_i.

Problema 37. *Keresd meg az A matrix sajatertekeit es sajatvektorait! Keresd meg azt as S hasonlosagi transzformaciot, ami dagonalizalja A-t, vagyis $D = S^{-1}AS$, ahol D diagonalis! Ird fel a v vektort a sajatvektorok linearis kombinaciojakent! Mennyi $A^{13}v$?*

$$a)\ (7) \qquad b)\ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \qquad d)\ \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad e)\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$f)\ \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \qquad g)\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad h)\ \begin{pmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Itt a v vektor erteke:

$$a)\ v = (8), \qquad b-f)\ v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \qquad g-h)\ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Megoldas 31. *b)*

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Sajatertekek: $\lambda_1 = 3, \qquad \lambda_2 = 2$,
Sajatvektorok:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Mivel A eleve diagonalis volt, ez a feladat trivialis, a sajatertekek a diagonalis elemek, a sajatvektorok pedig a standard bazis.

d)

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Sajatertekek egyenlete:

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$$

Sajatertekek: $\lambda_1 = 3, \qquad \lambda_2 = 2$.
Mivel a diagonalis alatt csupa 0 *all, igy a sajatertekek automatikusan a diagonalis elemek.*

Sajatvektorok egyenlete ($\lambda_1 = 3$*-ra):*

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix},$$

vagy

$$\begin{pmatrix} 2-3 & 1 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Innen $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$. *Ezek kozul valasztunk egy nem nulla vektort, pl. legyen* $x = 1$.

Sajatvektorok:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad\qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Az A *matrixot diagonalizalo hasonlosagi transzformacio:*

$$D = S^{-1}AS = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Itt S *a* v_1 *es a* v_2 *oszlopvektorokbol allo matrix, illetve*

$$S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Mennyi $A^{13}v$ *?*

$$A^{13}v = (SDS^{-1})^{13}v = SD^{13}S^{-1}v$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3^{13} & 0 \\ 0 & 2^{13} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

vagy

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \alpha v_1 + \beta v_2,$$

ahol

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S^{-1} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix},$$

tehat

$$A^{13}v = A^{13}(\alpha v_1 + \beta v_2) = \alpha\lambda_1^{13}v_1 + \beta\lambda_2^{13}v_2 = 4 \cdot 3^{13} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \cdot 2^{13} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

f)

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

Sajatertekek: $\lambda_1 = 2 + 3i, \qquad \lambda_2 = 2 - 3i$,
Sajatvektorok:

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}$$

g) Mivel A a d) es az a) blokkok kombinacioja, igy ezen ket feladat eredmenyeit felhasznalva a kovetkezoket kapjuk:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Sajatertekek: $\lambda_1 = 3, \qquad \lambda_2 = 2, \qquad \lambda_3 = 7, \qquad ,$
Sajatvektorok:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$D = S^{-1}AS$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Jegyzet a sajatertekekproblemarol: BME kurzus: Sajatertek, sajtvektor

7.2 Idofuggetlen homogen problemak

Problema 38. *Oldd meg az elozo feladatban szereplo A matrixokra az*

$$\frac{d}{dx}y = Ay, \;\; y(0) = v$$

DE-t! Ird fel az altalanos, illetve a partikuralis megoldast! Vizsgald meg az $y = 0$ *fixpont stabilitasat!*
Mennyi $\exp(xA)$ *? Ird fel a partikularis megoldast* e^{xA} *segitsegevel!*

Megoldas 32. *d) Feladat:*

$$\frac{d}{dt}\bar{y} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \bar{y}$$

Megoldas: Sajatertekek: $\lambda_1 = 3, \quad \lambda_2 = 2$ *. Sajatvektorok:*

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Az altalanos megoldas:

$$y_{alt}(x) = \sum_i C_i e^{\lambda_i x} v_i = C_1 e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Ha

$$\bar{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

vagyis $C_1 = 4,\ C_2 = -1$, *akkor a partikularis megoldas*

$$y_{part}(x) = 4e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Mivel mindket sajatertek valos resze pozitiv, igy az $y = 0$ *fixpont instabil. Az* xA *matrix exponencialis fuggvenye:*

$$e^{xA} = e^{xSDS^{-1}} = Se^{xD}S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{3x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Ennek segitsegevel a partikularis megoldas az

$$y_{part}(x) = e^{xA} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

alakban irhato fel.

Problema 39. $y'' = -y$. *Ird fel a DE karakterisztikus egyenletet, illetve a DE altalanos megoldasat! Ird at a DE-t egy elsorendu DE-rendszerre es oldd meg az elozo feladathoz hasonloan! Hasonlitsd ossze a ket megoldasi modszert!*

Megoldas 33. *1. A karakterisztikus egyenlet es annak gyokei:*

$$y'' = -y \quad \Longrightarrow \quad \lambda^2 = -1 \quad \Longrightarrow \quad \lambda_1 = 0 + 1 \cdot i,\ \ \lambda_2 = 0 - 1 \cdot i,$$

tehat az altalanos megoldas:

$$y = C_1 e^{(0+i)x} + C_2 e^{(0-i)x} = e^{0 \cdot x} \left(\widetilde{C_1} \cos(1 \cdot x) + \widetilde{C_2} \sin(1 \cdot x) \right)$$

Itt $C_1 = \widetilde{C_1}/2 + \widetilde{C_2}/(2i),\ C_2 = \widetilde{C_1}/2 - \widetilde{C_2}/(2i)$.

2. Ugyanez elsorendu DE rendszerkent:

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} = A \begin{pmatrix} y \\ v \end{pmatrix}$$

A sajatertekei es sajatvektorai:

$$\lambda_1 = i,\ \ v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \lambda_1 = -i,\ \ v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Tehat az altalanos megoldas:

$$\begin{pmatrix} y \\ v \end{pmatrix} = C_1 e^{ix} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-ix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Az eredeti harmonikus oszcillator problema csak valos szamokat tartalmaz, sajnos a megoldas (a komplex gyokok miatt) komplex szamokkal van kifejezve. Ezeknek a komplex szamoknak el kell tunniuk valos kezdeti ertek problema eseteben:

$$y \in \mathbb{R} \qquad \Rightarrow \qquad \overline{C}_2 = C_1, \qquad y(t) = 2|C_1|\cos(t + Arg(C_1)),$$

ahol $C_1 = |C_1|e^{i \cdot Arg(C_1)}$.

7.3 Jordan normal forma

Sajnos elofordulhat, hogy a fuggetlen sajatvektorok nem alkothatnak bazist.

Problema 40. *Keresd meg az A matrix sajaterteket es sajatvektorait!*

$$a)\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b)\ \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad d)\ \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Vajon mennyi lehet $\exp(xA)$ *?*

Megoldas 34. *c) Sajatertek:*

$$0 = \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} \qquad \Longrightarrow \qquad \lambda_1 = 2.$$

Az egyetlen fuggetlen sajatvektor:

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \textit{vagyis} \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Viszont ki tudjuk szamolni $\exp(xA)$*-t:*

$$\begin{aligned}
\exp(xA) &= \exp\left[\begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}\right] \\
&= \exp\begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 + \cdots\right] \\
&= \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix}\begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2x} & 3xe^{2x} \\ 0 & e^{2x} \end{pmatrix}
\end{aligned}$$

Itt az elso, $\exp(C + D) = \exp(C) \cdot \exp(D)$ *tipusu atalakitast azert lehetett elvegezni, mert esetunkben* $[C, D] = CD - DC = 0$ *volt:*

$$\begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix}\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Az utolso elotti atalakitasnal pedig azt hasznaltuk ki, hogy

$$\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 = (3x)^2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Problema 41. *Oldd meg a*

$$\frac{d}{dt}\bar{y} = \begin{pmatrix} 5 & 6 \\ 0 & 5 \end{pmatrix} \bar{y}, \qquad \bar{y}(0) = \begin{pmatrix} 77 \\ 88 \end{pmatrix}$$

DE-t!

Megoldas 35.

$$\begin{aligned} \bar{y}(0) = \exp\left[t\begin{pmatrix} 5 & 6 \\ 0 & 5 \end{pmatrix}\right]\begin{pmatrix} 77 \\ 88 \end{pmatrix} &= \left[\exp\begin{pmatrix} 5t & 0 \\ 0 & 5t \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & 6t \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} 77 \\ 88 \end{pmatrix} \\ &= \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} 1 & 6t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 77 \\ 88 \end{pmatrix}. \end{aligned}$$

Tetel 6. *Jordan normal forma: Minden A complex matrixhoz letezik olyan S invertalhato matrix, hogy $SAS^{-1} = N$, ahol N egy blokk diagonalis matrix, amelyben a diagonalis blokkok alakja (itt a haromdimezios illusztrativ esetet prezentaljuk):*

$$J_3 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

J_3 exponencialis fuggvenye:

$$\begin{aligned} e^{tJ_3} &= \exp\begin{pmatrix} \lambda t & 0 & 0 \\ 0 & \lambda t & 0 \\ 0 & 0 & \lambda t \end{pmatrix} \cdot \exp t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} 1 & t & t^2/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Mi a standard bazis tulajdonsaga egy ilyen J_3 matrix eseteben? (Itt $e_1 = (1,0,0)^T, e_2 = (0,1,0)^T, e_3 = (0,0,1)^T$.)

$$J_3 e_1 = \lambda e_1, \quad J_3 e_2 = \lambda e_2 + e_1, \quad J_3 e_3 = \lambda e_3 + e_2.$$

Tehat az A matrix λ sajaterteku 3d Jordan blokkjahoz tartozo specialis vektorokat a kovetkezo egyenletrendszer megoldasa detektalja:

$$Av_1 = \lambda v_1, \quad Av_2 = \lambda v_2 + v_1, \quad Av_3 = \lambda v_3 + v_2,$$

ahol azt, hogy a $3d$ block nem resze egy nagyobb blokknak, az garantalhatna, hogy az $Av_4 = \lambda v_4 + v_3$ egyenletnek mar nincs megoldasa.

Problema 42. *Csillapitott harmonikus oszcillator: $y'' = -y - \alpha y'$. Ird fel a DE karakterisztikus egyenletet, es hatarozd meg, hogy milyen α ertek eseten esnek egybe a gyokei! Ird fel a DE altalanos megoldasat!*
Ird at a DE-t egy elsorendu renszerre, es vizsgald meg az egyutthatomatrix Jordan dekompoziciojat!

Megoldas 36. • *Karakterisztikus egyenlet:*

$$y'' = -y - \alpha y' \quad \Longrightarrow \quad \lambda^2 = -1 - \alpha\lambda \quad \Longrightarrow \quad \lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

Egy gyok van, ha $\alpha = \pm 2$. Mi a $\alpha = 2$ esetet vizsgaljuk, ekkor $\lambda = -1$. Az altalanos megoldas:

$$y_{alt} = C_1 e^{-t} + C_2 t e^{-t}.$$

- *Ugyanez elsorendu DE rendszerkent:*

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = A\begin{pmatrix} y \\ v \end{pmatrix}$$

A egyetlen fuggetlen sajaterteke, sajatvektora:

$$\lambda = -1, \qquad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

A Jordan normalforma:

$$Av_1 = \lambda v_1,$$
$$Av_2 = \lambda v_2 + v_1$$
$$\textit{tehat}$$
$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$J = S^{-1}AS = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Ennek alapjan

$$\exp(tA) = \exp(tSJS^{-1}) = S\exp(tJ)S^{-1} = S\begin{pmatrix} e^{-x} & xe^{-x} \\ 0 & e^{-x} \end{pmatrix}S^{-1}.$$

A megoldas

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \exp(tA)\begin{pmatrix} y(0) \\ v(0) \end{pmatrix}.$$

A csillapitott harmonikus oszcillator

$$y'' = -y - \alpha y', \qquad \frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\alpha \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}$$

fazisportrei: A nyilak a ket utolso abran a kovetkezo matrixok sajatvektorait es sajatertekeit jelolik:

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ -1 & -2.2 \end{pmatrix}.$$

Az elso abran, az alulcsillapitott oszcillatornal sajnos egyszerre ket dolog is tortenik. A rendszer oszcillal az origo korul, de a csillapitas miatt egyre kozeledik is az origohoz. Hogy jobban attekinthessuk a helyzetet, elimenilaljuk a csillapitas, vagyis az egyutthatomatrixat a DE-nek megvaltoztatjuk:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -0.5 \end{pmatrix} \quad \to \quad A' = \begin{pmatrix} 0 & 1 \\ -1 & -0.5 \end{pmatrix} - \begin{pmatrix} -0.25 & 0 \\ 0 & -0.25 \end{pmatrix},$$

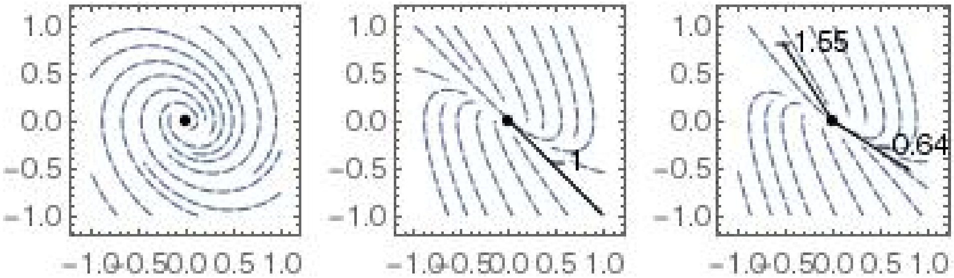


Figure 7.2: Harom fajta harmonikus oszcillator $y'' = -y - \alpha y'$: alul, kritikusan es tul csillapitott. A csillapitasi parameter $\alpha = 0.5$, 2, 2.2. A masodik es a harmadik abran a megoldasgorbek a nyilak iranyaba mozognak az origo fele $e^{\lambda t}$ faktorokkal szorozva. Mivel az abrakon feltuntetett λ fakorok negativak, igy a megoldasok az origo fele aramlanak.

vagyis A-bol levonjuk a sajaterteke valos reszet megszorozva az egysegmatrixszal. Ez ugyanazt a keringest irja le az origo korul, csak eppen nulla csillapitassal. A' sajatvektorai ugyanazok, mint A-nak, a sajatertekek viszont megnovekedtek 0.25-tel, vagyis a valos reszuk nulla lett. A' sajatrendszere:

$$\text{sajatertekek: } 0 \pm 0.96i, \quad \bar{v}_1 = \begin{pmatrix} 0.70 \\ -0.17 + 0.68i \end{pmatrix}, \quad \bar{v}_2 = \begin{pmatrix} 0.70 \\ -0.17 - 0.68i \end{pmatrix}.$$

Mivel az eredeti problema csak valos szamokat tartalmazott, igy a sajatertekek es sajatvektorok komplex konjugalt parokban erkeznek. Mi koze a csillapitott oszcillator elso abrajanak ezekhez a sajatvektorokhoz? A valos megoldasa

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = A' \begin{pmatrix} y \\ v \end{pmatrix}$$

a DE-nek

$$\begin{aligned} \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} &= Ce^{0.96it} \begin{pmatrix} 0.70 \\ -0.17 + 0.68i \end{pmatrix} + \bar{C}e^{-0.96it} \begin{pmatrix} 0.70 \\ -0.17 - 0.68i \end{pmatrix} \\ &= 2|C| \cos(0.96t + Arg(C)) \begin{pmatrix} 0.70 \\ -0.17 \end{pmatrix} + 2|C| \sin(0.96t + Arg(C)) \begin{pmatrix} 0 \\ 0.68 \end{pmatrix}, \end{aligned}$$

ahol $C = |C|e^{i \cdot Arg(C)}$. Tehat a megoldasgorbek ellipszisek a $\Re \bar{v}_1$, $\Im \bar{v}_1$ vektorok altal generalt koordinatarendszerben.

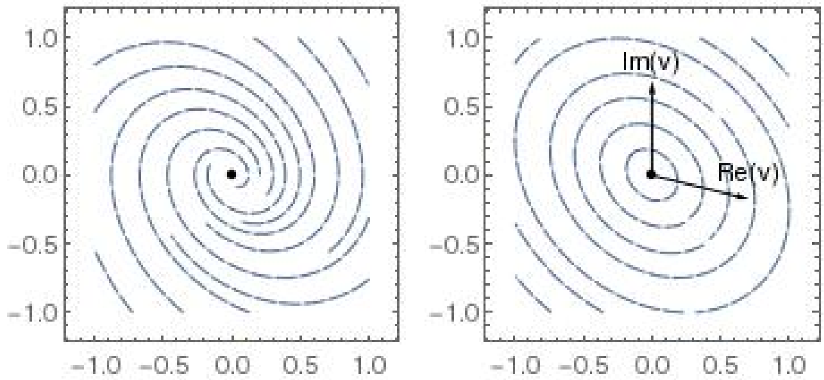


Figure 7.3: Ha a csillapitott harmonikus oszcillatorbol (elso abra) eliminaljuk a csillapitast (masodik abra), akkor periodikus, elliptikus trajektoriakat kapunk. A valos es a kepzetes reszei A (vagy A') sajatvektorainak jelolik ki az ellipszisek tengelyeit.

Chapter 8

Homogeneous linear systems

Homogeneous and inhomogeneous linear DE:

$$\text{hom:} \quad \frac{d}{dt}\bar{y} = A(t)\bar{y}, \qquad \text{inhom:} \quad \frac{d}{dt}\bar{y} = A(t)\bar{y} + \bar{f}(t),$$

($\bar{y}$ vector, A matrix.)

1. Superposition principle: If

   $$\frac{d}{dt}\overline{y_1} = A(t)\overline{y_1} \quad \text{es} \quad \frac{d}{dt}\overline{y_2} = A(t)\overline{y_2}$$

   then

   $$\frac{d}{dt}\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right) = A(t)\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right),$$

   So linear combinations of solutions are solutions, too

2. General solution of inhom. lin. DE: If

   $$\frac{d}{dt}\overline{y}_{hom} = A(t)\overline{y}_{hom} \quad \text{es} \quad \frac{d}{dt}\overline{y_{in}} = A(t)\overline{y_{in}} + \bar{f}(t),$$

   then

   $$\frac{d}{dt}\left(\overline{y}_{hom} + \overline{y_{in}}\right) = A(t)\left(\overline{y}_{hom} + \overline{y_{in}}\right) + \bar{f}(t).$$

   So the general solution can be written as a sum of the general solution $\overline{y}_{hom}$ of the hom. equation and a particular solution $\overline{y_{in}}$ of the inhom. equation: $\overline{y}_{hom} + \overline{y_{in}}$.

3. Linear input-output relation: If

   $$\frac{d}{dt}\overline{y}_1 = A(t)\overline{y}_1 + \bar{f}_1(t) \quad \text{es} \quad \frac{d}{dt}\overline{y}_2 = A(t)\overline{y}_2 + \bar{f}_2(t)$$

   then

   $$\frac{d}{dt}\left(\alpha_1\overline{y}_1 + \alpha_2\overline{y}_2\right) = A(t)\left(\alpha_1\overline{y}_1 + \alpha_2\overline{y}_2\right) + \left(\alpha_1\bar{f}_1(t) + \alpha_2\bar{f}_2(t)\right),$$

   so the $(\alpha_1\bar{f}_1(t) + \alpha_2\bar{f}_2(t))$ "input" generates $(\alpha_1\overline{y}_1 + \alpha_2\overline{y}_2)$ "output".

8.1 Time independent homogeneous systems

Problema 43. *Radioactive decay: $A \to B \to \emptyset$.*

$$\frac{d}{dt}\bar{y} = \frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2a \\ 2a-3b \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

This excercise is easy if $a(0) = 0, b(0) = 1$. Why is the case $a(0) = 1, b(0) = 0$ harder?

Megoldas 37. *if*

$$\bar{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ akkor } \dot{\bar{y}}(0) = -3 \cdot \bar{y}(0),$$

i.e. the position vector $\bar{y}$ and the velocity vector $\dot{\bar{y}}$ have the same direction, then the solution is

$$\bar{y}(t) = e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

They point into different directions in the case of $\bar{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so the solution curve $\bar{y}(t)$ is going to be a curve, not a straight line. However if

$$\bar{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ ekkor } \dot{\bar{y}}(0) = \begin{pmatrix} -2 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2 \cdot \bar{y}(0),$$

i.e. if $\bar{y}$ and $\dot{\bar{y}}$ are proportional vectors, then the evolution takes place on the line of the vector $\bar{y}(0)$, so the solution is

$$\bar{y}(t) = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

As the DE hom.lin., the general solution is

$$\bar{y}(t) = C_1 e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Tetel 7. *If the vectors $\bar{v}_1, \dots, \bar{v}_n$ form a basis and are eigenvectors of A (i.e. $A\bar{v}_i = \lambda_i \bar{v}_i$), then the general solution of the*

$$\frac{d}{dt}\bar{y} = A\bar{y}$$

DE is

$$\bar{y}(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t}\bar{v}_i.$$

Indeed both sides evaluate to the same

$$\frac{d}{dt}\left(\sum_{i=1}^{n} C_i e^{\lambda_i t}\bar{v}_i\right) = \sum_{i=1}^{n} C_i \lambda_i e^{\lambda_i t}\bar{v}_i,$$

$$A \cdot \left(\sum_{i=1}^{n} C_i e^{\lambda_i t}\bar{v}_i\right) = \sum_{i=1}^{n} C_i e^{\lambda_i t}\lambda_i \bar{v}_i.$$

8.1.1 Eigenvalues, eigenvectors

Eigenvalue (characteristic) equation:

$$\begin{aligned} A\bar{v} &= \lambda\bar{v}, \qquad \bar{v} \neq \bar{0} \\ A\bar{v} &= \lambda E\bar{v}, \\ (A-\lambda E)\bar{v} &= \bar{0}. \\ \text{However} \quad (A-\lambda E)\bar{0} &= \bar{0}, \end{aligned}$$

so the linear transformation $A - \lambda E$ is not one to one, it can not be inverted, so there is a nontrivial solution $\bar{v} \neq \bar{0}$ if and only if

$$\det(A-\lambda E) = 0.$$

This equation determines the possible values of λ. Then $\bar{v}$ can be found by solving a linear equation.

Tetel 8. *Assume that $\bar{v}_1, \ldots, \bar{v}_n$ form a basis and are eigenvectors of A (i.e. $A\bar{v}_i = \lambda_i\bar{v}_i$). Let S be the matrix formed by the column vectors $\bar{v}_i$. then*

$$S^{-1}AS = diag(\lambda_1, \ldots, \lambda_n) = D.$$

Here $diag(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix (i.e. the off diagonal elements are 0), and the diagonal element in the i^{th} row is λ_i.

Problema 44. *Find the eigenvectors and eigenvalues of A ! find the similarity transformation S which diagonalize A-t, i.e. $D = S^{-1}AS$, where D is diagonal! Express the vector v as a linear combination of the eigenvectors!*
Compute $A^{13}v$!

$$a)\ (7) \qquad b)\ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \qquad d)\ \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad e)\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$f)\ \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \qquad g)\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad h)\ \begin{pmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

here v is:

$$a)\ v = (8), \qquad b-f)\ v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \qquad g-h)\ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Megoldas 38. *b)*

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 3, \qquad \lambda_2 = 2$,
Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

This was a trivial problem, as A was diagonal.

d)

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Characteristic equation:

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$$

Eigenvalues: $\lambda_1 = 3, \qquad \lambda_2 = 2$.
Since A is triangular (the elements under the main diagonal are all 0), the eigenvalues are the diagonal elements.
Eigenvectors (for $\lambda_1 = 3$):

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix},$$

or

$$\begin{pmatrix} 2-3 & 1 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This yields $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$. *Pick a nonzero vector, for example let* $x = 1$.
Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Diagonalization:

$$D = S^{-1}AS = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Here S is comprised of the v_1 and v_2 column vectors, and

$$S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

How much is $A^{13}v$?

$$A^{13}v = (SDS^{-1})^{13}v = SD^{13}S^{-1}v$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3^{13} & 0 \\ 0 & 2^{13} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

or

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \alpha v_1 + \beta v_2,$$

where

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S^{-1} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix},$$

consequently

$$A^{13}v = A^{13}(\alpha v_1 + \beta v_2) = \alpha\lambda_1^{13}v_1 + \beta\lambda_2^{13}v_2 = 4 \cdot 3^{13} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \cdot 2^{13} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

f)

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2 + 3i, \qquad \lambda_2 = 2 - 3i$,
Eigenvectors:

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}$$

g) Since the block diagonal matrix A consists of the blocks d) and a), the combination of these two exercise yields

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 3, \qquad \lambda_2 = 2, \qquad \lambda_3 = 7, \qquad ,$
Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$D = S^{-1}AS$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

8.2 Time independent homogeneous systems

Problema 45. *Solve the DE*

$$\frac{d}{dx} y = Ay, \;\; y(0) = v$$

for the matrices and vectors of the previous problem! Find the general and particular solutions! Study the stability of the $\bar{y} = \bar{0}$ *fixed point! Compute* $\exp(xA)$ *! Express the particular solution with* e^{xA} *!*

Megoldas 39. *d) Exercise:*

$$\frac{d}{dt}\bar{y} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \bar{y}$$

Solution: Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = 2$. *Eigenvectors:*

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The general solution:

$$y_{alt}(x) = \sum_i C_i e^{\lambda_i x} v_i = C_1 e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

If

$$\bar{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then $C_1 = 4,\ C_2 = -1$, *and the particular solution is*

$$y_{part}(x) = 4e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Since there is an eigenvalue with positive real part, the $\bar{0}$ *fixed point is unstable The exponential function of* xA *:*

$$e^{xA} = e^{xSDS^{-1}} = Se^{xD}S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{3x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Then the particular solution is

$$y_{part}(x) = e^{xA} \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Problema 46. $y'' = -y$. *Write down the characteristic equation of the DE and find the general solution! Rewrite the DE as a first order system and solve it! Compare the two solutions!*

Megoldas 40. *1. The characteristic equation and its roots:*

$$y'' = -y \quad \Longrightarrow \quad \lambda^2 = -1 \quad \Longrightarrow \quad \lambda_1 = 0 + 1 \cdot i,\ \ \lambda_2 = 0 - 1 \cdot i,$$

so the general solution is

$$y = C_1 e^{(0+i)x} + C_2 e^{(0-i)x} = e^{0 \cdot x} \left(\widetilde{C_1} \cos(1 \cdot x) + \widetilde{C_2} \sin(1 \cdot x) \right)$$

Here $C_1 = \widetilde{C_1}/2 + \widetilde{C_2}/(2i),\ C_2 = \widetilde{C_1}/2 - \widetilde{C_2}/(2i)$.

2. The same as a first order DE:

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} = A \begin{pmatrix} y \\ v \end{pmatrix}$$

The eigenvalues and eigenvectors of A*:*

$$\lambda_1 = i,\ \ v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \lambda_1 = -i,\ \ v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

The general solution:

$$\begin{pmatrix} y \\ v \end{pmatrix} = C_1 e^{ix} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-ix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

The original harmonic oscillator problem contains only real numbers. So the imaginary part of the solution must be zero:

$$y \in \mathbb{R} \qquad \Rightarrow \qquad \overline{C}_2 = C_1, \qquad y(t) = 2|C_1| \cos(t + Arg(C_1)),$$

where $C_1 = |C_1| e^{i \cdot Arg(C_1)}$.

8.3 Jordan normal form

Unfortunately it is possible that there is no basis consisting of eigenvectors.

Problema 47. *Find the eigenvectors and eigenvalues of A !*

$$a)\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b)\ \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad d)\ \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

How much is $\exp(xA)$ *?*

Megoldas 41. *c) Eigenvalues:*

$$0 = \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} \quad \Longrightarrow \quad \lambda_1 = 2.$$

There is a single independent eigenvector:

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad so \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Nevertheless we can compute $\exp(xA)$ *:*

$$\begin{aligned} \exp(xA) &= \exp\left[\begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}\right] \\ &= \exp\begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 + \cdots\right] \\ &= \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2x} & 3xe^{2x} \\ 0 & e^{2x} \end{pmatrix} \end{aligned}$$

The $\exp(C+D) = \exp(C) \cdot \exp(D)$ *identity can be applied since* $[C,D] = CD - DC = 0$ *:*

$$\begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

We also used the nilpotency of the matrix

$$\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 = (3x)^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Problema 48. *Solve the*

$$\frac{d}{dt}\bar{y} = \begin{pmatrix} 5 & 6 \\ 0 & 5 \end{pmatrix} \bar{y}, \qquad \bar{y}(0) = \begin{pmatrix} 77 \\ 88 \end{pmatrix}$$

DE!

Megoldas 42.

$$\begin{aligned} \bar{y}(0) = \exp\left[t\begin{pmatrix} 5 & 6 \\ 0 & 5 \end{pmatrix}\right] \begin{pmatrix} 77 \\ 88 \end{pmatrix} &= \left[\exp\begin{pmatrix} 5t & 0 \\ 0 & 5t \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & 6t \\ 0 & 0 \end{pmatrix}\right] \begin{pmatrix} 77 \\ 88 \end{pmatrix} \\ &= \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 6t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 77 \\ 88 \end{pmatrix}. \end{aligned}$$

Chapter 9

Onadjungalt matrixok

Legyen ket komplex $\mathbb{C}^n$-beli vektor belso (valos esetben skalaris) szorzata

$$(\vec{u}, \vec{v}) = \sum_k \bar{u}_k v_k = \overline{\vec{u}^T} \vec{v}.$$

Egy A matrix elemeit a kovetkezokeppen nyerhetjuk ki:

$$A_{ij} = (\vec{e}_i, A\vec{e}_j).$$

Itt A-t a masodik vektorhoz kapcsoltuk, de persze megprobalhattuk volna ezt az elso $\vec{e}_i$ vektorral is. Definialjuk az A matrix A^* adjungaltjat a kovetkezokeppen:

$$(\vec{u}, A\vec{v}) = (A^*\vec{u}, \vec{v}) \qquad \forall \vec{u}, \vec{v}.$$

A^* matrixelemei:

$$(A^*)_{ij} = (\vec{e}_i, A^*\vec{e}_j) = \overline{(A^*\vec{e}_j, \vec{e}_i)} = \overline{(\vec{e}_j, A\vec{e}_i)} = \bar{A}_{ji}.$$

Tehat $A^* = \bar{A}^T$. A onadjungalt (hermitikus), ha $A = A^*$. Az ilyen matrixokra teljesul a kovetkezo tetel:

Tetel 9. *Ha $A = A^*$, akkor A sajatertekei valosak, tovabba letezik sajatvektorokbol allo ortonormalt bazis.*

1.

$$Av = \lambda v \quad \Longrightarrow \quad \lambda \in \mathbb{R}.$$
$$(v, Av) = (v, \lambda v) = \lambda(v, v),$$
$$(v, Av) = (A^*v, v) = (Av, v) = (\lambda v, v) = \bar{\lambda}(v, v).$$

   Mivel $(v, v) \neq 0$, igy $\lambda = \bar{\lambda}$, tehat a sajtertekek valosak.

2.

$$Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_2 v_2,\ \lambda_1 \neq \lambda_2 \quad \Longrightarrow \quad (v_1, v_2) = 0.$$
$$(v_1, Av_2) = (v_1, \lambda_2 v_2) = \lambda_2(v_1, v_2),$$
$$(v_1, Av_2) = (Av_1, v_2) = (\lambda_1 v_1, v_2) = \bar{\lambda}_1(v_1, v_2) = \lambda_1(v_1, v_2).$$

   Tehat $\lambda_2(v_1, v_2) = \lambda_1(v_1, v_2)$, igy $(v_1, v_2) = 0$. Vagyis kulonbozo sajatertekehez tartozo sajatvektorok ortogonalisak egymasra.

3. Az ortonormalt bazis konstrukcioja:
   Egy v_1, λ_1 sajatvektor-sajatertek par mindig van, mivel a $\det(A-\lambda E) = 0$ egyenletnek mindig van lagalabb egy λ_1 gyoke. Legyen W_1 a v_1 vektorra meroleges vektorok altere. Azt allitjuk, hogy ekkor $AW_1 \subset W_1$. Valoban, ha $w_1 \in W_1$ (vagyis $(w, v_1) = 0$), akkor

$$(Aw_1, v_1) = (w, Av_1) = (w_1, \lambda_1 v_1) = \lambda_1 (w_1, v_1) = 0 \cdot \lambda_1 = 0.$$

   Ezutan vegyuk A megszoritasat W_1-re (ahol W_1 dimezioja mar eggyel kevesebb mint az eredeti vektortere). W_1-ben talalhatunk egy uj, v_1-re ortogonalis v_2 sajatvektort. Ezt az eljarast folytatva megkonstrualhatjuk a keresett ortonormalt sajatvektor bazist.

Problema 49. *Mennyi a kovetkezo matrixok adjungaltja?*

$$\begin{pmatrix} 77 \end{pmatrix}, \quad \begin{pmatrix} -i \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} -3i & 4i \\ 5i & -6i \end{pmatrix}, \quad \begin{pmatrix} 2-i & 5+7i \\ 2+i & 5+7i \end{pmatrix}.$$

A linearis homogen rendszereknel a ket motivacios pelda a radioaktiv bomlas, illetve a csillapitott harmonikus oszcillator volt. Sajnos az egyutthatomatrix egyik esetben sem volt onadjungaltat. Igazabol a gyakorlatban inkabb az antihermitikus $A^* = -A$ matrixok fordulnak elo. Viszont ekkor $(iA)^* = iA$, tehat iA mar onadjungalt.

Problema 50. *Legyen*

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = A\bar{y}(t), \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix}.$$

Oldd meg a DE-t, es ellenorizd, hogy

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = R(t) \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix},$$

tehat $\bar{y}(t)$-t a kezdeti feltetelnek t szogu (radianban mert) $R(t)$ elforgatasaval kapjuk meg.
Mennyi A^2 ? Probald ki ennek alapjan kiszamolni $R(t) = e^{tA}$-t! Hasonlitsd ezt ossze $e^{it} = \cos(t) + i\sin(t)$ Taylor soros kiszamitasaval!

Problema 51. *Forogjon az $\bar{y}$ vektor egysegnyi szogsebesseggel az $\bar{n}$ egysegvektor altal meghatarozott tengely korul az oramutato jarasaval szemben. Mennyi A, ha*

$$\frac{d}{dt}\bar{y}(t) = A\bar{y}(t).$$

Megoldas 43. *1. Egy kis Δt idotartam alatt az $\bar{y}$ vektor megvaltozasa*

$$\Delta\bar{y} = \Delta t \cdot \bar{n} \times \bar{y}.$$

2. Tehat

$$\frac{d}{dt}\bar{y} = \bar{n} \times \bar{y} = \begin{pmatrix} n_2 y_3 - n_3 y_2 \\ n_3 y_1 - n_1 y_3 \\ n_1 y_2 - n_2 y_3 \end{pmatrix} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Megjegyzes: Ha $\bar{n}$ nem lenne egysegvektor, akkor is egy elforgatas csoportot kapnank az $\bar{n}$ tengely korul, de ekkor a szogsebesseg $||\bar{n}||$ lenne. Igy parba tudjuk allitani a 3d vektorokat es az antiszimmetrikus matrixokat

$$\bar{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \Longleftrightarrow \qquad A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \tag{9.1}$$

Problema 52. *Mi tortenne vegul az $\bar{y}$ vektorral, ha a kovetkezoket csinaljuk vele:*

1. *Forgassuk el $\bar{y}$-t az $\bar{B}$ vektor korul $-t||\bar{B}||$ szoggel,*
2. *Forgassuk el $\bar{y}$-t az $\bar{A}$ vektor korul $-t||\bar{A}||$ szoggel,*
3. *Forgassuk el $\bar{y}$-t az $\bar{B}$ vektor korul $t||\bar{B}||$ szoggel,*
4. *Forgassuk el $\bar{y}$-t az $\bar{A}$ vektor korul $t||\bar{A}||$ szoggel.*

Ha $t \approx 0$, akkor milyen (nagyon kicsi, t^2 nagysagrendu) elforgatast kapunk, es mi koze az eredmenynek az $\bar{A} \times \bar{B}$ vektorialis szorzashoz?

Problema 53. *Legyen*

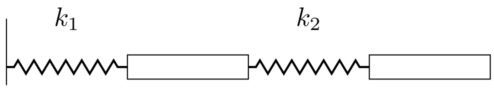
$$\frac{d}{dt}\bar{y}(t) = A\bar{y}(t),$$

es tegyuk fel hogy $\bar{y}(t)$ normaja (vagyis $||\bar{y}(t)|| = (\bar{y}(t), \bar{y}(t))^{1/2}$) allando:

$$\frac{d}{dt}||\bar{y}(t)|| = 0.$$

Mutasd meg, hogy ekkor $A^* = -A$, illetve a valos esetben $A^T = -A$.

Problema 54. *Vegyuk a kovetkezo, ket egysegnyi tomegbol allo, $k_{1,2} > 0$ rugoallandoju csillapitatlan tomeg-rugo rendeszert:*



1. *A rendszer energiaja (kinetikus+potencialis) :*

   $$H(y_1, y_2, \dot{y}_1, \dot{y}_2) = \frac{1}{2}\left(\dot{y}_1^2 + \dot{y}_2^2\right) + \frac{1}{2}\left(k_1 y_1^2 + k_2(y_1 - y_2)^2\right)$$

   Ird fel a Hamiltonikus

   $$\frac{d}{dt}y_i = \frac{\partial H}{\partial \dot{y}_i}, \qquad \frac{d}{dt}\dot{y}_i = -\frac{\partial H}{\partial y_i}$$

   egyenleteket, majd ellenorizd, hogy Newton harmadik torvenye ugyanezeket az egyeleteket adja!

2. *Mennyi A, ha*

   $$\frac{d^2}{dt^2}\bar{y} = -A\bar{y} \qquad ?$$

Mutasd meg, hogy esetunkben A onadjungalt! (Esetunkben A sajatertekei pozitivak lesznek.)
Legyen v_i egy A-nak a λ_i sajatertekeihez tartozo bazisvektorok rendszere.
Mutasd meg, hogy ekkor ennek a masodrendu DE-nek az altalanos megoldasa

$$\bar{y}(t) = \sum_i C_i \cos\left(\sqrt{\lambda_i} t\right) v_i + S_i \sin\left(\sqrt{\lambda_i} t\right) v_i.$$

3. *Oldd meg a DE-t es abrazol a megoldast, ha*

$$k_1 = 11,\ k_2 = 22,\ y_1(0) = 33,\ y_2(0) = 44,\ \dot{y}_1(0) = 55,\ \dot{y}_2(0) = 66.$$

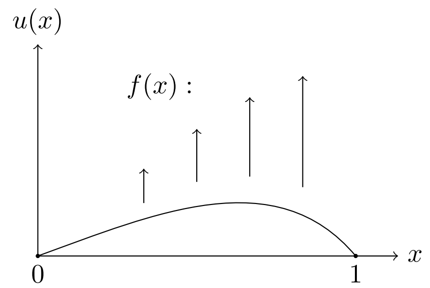
4. *Hogyan kellene modositani ezeket a szamitasokat, ha a feladat ket kulonbozo m_1 es m_2 tomegu test mozgasarol szolna?*

Chapter 10

Inhomogen linearis rendszerek

10.1 Egy kifeszitett ruhaszaritodrot alakjarol

Mennyi egy erosen elofeszitett (ezt a feszultseget hasznaljuk majd egysegkent) vizszintes hur $u(x)$ vertikalis elmozdulasa egy helyfuggo $f(x)$ vertikalis eroterben? A hur ket vege rogzitett, tehat $u(0) = u(1) = 0$.



A kovetkezo problemakat fogjuk vizsgalni:

1.
$$\text{Eroegyensuly} \quad \Longleftrightarrow \quad \Delta u = u''(x) = -f(x). \tag{10.1}$$

2. A hur gravitacios es elasztikus energiaja:
$$\text{Energia}[u] = \int_0^1 \frac{1}{2}[u'(x)]^2 - f(x)u(x)\,dx, \tag{10.2}$$
tehat (10.1) megoldasa ekvivalens (10.2) minimalizalasaval.

3. Hogyan tudjuk megoldani (10.1)-t, ha a derivaltakat differenciakkal helyetesitjuk (veges differenciak modszere) ? (Mas szoval az u fuggvenyt kozelitjuk egy, u-nak nehany x_i pontban kiszamitott ertekeibol allo vektorral.) Hogyan fejezhetjuk ki (10.1) megoldasat, ha tudjuk u-t abban az esetben, ha az f ero egyetlen pontra koncentralodik vegtelen erosuruseggel (impulzusvalasz, Green-fuggveny) ?

4. Hogyan tudjuk (10.2)-t kozelitoleg minimalizalni, ha feltesszuk, hogy u egy darbonkent egyenes (affin) folytonos fuggveny? (Veges elem modszer.)

5. Hogyan mukodik a (10.1) es (10.2) egyenletek kozotti ekvivalancia altalanosabban, azaz hogyan kereshetjuk meg egy

$$\mathrm{S}[u] = \int_0^1 L(x, u, u')\, dx$$

funkcional kritikus pontjait? (Variacioszamitas, Euler-Lagrange egyenletek.)

6. Hogyan magyarazza a feladat szimmetriaja (eltolas es elforgatas (tukrozes egy dimenzioban)) a Δ Laplace operator felbukkanasat? Mi egy szimmetria definicioja es hogyan tudjuk a szimmetriat a gyakorlatban kihasznalni?

10.1.1 Poisson egyenlet

Eroegyensuly:

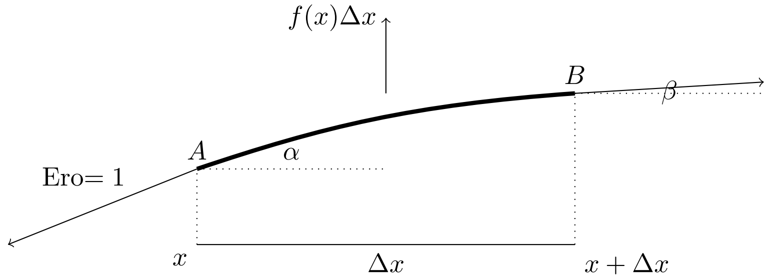


Figure 10.1: Az A es B pontokban hato egysegnyi ero fuggoleges komponenseit ellensulyozza az $f(x)\Delta x$ fuggoleges terheles.

$$\begin{gathered} f(x)\Delta x - 1 \cdot \sin(\alpha) + 1 \cdot \sin(\beta) = 0, \\ \cos(\alpha) \approx 1, \quad \alpha \approx \sin(\alpha) \approx \tan(\alpha), \quad \text{stb.} \\ \tan(\alpha) = u'(x), \quad \tan(\beta) = u'(x + \Delta x) \approx u'(x) + u''(x)\Delta x. \end{gathered}$$

Tehat

$$u''(x) = -f(x), \tag{10.3}$$

ami az egy dimenzios Poisson egyenlet. Magasabb dimenzioban:

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}, \qquad \Delta\phi(\bar{x}) = -f(\bar{x}).$$

10.1.2 Dirichlet elv

Mennyi az energiavaltozasa az $\overline{AB}$ hurdarabkanak a nyugalmi $u = 0$ helyzethez kepest?

1. A vertikalis $f(x)\Delta x$ ero $f(x)\Delta x \cdot u(x)$ munkat vegez.
2. A hurdarabkanak Δx hossza megno:

$$|\overline{AB}| = \sqrt{\Delta x^2 + [u'(x)\Delta x]^2} = \Delta x\sqrt{1 + [u'(x)]^2} \approx \Delta x + \frac{[u'(x)]^2}{2}\Delta x.$$

   Mindez egysegnyi hurfeszultseg elleneben tortenik.

Thehat a hur energiavaltozasa

$$\text{Energia}[u] = \int_0^1 \frac{1}{2}[u'(x)]^2 - f(x)u(x)\,dx. \tag{10.4}$$

$E[u]$ minimalizalasa az $u(0) = u(1) = 0$ feltetelek mellett ekvivalens az

$$u''(x) = -f(x), \quad , u(0) = u(1) = 0 \tag{10.5}$$

peremertekfeledat megodasaval.

10.2 Variacioszamitas, Euler-Lagrange egyenletek

Hogyan minimalizaljuk az

$$E[u] = \int_0^1 \frac{1}{2}[u'(x)]^2 - f(x)u(x)\,dx = \int_0^1 \frac{1}{2}L(x, u, u')\,dx$$

energia funkcionalt az $u(0) = u(1) = 0$ peremfeltetelek mellett?

Egy kritikus u pontban

$$\begin{aligned} E[u + \delta u] - E[u] &\approx \int_0^1 \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}(\delta u)'\,dx \\ &= \left.\frac{\partial L}{\partial u'}\right|_0^1 + \int_0^1 \left[\frac{\partial L}{\partial u} - \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right)\right] \delta u\,dx \end{aligned}$$

Mivel ez teljesul barmely $\delta u : \delta u(0) = \delta u(1) = 0$ variaciora, igy

$$\frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) - \frac{\partial L}{\partial u} = 0.$$

Ez az Euler-Lagrange egyenlet.

Ugyanez tobbdimenzioban: $L = L(\bar{x}, \phi^\mu(\bar{x}), \partial_{x_i}\phi^\mu(\bar{x}))$,

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial(\partial_{x_i}\phi^\mu)}\right) - \frac{\partial L}{\partial \phi^\mu} = 0.$$

Problema 55. *Ird fel az EL egyenleteket a kovetkezo L Lagrange fuggvenyekre:*

1. *Egy $V(\bar{x})$ potencialmezoben mozgo pont $\bar{x}(t)$ palyaja:*

$$L = \frac{1}{2}\sum_i \dot{x}_i^2 - V(\bar{x}).$$

2. *Egy* $\bar{A}(\bar{x})$ *magneses vektorpotencial mezojeben mozgo pont* $\bar{x}(t)$ *palyaja:*

$$L = \frac{1}{2}\sum_i \dot{x}_i^2 + \bar{A}(\bar{x}) \cdot \dot{\bar{x}}.$$

3. *Egy monocikli* $x(t), y(t), \phi(t)$ *koordinatai:*

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\phi}^2\right) + \lambda(t) \cdot (\dot{x}\sin\phi - \dot{y}\cos\phi)\,.$$

Itt a λ*-hoz tartozo EL egyenlet biztositja hogy a sebessegvekor* $(\dot{x}, \dot{y})$ *iranya ugyanaz mint a* $(\cos(\phi), \sin(\phi))$ *vektore.*

10.2.1 Veges differenciak

Probaljuk a vegtelen dimenzios $Fun([0,1])$ vektorterben lako u fuggvenyt a $x_i = i\Delta x$, $i = 1, \ldots, N-1$ helyeken (itt $N = 1/\Delta x$) megmert ertekeibol allo

$$\vec{u} = (u(1 \cdot \Delta x, \ldots, u(i \cdot \Delta x), \ldots, u((N-1)\Delta x))^T = (u_1, \ldots, u_{N-1})^T$$

vektorral kozeliteni, amelynek a komponensei $u_i = u(x_i) = u(i \cdot \Delta x)$.

Skalar (belso) szorzat: Legyen az igy kapott $\mathbb{R}^{N-1}$ Euklideszi vektorteren a skalarszorzat

$$(\vec{u}, \vec{v}) = \sum_{i=1}^{N-1} u_i v_i\,\Delta x. \tag{10.6}$$

Ekkor

$$\lim_{N\to\infty} \sum_{i=1}^{N-1} u_i v_i\,\Delta x = \int_0^1 u(x)v(x)\,dx, \tag{10.7}$$

tehat definialjuk a skalarszorzatat az u es v fuggvenyeknek:

$$(u, v) = \int_0^1 u(x)v(x)\,dx. \tag{10.8}$$

Megjegyzes: Ha a vektorok es a fuggvenyek komplexek, akkor a helyes (pozitiv definite) definicio:

$$(\vec{u}, \vec{v}) = \sum_{i=1}^{N-1} \bar{u}_i v_i\,\Delta x, \qquad (u, v) = \int_0^1 \overline{u(x)}v(x)\,dx. \tag{10.9}$$

A skalarszorzat segitsegevel bevezethetjuk a fuggvenyek normajat es tavolsagat:

$$||u||_2 = (u, u)^{1/2} = \left(\int_0^1 \overline{u(x)}u(x)\,dx\right)^{1/2} = \left(\int_0^1 |u(x)|^2\,dx\right)^{1/2},$$
$$d(u, v) = ||u - v||_2.$$

Az igy kapott ter "lezartjat" nevezzuk $\mathcal{H} = L^2([0,1], dx)$ Hilbert ternek, vagyis a $[0,1]$ intervallumon negyzetesen integralhato fuggvenyek terenek.

Megjegyzes: Eddig nem specifikaltuk azt, hogy mit is ertunk a Fun fuggvenyter alatt. Esetunkben ez lehetne $C^2([0,1])$ vagyis a ketszer derivalhato fuggvenyek tere, hiszen ebben laknanak a Poisson egyenlet klasszikus (vagyis a DE teljesul minden x-re) megoldasai. Vagy valaszthatnank Fun-nak pl. a darabonkent konstans, vagy darabonkent affine es folytonos fuggvenyek tereit. Az igy kapott terek nem lennenek teljesek, lennenek bennuk olyan $u^1, u^2, u^3, \ldots$ fuggvenysorozatok, hogy $||u^n - u^m|| \to 0$ ahogy $n, m \to \infty$ (az ilyen sorozatokat Cauchy sorozatoknak nevezzuk), de megsincs olyan $u \in Fun$, hogy $\lim_{n\to\infty} ||u^n - u|| = 0$. Ezert ezen terek lezartjait ugy definialjuk, mint a Cauchy sorozataiknak ekvilenciaosztalyait, ahol az u^n es a v^n sorozat akkor ekvivalens egymassal, ha $\lim_{n\to\infty} ||u^n - v^n|| = 0$.

Megjegyzes: Egy vektor vagy fuggveny normajat persze maskeppen is definialhatjuk, talan a ket legfontosabb varians:

$$||\vec{u}\,||_1 = \sum_i |v_i|, \qquad ||u||_1 = \int_0^1 |u(x)|\,dx,$$
$$||\vec{u}\,||_\infty = \max_i |u_i| \qquad ||u||_\infty = \max_x |u(x)|.$$

Azonban az igy kapott terekben nem igazan ertelmezheto az ortonormalt bazis fogalma, igy hianyozna a DE-k elmeletenek a kovetkezo alapveto segedeszkoze:

Fourier transzformacio: Ortogonalis sorfejtes:

Tetel 10. *Legyen $\vec{n}_0, \ldots, \vec{n}_{N-1}$ egy ortonormalt bazis egy V vektorterben. Ekkor ha*

$$\vec{v} = \alpha_0\vec{n}_0 + \cdots + \alpha_{N-1}\vec{n}_{N-1},$$

akkor

$$\alpha_i = (\vec{n}_i, \vec{v}).$$

Alkalmazzuk formalisan ezt a kepletet vegtelen dimenzioban.

1. Klasszikus Fourier transzformacio. A $c_0, c_1, c_2, \ldots, s_1, , s_2, \ldots$ fuggvenyek egy ortonormalt bazisat alkotjak $L^2([-\pi, \pi], dx)$-nek:

   $$c_0(x) = \frac{1}{\sqrt{2\pi}}, \quad c_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx), \quad s_n(x) = \frac{1}{\sqrt{\pi}}\sin(nx).$$

   Mivel

   $$-\frac{d^2}{dx^2}c_n(x) = n^2 c_n(x), \qquad -\frac{d^2}{dx^2}s_n(x) = n^2 s_n(x), \quad \forall n \in \mathbb{Z},$$

   a c_n, s_n fuggvenyek egy, a $D^2 = -\frac{d^2}{dx^2}$ operator sajatvektoraibol allo bazist alkotnak. D^2 (formalisan) onadjungalt, mivel $(f, D^2 g) = (D^2 f, g)$ periodikus sima f, g fuggvenyekre.

   Ha

   $$f(x) = \alpha_0 c_0(x) + \sum_{n=1}^{\infty} \alpha_n c_n(x) + \beta_n s_n(x)$$
   $$= \alpha_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \alpha_n \frac{1}{\sqrt{\pi}}\cos(nx) + \beta_n \frac{1}{\sqrt{\pi}}\sin(nx),$$

akkor

$$c_0 = (c_0, f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, dx,$$
$$c_n = (c_n, f) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(nx) f(x)\, dx,$$
$$s_n = (s_n, f) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin(nx) f(x)\, dx$$

2. Exponencialis forma. Az $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$ fuggvenyek egy ortonormalt bazisat alkotjak $L^2([-\pi, \pi], dx)$-nek. Ezek a fuggvenyek sajatvektorai a $D = -i\frac{d}{dx}$ (formalisan) onadjungalt operatornak:

$$(De_n)(x) = -i\frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}} e^{inx}\right) = n\left(\frac{1}{\sqrt{2\pi}} e^{inx}\right).$$
$$(f, Dg) = (Df, g)$$
$$= \int_{-\pi}^{\pi} \overline{f(x)}(-ig'(x))\, dx = \overline{f(x)}(-ig(x)) \mid_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \overline{if'(x)} g(x)\, dx$$
$$= \int_{-\pi}^{\pi} \overline{-if'(x)} g(x)\, dx,$$

ha f, g periodikus sima fuggvenyek.

Ha
$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}_n e_n(x) = \sum_{n \in \mathbb{Z}} \hat{f}_n \frac{1}{\sqrt{2\pi}} e^{inx},$$
akkor
$$\hat{f}_n = (e_n, f) = \int_{-\pi}^{\pi} \overline{e_n(x)} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

Itt a kalap a Fourier-tr. egyik tradicionalis jelolese.

$$||f||_2 = \left(\sum_{n \in \mathbb{Z}} \hat{f}_n e_n, \sum_{m \in \mathbb{Z}} \hat{f}_m e_m\right) = \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2 (e_n, e_n) = \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2,$$

(az ortonormaltsag miatt $(e_n, e_m) = 0$, ha $n \neq m$) tehat a tr. a negyzetesen integralhato fuggvenyeket a negyzetesen osszegezheto sorozatok ℓ_2 Hilbert terere kepezi le.

3. Szinusz transzformacio. Az $s_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \ldots$ fuggvenyek egy ortonormalt bazisat alkotjak $L^2([0, \pi], dx)$-nek. Tehat, ha

$$f(x) = \sum_{n=1}^{\infty} \alpha_n s_n(x),$$

akkor

$$\alpha_n = (s_n, f) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin(nx) f(x)\, dx.$$

Problema 56. *1. Mi a szinusz tr. koszinusz verzioja?*

2. *Irj fel hasonlo kepleteket egy tetszoleges $[a, b]$ intervallumra!*
3. *Vizsgald meg ez exponencialis tr. viselkedest az $[-\pi L, \pi L]$ intervallumon, ha $L \to \infty$!*
4. *Hogyan lehetne ezt a hataratmenetet alkalmazni arra, hogy egy nem periodikus $f(x)$ fuggvenyt kifejezzunk az e^{ipx}, $p \in \mathbb{R}$ fuggvenyek segitsegevel?*
5. *A jelfeldolgozasban jobban szeretnek az $L^2([0,1], dx)$ teren dolgozni. Ird fel itt a Fourier transzformaciokat!*

Megoldas 44. *3: $L^2([-\pi L, \pi L], dx)$ egy ortonormalt bazisa:*

$$e_n = \frac{1}{\sqrt{2\pi L}} e^{i\frac{n}{L}x}.$$

Fourier tr: ha

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n e_n(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{1}{\sqrt{2\pi L}} e^{i\frac{n}{L}x},$$

akkor

$$\hat{f}_n = (e_n, f) = \int_{-\pi L}^{\pi L} \overline{e_n(x)} f(x)\, dx = \frac{1}{\sqrt{2\pi L}} \int_{-\pi L}^{\pi L} e^{-i\frac{n}{L}x} f(x)\, dx$$

4: Legyen $p_n = \frac{n}{L}$, $\Delta p = p_{n+1} - p_n = \frac{1}{L}$, tovabba normalizaljuk egy kicsit maskepp a tr.-t, tovabba tekintsuk a $\hat{f}$ sorozatot ugy, mint egy diszkret pontokban adott $\tilde{f}$ fuggvenyt:

$$\tilde{f}(p_n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi L}^{\pi L} e^{-ip_n x} f(x)\, dx,$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{p_n \in \Delta p\cdot\mathbb{Z}} \tilde{f}(p_n) e^{ip_n x} \Delta p.$$

(Itt az egyetlen Δp faktorba olvaszottuk bele a ket $\sqrt{L}$ tenyezot.) Tehat az $L \to \infty$ hataresetben, vagyis a teljes valos tengelyen a Fourier tr:

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx,$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p) e^{ipx}\, dp.$$

Tobbdimenzios Fourier tr.

1. Vektorterek tenzor szorzata. Legyen az U vektorter egy ortonormalt bazisa $\bar{e}_1, \ldots, \bar{e}_n$, mig ugyanez V-re $\bar{f}_1, \ldots, \bar{f}_m$. Ekkor legyen U es V tenzor szorzata az $U \otimes V$ vektorter, amelynek ortonormalt bazisa

$$\bar{g}_{ij} = \bar{e}_i \otimes \bar{f}_j, \qquad (i,j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}.$$

Itt a $\otimes$ jeloles egy muveletet is jelol:

$$\otimes : U \times V \to U \otimes V,$$
$$\text{pl.:} \qquad (2\bar{e}_1 + 3\bar{e}_2) \otimes (4\bar{f}_1 + 5\bar{f}_2)$$
$$= 8\bar{e}_1 \otimes \bar{f}_1 + 10\bar{e}_1 \otimes \bar{f}_2 + 12\bar{e}_2 \otimes \bar{f}_1 + 15\bar{e}_2 \otimes \bar{f}_2.$$

Az eredmenyt hivhatjuk a ket vektor tenzor vagy kulso (*outer*, de nem *exterior, ek, wedge*) szorzatanak.

Problema 57. *Az, hogy a g_{ij} vektorok egy ortonomalt bazis alkotnak, azt jelenti, hogy*

$$(\bar{g}_{ij}, \bar{g}_{kl}) = (\bar{e}_i \otimes \bar{f}_j, \bar{e}_k \otimes \bar{f}_l) = \delta_{ik}\delta_{jl},$$

ahol a δ_{ij} Kronecker delta szimbolum, ami 1, ha $i = j$, amugy nulla.
Mutasd meg, hogy

$$(\bar{u}_1 \otimes \bar{v}_1, \bar{u}_2 \otimes \bar{v}_2) = (u_1, v_1) \cdot (u_2, v_2).$$

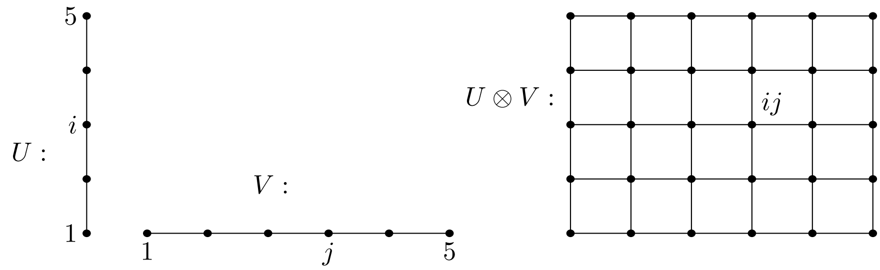


Figure 10.2: Az otdimenzios U vektorteret az ot pontban ertelmezett fuggvenyek alkotjak, mig V elemei hatdimenzios vektorok. Az i cimke az $\bar{e}_i$ bazisvektort reprezentalja. Az $U \otimes V$ vektorter $5 \times 6 = 30$ dimenzios, az ij cimke az $\bar{e}_i \otimes \bar{f}_j$ vektort reprezentalja.

2. Ket dimenzios Fourier tr.

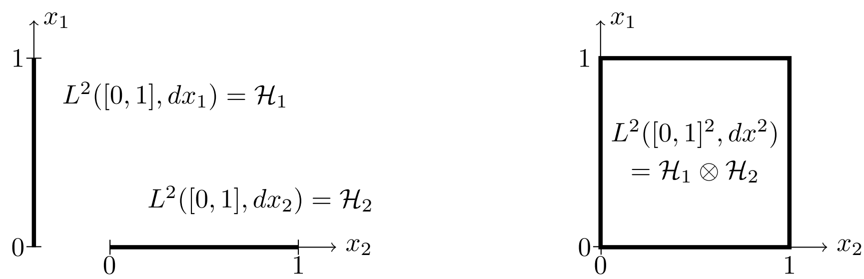


Figure 10.3:

A bazisvektorok $\mathcal{H}_1$-ben es $\mathcal{H}_2$-ben:

$$e_{n_1}(x) = e^{2in_1\pi x_1} \quad \text{es} \quad f_{n_2}(x) = e^{2in_2\pi x_2}, \qquad n_{1,2} \in \mathbb{Z}.$$

$\mathcal{H}_1 \otimes \mathcal{H}_2$ bazisvektorai:

$$g_{nm}(x_1, x_2) = e_n(x_1) f_m(x_2) = e^{2in\pi x_1} e^{2im\pi x_2} = e^{2i\pi(nx_1 + mx_2)}.$$

Tehat a kisse szisztematikusabb

$$e_{(n_1,n_2)}((x_1, x_2)) = e_{\bar{n}}(\bar{x}) = e^{2i\pi(n_1x_1 + n_2x_2)} = e^{2i\pi(\bar{n},\bar{x})}$$

jelolessel egy $f(x_1, x_2) = f(\bar{x})$ fuggvenyre a Fourier tr:

$$\hat{f}_{\bar{n}} = \int_{x_1=0}^{x_1=1} \int_{x_2=0}^{x_2=1} e^{-2i\pi(n_1x_1+n_2x_2)} f(x_1, x_2)\, dx_1 dx_2 = \int_D e^{-2\pi i(\bar{n},\bar{x})} f(\bar{x})\, d^2x,$$

$$f(\bar{x}) = \sum_{\bar{n}\in\mathbb{Z}^2} \hat{f}_{\bar{n}} e^{2i\pi(\bar{n},\bar{x})}.$$

(Itt D a $[0,1]\times[0,1]$ egysegnegyzet.)

Problema 58. *Hogyan mukodik az $\mathbb{R}^N$-en adott fuggvenyeken a Fourier integralis transzformacio?*

10.3 A Fourier transzformacio konvergenciajarol

Nehany fuggveny

Az $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$ fuggvenyek egy ortonormalt bazisat alkotjak $L^2 = L^2([-\pi,\pi], dx)$-nek. Legyen

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n e_n(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{1}{\sqrt{2\pi}} e^{inx},$$

es

$$\hat{f}_n = (e_n, f) = \int_{-\pi}^{\pi} \overline{e_n(x)} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx,$$

tovabba

$$f_N(x) = \sum_{n=-N}^{N} \hat{f}_n \frac{1}{\sqrt{2\pi}} e^{inx}.$$

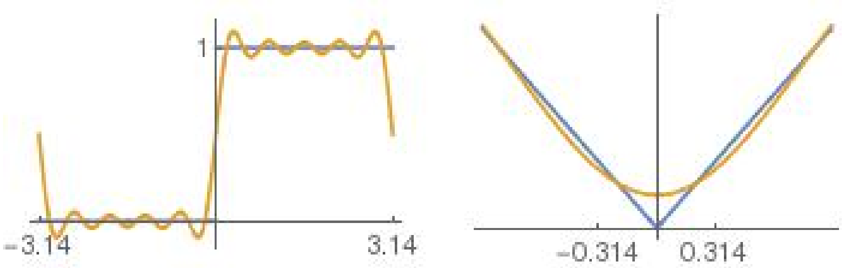


Figure 10.4: Az $N = 10$ kozelitese az egysegugras fuggvenynek, illetve az $N = 3$ kozelitese az abszolutertek $|x|$ fuggvenynek. Minel simabb egy f fuggveny, annal gyorsabban konvergal hozza f_N.

```
f1 = HeavisideTheta[x];
Ff1 = FourierSeries[f1, x, 10];
f2 = Abs[x];
Ff2 = FourierSeries[f2, x, 3];
res = GraphicsRow[{
   Plot[{f1, Ff1}, {x, -Pi, Pi}, PlotRange -> All, Ticks -> {{-3.14, 3.14}, {0, 1}}],
   Plot[{f2, Ff2}, {x, -Pi*0.3, Pi*0.3},
                  PlotRange -> All, Ticks -> {{-0.1*3.14, 0.1*3.14}, {}}]
  }]
Export["fourierseries.jpg", res]
```

A kovetkezo eredmenyek teljesulnek:

1. $$f \in L^2 \quad \Longrightarrow \quad \lim_{N\to\infty} ||f - f_N|| = \lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - f_N(x)|^2 \, dx = 0.$$

   Sajnos ez semmilyen garanciat nem tartalmaz $\lim_{N\to\infty} f_N(x) = f(x)$ teljesulesere.

2. Legyen f egy veges sok sima darabbol allo fuggveny. Ekkor

   (a) Ha f folytonos az x_0 pontban, akkor $\lim_{x\to x_0} f_N(x_0) = f(x_0)$.

   (b) Amugy egy szakadasi x_0 helynel

   $$\lim_{x\to x_0} f_N(x_0) = \frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right)$$

3. Ha f sokszor derivalhato, akkor a $\hat{f}_n$ Fourier egyutthatok gyorsan csokkennek. Pl. ha $f^{(3)} \in L^2$, akkor

   $$f^{(3)}(x) = \frac{d^3}{dx^3} \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{1}{\sqrt{2\pi}} e^{inx} = \sum_{n\in\mathbb{Z}} \left[(in)^3 \hat{f}_n\right] \frac{1}{\sqrt{2\pi}} e^{inx},$$

   tehat

   $$\sum_{n\in\mathbb{Z}} \left|(in)^3 \hat{f}_n\right|^2 = \sum_{n\in\mathbb{Z}} n^6 \left|\hat{f}_n\right|^2 < \infty.$$

4. Dirichlet mag.

   $$f_N(0) = \sum_{n=-N}^{N} \hat{f}_n \frac{1}{\sqrt{2\pi}} e^{in0}$$

   $$= \sum_{n=-N}^{N} \left[\int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} f(x) \, dx\right] \cdot \frac{e^{-in\cdot 0}}{\sqrt{2\pi}} = \int_{-\pi}^{\pi} D(x) f(x) \, dx,$$

   ahol

   $$D_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{-inx} = \frac{1}{2\pi} \frac{\sin\left[\left(N + \frac{1}{2}\right) x\right]}{\sin\left[\frac{1}{2} x\right]}.$$

```
dirichlet = Sum[Exp[-I n x], {n, -n1, n1}] // Simplify
dirichlet - Sin[(n1 + 1/2) x]/Sin[1/2 x] // Simplify
fejer = Sum[(1/(n2 + 1)) s, {n1, 0, n2}] // Simplify
fejer - (1/(n2 + 1)) (1 - Cos[(n2 + 1) x])/(1 - Cos[x]) // Simplify
res = GraphicsRow[{Plot[dirichlet /. n1 -> 25, {x, -Pi, Pi},
                        PlotRange -> All, Ticks -> {{-3.14, 3.14}, {-15, 50}}],
                   Plot[fejer /. n2 -> 25, {x, -Pi, Pi},
                        PlotRange -> All, Ticks -> {{-3.14, 3.14}, {0, 25}}  ]}]
Export["dirichlet.jpg", res]
```

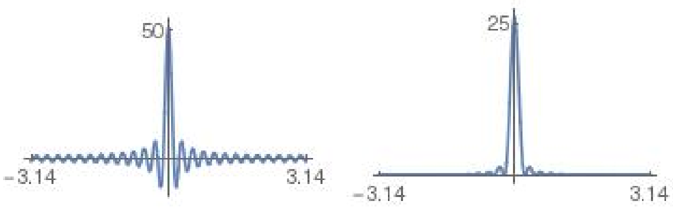


Figure 10.5: A D_N Dirichlet es F_N Fejer magok $N = 25$-re. Mindkettojuk konvergal a $\delta(x)$ Dirac delta fuggvenyhez, vagyis ha $f(x)$ sima, akkor $\int_{-\pi}^{\pi} D_N(x)f(x)\,dx = \int_{-\pi}^{\pi} F_N(x)f(x)\,dx \to f(0)$. Azonba ha f csak folytonos, akkor a Dirichlet mag eseteben nem feltetlenul teljesul a konvergencia.

5. Fejer (Lipot) mag. Atlagoljuk az $f_N(0)$ sorozat elso $K+1$ tagjat:

$$s_K(0) = \frac{1}{K+1}\sum_{N=0}^{K} f_N(0) = \int_{-\pi}^{\pi} F_K(x)f(x)\,dx,$$

ahol

$$F_K(x) = \frac{1}{K+1}\sum_{N=0}^{K} D_N(x) = \frac{1}{2\pi}\frac{1}{K+1}\frac{1-\cos[(K+1)x]}{1-\cos[x]}.$$

$F_K(x)$ nemnegativ, az integralja az $x = 0$ pont kornyezetere koncentralodik es egyenlo 1-gyel. Igy a kiatlagolt Fourier sora egy folytonos $f$fuggvenynek pontonkent konvergal $f(x)$-hez.

6. Haar Alfred ortogonalis rendszere (Haar wavelet). A Fourier tr. nagy elonye, hogy az $e_n \sim e^{inx}$ fuggvenyek diagonalizaljak a derivalas operatorat. Viszont ezek a fuggvenyek nem lokalizaltak, nem koncentralodnak egy pont kore. Ez nyilvan gondot okoz, ha olyan fuggvenyeket akarunk leirni, amelyek csak egy rovid szakaszon nem nullak.

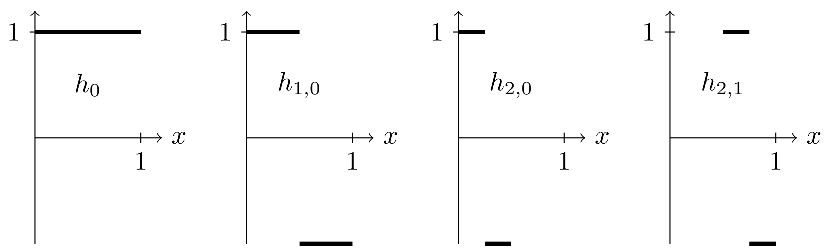


Figure 10.6: A h fuggvenyrenszer (az abran az elso negy szerepel) ortogonalis, de nincs normalizalva. A $h_{i,j}$ fuggvenyek mindegyike $h_{1,0}$ eltolt es az x iranyban osszenyomott variansa. Ha egy f fuggvenyt kifejtunk ezen bazis segitsegevel, akkor az abran lathato tagokat tartalmazo reszletoszeg az f fuggveny atlagait reprodukalja az $[0, 0.25], \ldots, [0.75, 1]$ szakaszokon. Igy, ha f folytonos, akkor ezen bazis szerinti ortogonalis sorfejtese f-nek pontonkent konvergal $f(x)$-hez.

A JPEG2000 standard ezt az otletet (ennek ketdimenzios, javitott valtozatat) hasznalja, mig a korabbi JPEG standard direkt modon a koszinusz transzformaciot hasznalta.

7. Shannon mintavetelezesi tetele: Legyen $\tilde{f}(p) = \int_{\mathbb{R}} e^{-2\pi i p t} f(t)\,dt$, es legyen $\tilde{f}(p) = 0$, ha $p \notin [-B, B]$. Ekkor az $f(k/(2B))$, $k \in \mathbb{Z}$ sorozatbol rekonstrualhato $f(t)$:

$$f(t) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2B}\right) sinc(2\pi Bt - \pi k),$$
$$\text{ahol} \quad sinc(t) = \begin{cases} \frac{\sin t}{t}, & \text{ha } t \neq 0 \\ 1 & \text{ha } t = 0. \end{cases}$$

Pl. ha egy mp3 fajlban a 22kHz frekvenciaig akarjuk megorizni a frekvenciakomponenseket, akkor masodpercenkent $44 \cdot 10^3$ mintaveltelre van szukseg.

Derivalas: A derivalas numerikus kozelitese:

$$u'(x) \approx \frac{1}{\Delta x}\left(u(x+\Delta x) - u(x)\right) \approx \frac{1}{\Delta x}\left(u(x) - u(x-\Delta x)\right)$$
$$\approx \frac{1}{2\Delta x}\left(u(x+\Delta x) - u(x-\Delta x)\right).$$

Problema 59. *Melyik a "legjobb" ezen kozelitesek kozul (probald meg grafikus eldonteni) ? Tipikusan egy ilyen kozelites hibajanak a viselkedese: hiba*$(\Delta x) \approx C \cdot \Delta x^{\alpha}$*, ha* $\Delta x \to 0$*. Mennyi* α *?*

Ha az $u(x)$ fuggvenyek altal alkotott vektorter vektorait az

$$\vec{u} = (u(0.25), u(0.5), u(0.75))^T = (u_1, u_2, u_3) \in \mathbb{R}^3$$

veges (esetunkben 3) dimenenzios vektorokkal kozelitjuk (itt $\Delta x = 1/4$), akkor u' approximacioi:

$$\frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$
$$\frac{1}{\Delta x}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$
$$\frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

(Ezek kozul a legjobb az utolso, mivel tipikusan pontosabb numerikusan, es ezenkivul antiszimmetrikus is.)

Derivalas a fuggvenyek vegtelen dimenzios vektortereben $\Longrightarrow$ *linearis transzformaciok, matrixok veges dimenzios vektorterekben.*

u'' kozelitese:

$$u''(x) \approx \frac{1}{\Delta x^2}(u(x+\Delta x) - 2u(x) + u(x-\Delta x)).$$

Tehat az

$$-u''(x) = f(x), \quad \text{vagy} \ -\frac{d^2}{dx}u = f, \qquad u(0) = u(1) = 0,$$

egyenlet numerikus kozelitese:

$$\frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \text{vagy} \quad \Delta x \cdot L\,\vec{u} = \vec{f}.$$

(Itt az extra Δx factort a skalarszorzat normalizacioja motivalja.) Ekkor

$$(\Delta x \cdot L)\,\vec{u} = \vec{f},$$
$$(\Delta x \cdot G)\,\vec{f} = \vec{u} = \Delta x \left(\Delta x^2 L\right)^{-1} \vec{f},$$

tehat

$$G = \left(\Delta x^2 L\right)^{-1} = \Delta x \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0.1875 & 0.125 & 0.0625 \\ 0.125 & 0.25 & 0.125 \\ 0.0625 & 0.125 & 0.1875 \end{pmatrix}.$$

Megjegyzes: Minek ez a szerencsetlenkedes a Δx faktorokkal? Kiszamolhatnank $\vec{u}$-t amikor $\vec{f}$-nek csak egyetlen *egysegnyi* nem nulla komponense lenne. Ekkor azonban az $\int_0^1 f(x)\,dx$ osszterheles csak $1 \cdot \Delta x$ lenne, ami a nullahoz tartana, ha $\Delta x \to 0$. Igy az $u(x)$ valasz is a nullahoz tartana. Ha viszont az egyetlen nem nulla komponens $1/\Delta x$, akkor az oszterheles $1/\Delta x \cdot \Delta x = 1$, tehat igy azt vizsgaljuk, hogy milyen valaszt ad a rendszer valamilyen egy pont kore koncentralodo egysegnyi osszterhelesre.

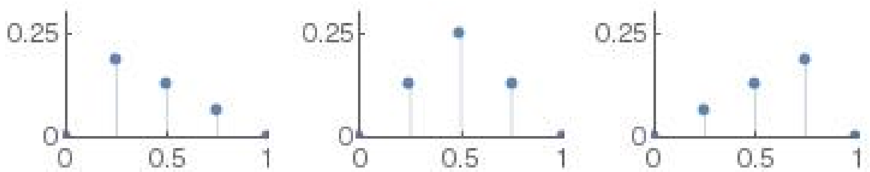


Figure 10.7: G harom oszlopvektora. Pl. a masodik abran a kozepso harom pont a $G_{i,2}$, $i = 1, 2, 3$ matrixelemek nagysagat jeloli.

Ugyanez az abra $N = 20$-ra:

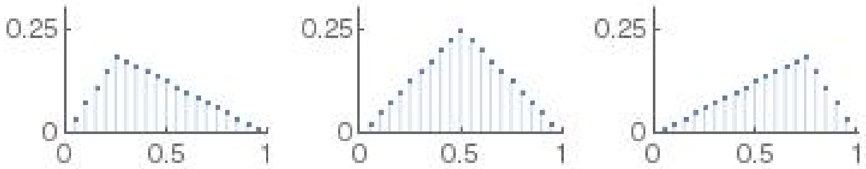


Figure 10.8: G harom (5,10,15-odik) oszlopvektora. Itt pl. az elso abra azt mondja meg, hogy ha az $\vec{f}$ vektornak minden komponense nulla kiveve az 5-odiket $1/\Delta x$ ertekkel, akkor mennyi $u_i = u(x_i)$. Ezek az abrak a $G(x,z)$ Green fuggvenyhez konvergalnak, ahol a z valtozo jeloli, hogy hol hat a concentralt egysegnyi erohatas (a fuggveny csucsanak a helye), mig x a vizszintes koordinata.

Ha $\Delta x \to 0$, akkor a G_{ij} matrixbol megalkothatjuk a (ugyanazzal a betuvel jelolt)

$$G(x_i, z_j) = G_{i,j}$$

fuggvenyt. Ekkor az $\vec{u} = (\Delta x \cdot G)\,\vec{f}$ osszefugges folytonos alakja

$$u(x) = \int_0^1 G(x,z) f(z)\, dx.$$

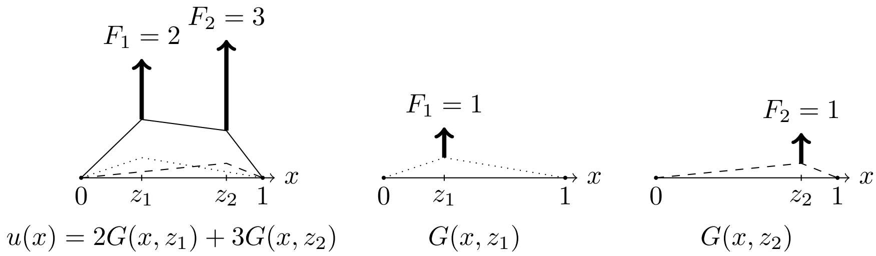


Figure 10.9: Az $-u''(x) = 2\delta(x - 1/3) + 3\delta(x - 4/5)$ egyenlet megoldasa. a megoldas linearis kombinacioja a $-G''_{xx}(x, z_i) = \delta(x - z_i)$ egyenletek megoldasainak.

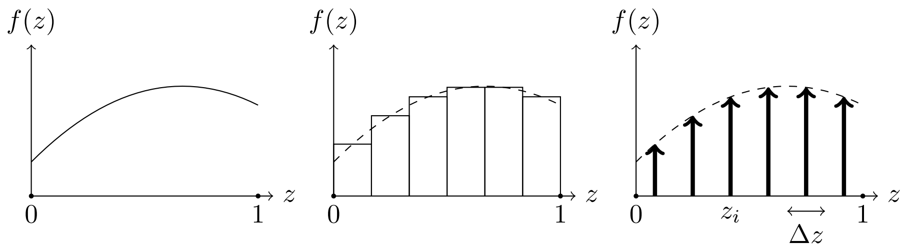


Figure 10.10: Egy $f(z)$ erosuruseg approximacioja Dirac delta fuggvenyek $\sum_i f(z_i)\Delta z \cdot \delta(z - z_i) \approx f(z)$ linearis kombinaciojaval.

Melyek G tulajdonsagai? Legyen $\delta_\epsilon^z(x)$ egy olyan (nemnegativ) fuggveny, amelyik csak a z pont $(z-\epsilon/2, z+\epsilon/2)$ kornyezeteben nem nulla, tovabba $\int_0^1 \delta_\epsilon^z(x)\,dx = 1$. Oldjuk meg az

$$-u''(x) = \delta_\epsilon^z(x), \qquad u(0) = u(1) = 0$$

DE-t! A z-tol fuggo u megoldas adja meg G_ϵ-t:

$$G_\epsilon(x,z) = u(x).$$

Ekkor a kovetkezo tulajdonsagai lesznek G_ϵ-nek:

1. u peremfeltetelei miatt
$$G_\epsilon(0,z) = G_\epsilon(1,z) = 0,$$
2. Ahol $\delta_\epsilon^z(x)$ nulla, ott $\partial_x^2 G_\epsilon(x,z) = 0$, vagyis ezeken az x helyeken $G_\epsilon(x,z)$ affin az x valtozoban.
3. Mivel $-u''(x) = \delta_\epsilon^z(x)$, igy
$$\int_{z-\epsilon}^{z+\epsilon} u''(x)\,dx = u'(z+\epsilon) - u'(z-\epsilon) = \int_{z-\epsilon}^{z+\epsilon} -\delta_\epsilon^z(x)\,dx = -1.$$

Tehat a $\epsilon \to 0$ hataresetben egy olyan $G(x,z)$ fuggvenyt keresunk, amelyre igazak a kovetkezok:

1. $G(0,z) = G(1,z) = 0$,
2. $G(x,z)$ affin a $[0,z]$ es $[z,1]$ intervallumokon,
3. $\lim_{x\to z+0} G(x,z)(x) = \lim_{x\to z-0} G(x,z)(x)$,
$\lim_{x\to z+0} G'_x(x,z) - \lim_{x\to z-0} G'_x(x,z) = -1$.

Grafikusan:

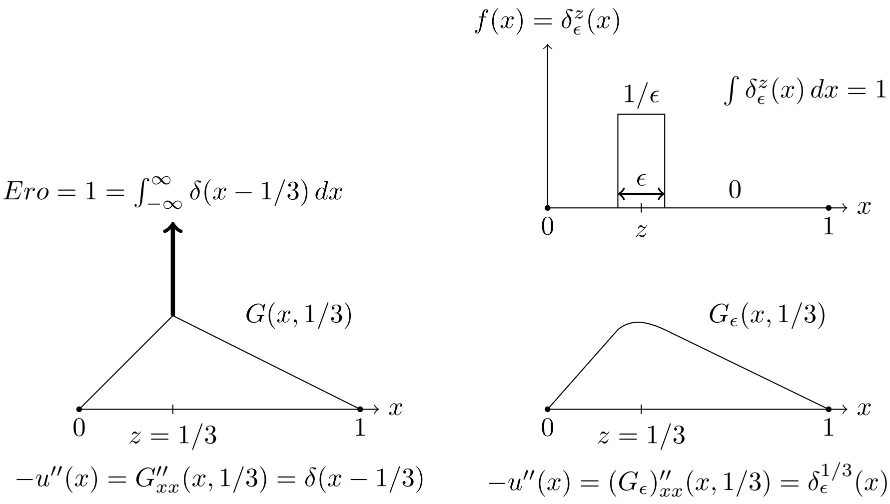


Figure 10.11: Az idealizalt, az $x = 1/3$ pontra koncentralodo egysegny osszintegralu $\delta(x-1/3) \approx \delta_\epsilon^{1/3}(x)$ Dirac delta fuggveny kozelitese egy $\epsilon \approx 0$ hosszusagu intervallumra koncentralodo $1/\epsilon$ nagysagu erosuruseggel.

Ezekbol kovetkezik, hogy

$$G(x,z) = \begin{cases} (1-z)x, & \text{ha } 0 < x < z, \\ -(x-1)z, & \text{ha } z < x < 1. \end{cases}$$

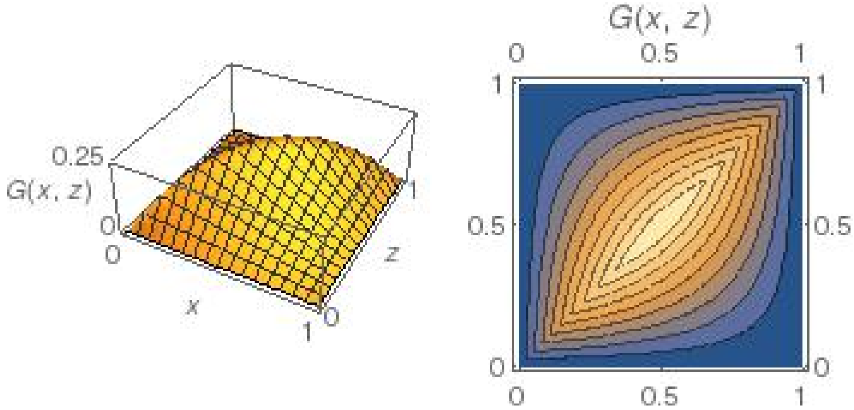


Figure 10.12: A $G(x,z)$ Green fuggveny.

Kesobb latni fogjuk, hogy G megoldja a

$$\frac{\partial^2}{\partial x^2} G(x,z) = -\delta(x-z)$$

egyenletet, ahol a $\delta(x)$ Dirac-delta "fuggveny" (pontosabban disztribucio) a $\delta_\epsilon^0(x)$ fuggvenyek $\epsilon \to 0$ limitje.

10.3.1 Veges elem modszer

Energia minimalizacio: Ahelyett, hogy az u fuggvenyt egy veges dimenzios vektorral kozelitjuk (lasd a veges sok pontot az .. abrakon), probaljuk meg a Fun fuggvenyek egy veges dimenzios V altereben megkeresni a megoldas legjobb kozeliteset!

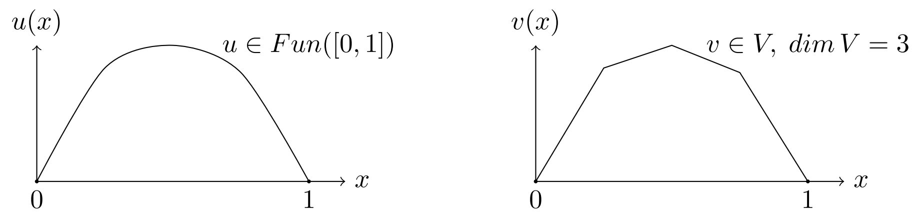


Figure 10.13: A v fuggvenyek egy haromdimenzios alteret alkotnak az "osszes" u fuggvenyek vegtelen dimenzios tereben.

A V alternek egy bazisa:

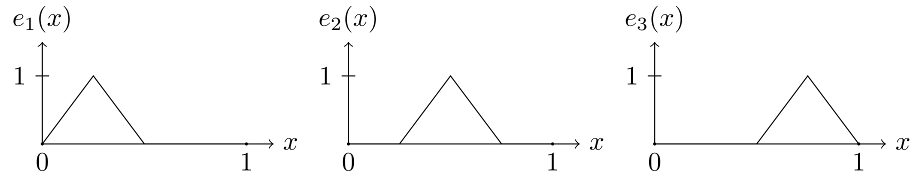


Figure 10.14: V egy bazisat alkotjak ezek a sator alaku fuggvenyek.

Tehat u kozelitese:

$$\begin{aligned} u(x) \approx v(x) &= u(0.25)e_1(x) + u(0.5)e_2(x) + u(0.75)e_3(x) \\ &= v_1e_1(x) + v_2e_2(x) + v_3e_3(x). \end{aligned}$$

Ennek semmi ertelme, ha a (10.1) egyenletet akarjuk megoldani, itt v'' mindenutt nulla, legalabbis ahol egyaltalan ertelmezve van. Viszont az energia funkcional (10.2)

$$\text{Energia}[v] = \int_0^1 \frac{1}{2}[v'(x)]^2 - f(x)v(x)\,dx$$

jol kozeliti Energia$[u]$-t, legalabbis ha nincs nagy kulonbseg u es v elso derivaltjai kozott. Ha ki akarjuk szamitani Energia$[v]$-t, akkor persze egy kozelito modszerben nincs ertelme az $f(x)v(x)$ tagot az integrandusban egzaktul kezelni, igy pl. valaszthatjuk f kozelitesenek a

$$\begin{aligned} f(x) &\approx f(0.25)e_1(x) + f(0.5)e_2(x) + f(0.75)e_3(x) \\ &= f_1e_1(x) + f_2e_2(x) + f_3e_3(x) \end{aligned}$$

fuggvenyt. Ekkor

$$\begin{aligned} E[\bar{v}] = E[(v_1, v_2, v_3)^T] &= \\ +4v_1^2 - 4v_2v_1 + 4v_2^2 + 4v_3^2 - 4v_2v_3 & \\ -\frac{f_1v_1}{6} - \frac{f_2v_1}{24} - \frac{f_1v_2}{24} - \frac{f_2v_2}{6} - \frac{f_3v_2}{24} - \frac{f_2v_3}{24} - \frac{f_3v_3}{6} & \\ = (v_1, v_2, v_3)\begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} & \\ +(f_1, f_2, f_3)\begin{pmatrix} -\frac{1}{6} & -\frac{1}{24} & 0 \\ -\frac{1}{24} & -\frac{1}{6} & -\frac{1}{24} \\ 0 & -\frac{1}{24} & -\frac{1}{6} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} & \\ = \bar{v}^T L\bar{v} + \bar{f}^T M\bar{v} = \bar{v}^T L\bar{v} + \bar{g}^T\bar{v}, & \end{aligned} \tag{10.10}$$

ahol $\bar{g} = M^T\bar{f}$. Ezt a kifejezest az $N = 4$ reszre osztott $[0, 1]$ intervallumon torteno integralas generalja, pl. a harmadik $[0.5, 0.75]$ szakaszon (elemen)

$$v(x) = v_2 + (v_3 - v_2)\frac{x - 0.5}{0.25}, \qquad f(x) = f_2 + (f_3 - f_2)\frac{x - 0.5}{0.25},$$

ennek a hozzajarulasa Energia[v]-hez

$$\int_{0.5}^{0.75} \frac{1}{2}\left(\frac{v_3 - v_2}{0.25}\right)^2 - \left(f_2 + (f_3 - f_2)\frac{x-0.5}{0.25}\right)\left(v_2 + (v_3 - v_2)\frac{x-0.5}{0.25}\right)\, dx.$$

Mathematica kod:

```
n = 4; dx = 1/n;
e[i_, x_] := Piecewise[{{0, x < (i - 1) dx}, {0, x > (i + 1) dx},
   {n (x - (i - 1) dx), (i - 1) dx < x && x < i dx},
   {1 - n (x - i dx), i dx < x && x < (i + 1) dx}}]
vfunc = Sum[v[i] e[i, x], {i, 1, n - 1}];
ffunc = Sum[f[i] e[i, x], {i, 1, n - 1}];
GraphicsRow[
 Append[Table[
   Plot[e[i, x], {x, 0, 1}, Ticks -> {{0, 1}, {0, 1}}], {i, 1, n - 1}],
  Plot[vfunc /. {v[1] -> 2, v[2] -> 3, v[3] -> 1}, {x, 0, 1}, Ticks -> {{0, 1}, {0, 3}}]]]
energy[vfunc_, ffunc_] :=
 Integrate[1/2 (D[vfunc, x])^2 - ffunc vfunc, {x, 0, 1}]
en = energy[vfunc, ffunc] // Expand
en /. (Table[{v[i] -> Subscript[v, i], f[i] -> Subscript[f, i]}, {i, 1, 3}]
    // Flatten) // TeXForm ;   (*   *)
vvCoeff = Table[If[i != j,
   Coefficient[en, v[i] v[j]]/2, Coefficient[en, v[i] v[j]]],{i, 1, n - 1}, {j, 1, n - 1}];
fvCoeff = Table[Coefficient[en, f[i] v[j]], {i, 1, n - 1}, {j, 1, n - 1}];
vvCoeff // TableForm
fvCoeff // TableForm
vVec = Table[v[i], {i, 1, n - 1}];
fVec = Table[f[i], {i, 1, n - 1}];
vVec.(vvCoeff.vVec) + fVec.(fvCoeff.vVec) - en // Simplify
```

A vegeselem modszer filozofiajahoz kozelebb alo kod:

```
n = 4; dx = 1/n;
abi = Integrate[((x1 - x)/(x1 - x0)*a0 + (x - x0)/(x1 - x0)*a1)
                ((x1 - x)/(x1 - x0)*b0 + (x - x0)/(x1 - x0)*b1), {x, x0, x1}]  ;
energy[vVec_, fVec_] := Module[{i, en, der, vv, ff},
  For[i = 0, i <= n, i++,
   vv[i] = If[0 < i && i < n, vv[i] = vVec[[i]], 0];
   ff[i] = If[0 < i && i < n, ff[i] = fVec[[i]], 0]];
  en = 0;
  For[i = 1, i <= n, i++,
   der = (vv[i] - vv[i - 1])/dx;
   en +=
    (1/2)*(der^2 ) dx
     -
     (abi /. {x0 -> (i - 1) dx, x1 -> i dx, a0 -> vv[i - 1],
        a1 -> vv[i], b0 -> ff[i - 1], b1 -> ff[i]})];
  Return[en]]
energy[Table[v[i], {i, 1, n - 1}], Table[f[i], {i, 1, n - 1}]] // Expand
```

Minimalizaljuk (13.2.1)-t! A kritikus pontban

$$0 = \frac{\partial}{\partial v_k}\left(\sum_{i,j} v_i L_{ij} v_j + \sum_i g_i v_i\right) = 2\sum_j v_i L_{ik} + g_k$$

barmely k-ra, vagyis

$$2\bar{v}^T L = -\bar{g}^T \quad \Leftrightarrow \quad 2L^T\bar{v} = -\bar{g} \quad \Leftrightarrow \quad \bar{v} = -\frac{1}{2}(L^T)^{-1}\bar{g} = -\frac{1}{2}(L^{-1})^T M^T \bar{f}.$$

A $\bar{v}$ vektor meghatarozza a kozelito megoldast.

Gyenge megoldas: Hogyan tudnank hasonlo egyenleteket generalni, ha nem ismerjuk a minimalizalando energiafunkcionalt? Legyen u az (10.1)

$$u''(x) + f(x) = 0, \;\; u(0) = u(1) = 0$$

DE megoldasa. Ekkor barmely $\phi(x): \phi(0) = \phi(1) = 0$ fuggvenyre

$$0 = \int_0^1 \phi(x)(u''(x) + f(x))\, dx = \int_0^1 -\phi'(x)u'(x) + \phi(x)f(x)\, dx \qquad (10.11)$$

mivel a parcialis integralasnal a $\phi(x)u'(x) \mid_0^1$ tag kiesik ϕ peremfeltetelei miatt. Az utolso alaknak ugyanaz az elonye, mint az Energia$[v]$ funkcionalnak, csak az elso derivaltra van szukseg.

Az $u(x)$ megoldas

$$u(x) \approx v(x) = v_1 e_1(x) + v_2 e_2(x) + v_3 e_3(x)$$

kozeliteset a kovetkezo keppen hatarozhatjuk meg. Megkoveteljuk, hogy a masodik integral (13.2)-ben nulla legyen, ha u helyere v-t, illetve ϕ helyebe $\phi(x) = c_1 e_1(x) + c_2 e_2(x) + c_2 e_2(x)$-t helyetesitjuk. Mivel $\bar{c}$ tetszoleges, igy a kovetkezo harom integralnak nullanak kell lennie:

$$0 = \int_0^1 -e_i'(x)v'(x) + e_i(x)f(x)\, dx, \qquad i = 1, 2, 3. \qquad (10.12)$$

Ha f-re az $f \approx f_1 e_1 + f_2 e_2 + f_3 e_3$ kozelitest hasznaljuk (ez perszen nem kotelezo), akkor a kovetkezo numerikus egyenletrendszert kapjuk:

$$\begin{aligned}
\frac{1}{6}f_1 + \frac{1}{24}f_2 + 0f_3 - 8v_1 + 4v_2 + 0v_3 &= 0\\
\frac{1}{24}f_1 + \frac{1}{6}f_2 + \frac{1}{24}f_3 + 4v_1 - 8v_2 + 4v_3 &= 0\\
0f_1 + \frac{1}{24}f_2 + \frac{1}{6}f_3 + 0v_1 + 4v_2 - 8v_3 &= 0.
\end{aligned}$$

Itt peldaul az elso egyenletet a kovekezokeppen kapjuk:

$$0 = \int_0^{0.25} -\frac{1}{0.25}\frac{v_1}{0.25} + \frac{x}{0.25}\frac{f_1 x}{0.25}\, dx$$

$$+ \int_{0.25}^{0.5} -\frac{-1}{0.25}\frac{v_2 - v_1}{0.25} + \left(1 - \frac{x - 0.25}{0.25}\right)\left(v_1 + (v_2 - v_1)\frac{x - 0.25}{0.25}\right)\, dx.$$

Adott $\bar{f}$ eseteben a $\bar{v}$ vektor szolgaltatja a kozelito $v(x)$ megoldast.

Mathematica kod:

```
n = 4; dx = 1/n;
weak[vVec_, fVec_] := Module[{w, i, vfunc, ffunc, eqs},
  e[i_, x_] := Piecewise[{{0, x < (i - 1) dx}, {0, x > (i + 1) dx},
                         {n (x - (i - 1) dx), (i - 1) dx < x && x < i dx},
                         {1 - n (x - i dx), i dx < x && x < (i + 1) dx}}];
  vfunc = Sum[vVec[[i]] e[i, x], {i, 1, n - 1}];
  ffunc = Sum[fVec[[i]] e[i, x], {i, 1, n - 1}];
  eqs = Table[
    Integrate[ -D[e[i, x], x] D[vfunc, x] + e[i, x]*ffunc  , {x, 0, 1}], {i, 1, 3}];
  Return[eqs]]
weak[Table[v[i], {i, 1, n - 1}], Table[f[i], {i, 1, n - 1}]] // Expand
```

Egyaltalan nem nyilvanvalo, hogy ez az eljaras mukodik. Amiben biztosak lehetunk, az legfeljebb az, hogy ha v egy jo kozelites, akkor a (13.3) egyenletek kozelitoleg teljesulnek. Azonban a kozelito egyenletek pontos megoldasa nem feltetlenul van kozel a DE pontos megoldsahoz. (Ha pl. ϕ-t egy negydimenzios alterbol valasztanak, akkor negy egyenletunk lenne a harom v_i ismeretlenre, igy valoszinuleg meg se tudnank oldani a numerikus egyenletrendszerunket.)

Probald kidolgozni a kovetkezo problemakat az energiaminimalizacio es a gyenge megoldasok szempontjabol is!

Problema 60. *Milyen valtoztatasokra van szukseg, ha a peremfeltetel $u(0) = 2$, $u(1) = 3$?*

Problema 61. *Eddig a haromdimenzios V alterben kerestuk a kozelito megoldast. Ennek elemei a folytonos, szakaszonkent affine fuggvenyet voltak. Milyen valtoztatasokra van szukseg, ha V folytonos, szakaszonkent kvadratikus fuggvenyekbol allna? Hany dimenzios lenne ez a vektorter? Mi lehetne egy kenyelmes bazisa? Hatranyt jelentene-e, ha azt is megkovetelnenk, hogy a vektorter fuggvenyei derivalhatoak legyenek a szakaszok talalkozasainal?*

Problema 62. *Vajon hogyan mukodhetne az magasabb, pl. ket dimenzioban? Egy tipikus problema a D egysegnegyzeten:*

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) u(x_1, x_1) = \Delta u(x_1, x_1) = -f(x_1, x_2),$$
$$u(x_1, 0) = u(x_1, 1) = u(0, x_2) = (1, x_2) = 0,$$

vagy minimalizald az

$$E[u] = \int_D \frac{1}{2}\left((u'_{x_1})^2 + (u'_{x_2})^2\right) - f u\, dx^2$$

funkcionalt! (A konkretsag kedveert legyen $f(x_1, x_2) = 1$, tehat a rogzitett oldalu negyzetre egyenletes terheles hat.)

1. *Vagd fel D-t eloszor 4×4 negyzetre, majd vagd felbe mindegyik kis negyzetet! Az igy kapott 32 haromszog jatsza az egydimenzios problema negy szakaszanak a szerepet! Mi lehetne a sator alaku $e_i(x)$ bazisfuggvenyek ketdimenzios variansa?*
2. *Ezen bazisfuggvenyek $v(x_1, x_2)$ linearis kombinacioja lesz u kozelitese. Ird fel, hogy mennyi $E[v]$!*
3. *Keresd meg a gyenge megoldas kozelito egyenleteit!*

10.4 Szimmetria

10.4.1 A Laplace operator szimmetriaja

Az egydimenzios egysegnyi erovel elofeszitett hur eseteben az kicsi $u(x)$ deformaciohoz tartozo (visszaterito) ero

$$(Ero[u])(x) = \frac{d^2}{dx^2} u(x)$$

volt. Ez a forma nehany nagyon altalanos elvbol is kovetkezik.

1. Tegyuk fel, hogy az $u \to Ero[u]$ lekepezes linearis, tovabba ha u nulla egy I intervallumon, akkor ott $Ero[u]$ is nulla. Ekkor Ero egy veges rendu differencialoperator segitsegevel szamithato ki:
$$(Ero[u])(x) = (Lu)(x), \qquad \text{ahol } L = \sum_{k=0}^{N} l_k(x) \frac{d^k}{dx^k}.$$
(Valami effelet mond ki Peetre tetele.)
2. Ha L eltolas invarians, akkor
$$L = \sum_{k=0}^{N} l_k \frac{d^k}{dx^k}.$$
3. Ha tovabba L tukrozes (1d elforgatas) invarians is, akkor
$$L = \sum_{k=0}^{M} l_k \left(\frac{d^2}{dx^2}\right)^k = p(\Delta),$$
vagyis L a Laplace operator valamely polinomja. Ez magasabb dimenzioban is igaz: Egy eltolas es elforgatas invarians skalaris differencialoperator Δ polinomja.
4. Tegyuk fel, hogy ha u-nak maximuma van x_{max}-nal, akkor $(Lu)(x)$ negativ, vagyis az ero a maximumot lefele huzza. Ekkor
$$L = c \cdot \frac{d^2}{dx^2} = c\Delta \qquad \text{valamely } c > 0 - \text{ra}.$$

Magyarazat:

3. Mit jelent az, hogy Δ eltolas es tukrozes invarians? Vezessunk be ket lin.op.-ot amelyek a $\psi(x)$ fuggvenyeken hatnak.
$$(T_a\psi)(x) = \psi(x-a), \qquad (P\psi)(x) = \psi(-x).$$
Ekkor a szimmetria jelentese, definicioja:
$$T_a\Delta = \Delta T_a, \qquad P\Delta = \Delta P.$$
Bizonyitsuk a masodikat:
$$\psi(x) \to (\Delta\psi)(x) = y''(x) \to (P\Delta\psi)(x) = y''(-x),$$
$$\psi(x) \to (P\psi)(x) = \psi(-x) \to (\Delta P\psi)(x) = \frac{d^2}{dx^2}\psi(-x) = (-1)^2 y''(-x).$$
Mivel ugyanazt kaptuk, Δ valoban invarians a P tukrozesre nezve.

Problema 63. *Hogy fugg ossze*
$$\frac{d}{dx} \cdot P \qquad es \qquad P \cdot \frac{d}{dx} \quad ?$$
Bizonyitsd Δ invarianciajat a T_a eltolasra nezve? Mennyi $T_a \cdot T_b$? Mennyi $(T_a)^{-1}$? Mennyi $(T_a)^*$?

4. Legyen a maximum helye $x_{max} = 0$ es kis $x \approx 0$-ra legyen pl. $u(x) \approx 2 - 3x^2 + \alpha x^4$. Ha pl.

$$L = 7 \cdot \Delta + \beta \Delta^2, \quad \text{akkor} \quad (Lu)(0) \approx 7 \cdot (-3 \cdot 2!) + \beta \cdot \alpha \cdot 4!$$

Ez viszont pozitiv lenne, ha $sign(\alpha) = sign(\beta)$ es $|\alpha|$ eleg nagy, vagyis nem teljesulne a $(Lu)(x_{max}) \leq 0$ feltetel.

Mi lenne, ha egy hajlekony hur helyett egy vekony, de merev, rugalmas rud eseten vizsgalnak a visszaterito erot kis $u(x)$ deformaciok eseteben? Tegyuk fel, hogy az ero aranyos az u fuggveny gorbuleti sugaraval:

$$Ero(x) \sim \kappa(x) = \frac{f''(x)}{\left(1 + [f'(x)]^2\right)^{3/2}} \approx f''(x) - \frac{3}{2} f''(x) \left[f'(x)\right]^2 + \ldots$$

A linearis tag (leszamitva esetleg egy pozitiv szorzo faktort) megegyezik a ket modelben.

Ugyanez a meggondolas alapjan a hovezetes linearis modelje:

$$\partial_t u(t, x) = c \cdot \partial_x^2 u(t, x), \qquad c > 0.$$

Szuksegszeruen ehhez az alakhoz jutunk, ha feltesszuk a kovetkezoket:

1. Ha egy szakaszon nulla a homerseklet, akkor ott a pillanatnyi homereklet-valtozas is nulla.

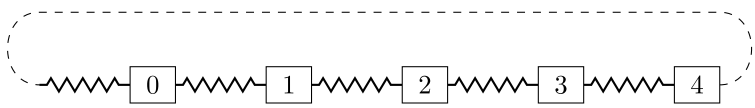
2. A model linearis, homogen es izotrop.

3. A maximalis homerseklet nem nohet.

Problema 64. *Probalj valamifele "fizikai" levezetest adni a hoegyenletre!*

10.4.2 Diszkret Fourier Transzformacio (DFT, FFT)

Problema 65. *Ot egysegnyi tomegu test periodikusan ossze van kotve egysegnyi rugoallandoju rugokkal. a testek poziciojat az $\vec{y} = (y_0, \ldots, y_4)^T$ vektor adja meg. Ird fel a rendszer mozgasegyenletet, es oldd is meg azt. Vegezd el ezt egy hasonlo, N tomegbol allo rendszerre is!*



Ket tetel bizonyitas nelkul:

Tetel 11. *Legyen A es B ket diagonalizalhato matrix, tovabba tegyuk fel, hogy $AB = BA$. Ekkor letezik A es B kozos sajatvektoraibol allo bazis.*

Tetel 12. *Legyen M egy normalis matrix, vagyis teljesuljon, hogy $MM^* = M^*M$. Ekkor letezik M sajatvektoraibol allo ortonormalt bazis.*

Az $N = 5$ testbol allo rendszer szimmetriaja a testek ciklikus T permutacioja. T normalis operator lesz, aminek letezik ortonormalt bazisa. A feladatban szereplo operatorok kommutalnak T-vel, igy (esetunkben) T sajatvektorai segitsegevel ki tudjuk fejezni a feladat megoldasat.

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_0 \end{pmatrix}.$$

Problema 66. *Mennyi T^{-1}, T^* es $TT^* - T^*T$?*

Mivel $T^N = E$, igy T lehetseges sajatertekei

$$\lambda_k = \varepsilon^k, \qquad \varepsilon = e^{i\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right).$$

A (nemnormalizalt, de ortogonalis) $\vec{v}_k$ sajatvektorok (ezek periodikus mertani sorok elso N tagjai λ_k hanyadossal):

$$\vec{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ \varepsilon^1 \\ \varepsilon^2 \\ \varepsilon^3 \\ \varepsilon^4 \end{pmatrix}, \quad \ldots, \quad \vec{v}_k = \begin{pmatrix} \varepsilon^{0\cdot k} \\ \varepsilon^{1\cdot k} \\ \varepsilon^{2\cdot k} \\ \varepsilon^{3\cdot k} \\ \varepsilon^{4\cdot k} \end{pmatrix}, \ldots, \qquad k = 0, 1, \ldots, N-1.$$

Jegyezzuk meg, hogy $\vec{v}_k = \vec{v}_{k'}$, ha $k - k'$ oszthato N-nel.

A $\vec{v}_k$ sajatvektor megkaphato a $[0,1]$-en ertelmezett $e^{2\pi i k x}$ fuggveny mintavetelezesekent az $x_0 = 0, \ldots, x_l = l/N, \ldots, x_{N-1} = (N-1)/N$ pontokban. Ez magyarazza az F betut a DFT-ben.

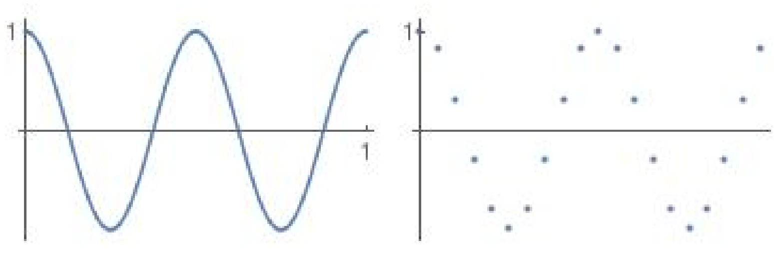


Figure 10.15: Az $e^{2\pi i k x}$, $k = 2$ fuggveny, illetve annak mintavetelezese $N = 1/20$ idotartamonkent, ami nem mas, mint $\vec{v}_k = \vec{v}_2$ komponenseinek abrazolasa. (Csak a valos reszeket abrazoltuk.)

```
k = 2; nn = 20; eps = Exp[I 2 Pi/nn];
res = GraphicsRow[{Plot[Exp[2 Pi I k x] // Re, {x, 0, 1}, Ticks -> {{0, 1}, {0, 1}}],
   ListPlot[Table[{n*( 1/nn), eps^(k n) // Re}, {n, 0, nn - 1}], Ticks -> {{0, 1}, {0, 1}}]}]
Export["dft.jpg", res]
```

Legyen az $\mathcal{F}$ DFT az ezen bazis szerinti sorfejtes:

$$\vec{F} = \mathcal{F}\vec{f} = \begin{pmatrix} (\vec{v}_0, \vec{f}) \\ (\vec{v}_1, \vec{f}) \\ (\vec{v}_2, \vec{f}) \\ (\vec{v}_3, \vec{f}) \\ (\vec{v}_4, \vec{f}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon^{-1} & \epsilon^{-2} & \epsilon^{-3} & \epsilon^{-4} \\ 1 & \epsilon^{-2} & \epsilon^{-4} & \epsilon^{-6} & \epsilon^{-8} \\ 1 & \epsilon^{-3} & \epsilon^{-6} & \epsilon^{-9} & \epsilon^{-12} \\ 1 & \epsilon^{-4} & \epsilon^{-8} & \epsilon^{-12} & \epsilon^{-16} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}.$$

```
F = Table[ Superscript[epsilon, -i j], {i, 0, 4}, {j, 0, 4}] /. Superscript[epsilon, 0] -> 1;
F // TeXForm
```

Az inverz $\mathcal{F}^{-1}$ tr:

$$\vec{f} = \mathcal{F}^{-1}\vec{F} = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon^{1} & \epsilon^{2} & \epsilon^{3} & \epsilon^{4} \\ 1 & \epsilon^{2} & \epsilon^{4} & \epsilon^{6} & \epsilon^{8} \\ 1 & \epsilon^{3} & \epsilon^{6} & \epsilon^{9} & \epsilon^{12} \\ 1 & \epsilon^{4} & \epsilon^{8} & \epsilon^{12} & \epsilon^{16} \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}.$$

(Mivel $\frac{1}{\sqrt{N}}\mathcal{F}$ uniter, igy ennek az inverze $\frac{1}{\sqrt{N}}\mathcal{F}^*$.) Az FFT (Gyors (Fast) Fourier Tr.) ezt a muveletet $O(N\log(N))$ ido alatt kepes kiszamitani.

Problema 67. *Terjunk vissza az $N = 5$ tomegpont problemajahoz.*

1. *Ellenorizd, hogy a mozgasegyenlete az $\vec{y}(t)$ elmozdulasvektornak:*

$$\frac{d^2}{dt^2}\vec{y} = -\left(-T^{-1} + 2E - T\right)\vec{y} = -A\vec{y}.$$

   Mik A sajatvektorai es sajatertekei?

2. *Ird fel az altalanos megoldast!*

Megoldas 45. *1.*

$$\frac{d^2}{dt^2}\vec{y}(t) = -A\vec{y}(t) = -\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}\vec{y}(t)$$
$$= -\left(-T + 2E - T^{-1}\right)\vec{y}(t).$$

Mivel A kifejezheto T-vel, igy a sajatvektoraik ugyanazok.

$$k = 0, \ldots, N-1, \qquad \lambda_k = -\varepsilon^k + 2 - \varepsilon^{-1} = 2\left(1 - \cos(2k\pi/N)\right),$$
$$\vec{v}_k = \begin{pmatrix} e^{ik\cdot 0\cdot(2\pi/N)} \\ \vdots \\ e^{ik\cdot(N-1)\cdot(2\pi/N)} \end{pmatrix}$$
$$= \begin{pmatrix} \cos(k\cdot 0\cdot(2\pi/N)) \\ \vdots \\ \cos(k\cdot(N-1)\cdot(2\pi/N)) \end{pmatrix} + i\begin{pmatrix} \sin(k\cdot 0\cdot(2\pi/N)) \\ \vdots \\ \sin(k\cdot(N-1)\cdot(2\pi/N)) \end{pmatrix}.$$

2. Az altalanos megoldas:

$$\vec{y}(t) = \sum_{k=0}^{N-1} \left(P_k e^{i\sqrt{\lambda_k} t} + M_k e^{-i\sqrt{\lambda_k} t} \right) \vec{v}_k$$
$$= \sum_{k=0}^{N-1} \left(C_k \cos\left(\sqrt{\lambda_k} t\right) + S_k \sin\left(\sqrt{\lambda_k} t\right)\right) \vec{v}_k$$

A megoldas idofuggo reszenek az $\omega_k = \sqrt{\lambda_k}$ korfrekvenciajanak az abrazolasa:

$$k = -\frac{N-1}{2}, \ldots, 0, \ldots, \frac{N-1}{2}, \qquad \vec{v}_k = \vec{v}_{k-N},$$

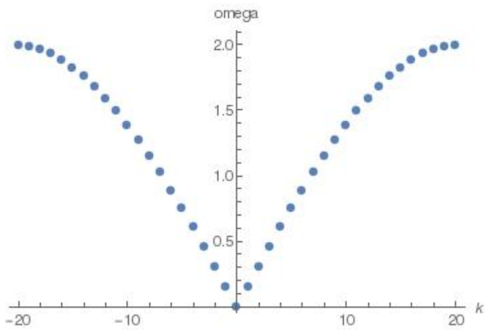


Figure 10.16: Az ω_k korfrekvencia a k hullamszam fuggvenyeben, $N = 41$.

3. A partikularis megoldas:

$$\vec{y}(0) = \vec{y}_0, \qquad \dot{\vec{y}}(0) = \vec{z}_0.$$
$$\frac{d}{dt}\vec{y}(t) = \sum_{k=0}^{N-1} \sqrt{\lambda_k} \left(-C_k \sin\left(\sqrt{\lambda_k} t\right) + S_k \cos\left(\sqrt{\lambda_k} t\right)\right) \vec{v}_k,$$
$$\vec{y}_0 = \sum_{k=0}^{N-1} C_k \vec{v}_k, \qquad \vec{z}_0 = \sum_{k=0}^{N-1} \sqrt{\lambda_k} S_k \vec{v}_k$$

Tehat

$$C_k = \frac{1}{N}(\vec{v}_k, \vec{y}_0), \quad \textit{vagy} \ \vec{C} = \frac{1}{N}\mathcal{F}\vec{y}_0, \qquad S_k = \frac{1}{\sqrt{\lambda_k} \cdot N}(\vec{v}_k, \vec{z}_0).$$

Ha a kezdeti feltetelek valosak, akkor

$$C_{-k} = C_{N-k} = \overline{C_k}, \qquad S_{-k} = S_{N-k} = \overline{S_k}.$$

A kovetkezo ket problema bizonyos eretelemben ennek a problemanak a "folytonos" verzioja.

Problema 68. *Hullamegyenlet $\mathbb{R}^2$-en.*

$$\partial_t^2 \phi(t, x) = \partial_x^2 \phi(t, x).$$

1. *Sikhullam megoldas: Legyen*

$$\phi(t,x) = e^{i(kx-\omega_k t)}.$$

Mennyi lehet ω_k *?*

2. *Mennyi lenne* ω_k*, ha a hullamegyenletunk* $\partial_t^2\phi(t,x) = 9\partial_x^2\phi(t,x)$ *lenne? Milyen sebesseggel "mozogna" ekkor az* $e^{i(kx-\omega_k t)}$ *sikhullam megoldas?*

3. *Sikhullamok szuperpozicioja:*

$$\begin{aligned}\phi(t,x) &= \int_{-\infty}^{\infty} dk\ \left(p(k)e^{i|k|t} + m(k)e^{-i|k|t}\right)e^{ikx}\\ &= \int_{-\infty}^{\infty} dk\ \left(c(k)\cos(|k|t) + s(k)\sin(|k|t)\right)e^{ikx}.\end{aligned}$$

Ha

$$\phi(0,x) = f(x), \qquad es \qquad \dot\phi(0,x) = g(x),$$

akkor mennyi $c(k)$ *es* $s(k)$ *?*

Problema 69. *Hullamegyenlet* $\mathbb{R}\times\mathcal{S}^1$*-en.*

$$\partial_t^2\phi(t,x) = \partial_x^2\phi(t,x), \qquad \phi(t,x) = \phi(t,x+1).$$

1. *Sikhullam megoldas: Legyen*

$$\phi(t,x) = e^{i(kx-\omega_k t)}.$$

Milyen ertekeket vehet fel k *? Mennyi lehet* ω_k *?*

2. *Sikhullamok szuperpozicioja:*

$$\begin{aligned}\phi(t,x) &= \sum_{k\in\frac{1}{2\pi}\mathbb{Z}} \left(p_k e^{i|k|t} + m_k e^{-i|k|t}\right)e^{ikx}\\ &= \sum_{k\in\frac{1}{2\pi}\mathbb{Z}} \left(c_k\cos(|k|t) + s_k\sin(|k|t)\right)e^{ikx}.\end{aligned}$$

Ha

$$\phi(0,x) = f(x), \qquad es \qquad \dot\phi(0,x) = g(x),$$

akkor mennyi c_k *es* s_k *?*

Chapter 11

Inhomogen evolucios linearis egyenletek

11.1 Input-output relacio

Homogen linearis egyenlet:

$$\bar{y} = A(t)\bar{y}.$$

($\bar{y}$ vektor, A matrix.)

Szuperpozicio elve: Ha

$$\frac{d}{dt}\overline{y_1} = A(t)\overline{y_1} \quad \text{es} \quad \frac{d}{dt}\overline{y_2} = A(t)\overline{y_2}$$

akkor

$$\frac{d}{dt}\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right) = A(t)\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right),$$

vagyis a homogen egyenletek megoldasainak linearis kombinacioja is megoldas.

Inhomogen linearis egyenlet:

$$\bar{y} = A(t)\bar{y} + \bar{f}(t). \tag{11.1}$$

Ekkor, ha

$$\frac{d}{dt}\overline{y}_{hom} = A(t)\overline{y}_{hom} \quad \text{es} \quad \frac{d}{dt}\overline{y}_{part} = A(t)\overline{y}_{part} + \bar{f}(t),$$

akkor

$$\frac{d}{dt}\left(\overline{y}_{hom} + \overline{y}_{part}\right) = A(t)\left(\overline{y}_{hom} + \overline{y}_{part}\right) + \bar{f}(t).$$

Tehat az inhomogen egyenlet altalanos megoldasa felirhato a homogen egyenlet $\overline{y}_{hom}$ altalanos megoldasa es az inhom. egyenlet egy $\overline{y}_{part}$ partikularis megoldasanak az $\overline{y}_{hom} + \overline{y}_{part}$ osszegekent.

Linearis input-output relacio: Ha

$$\frac{d}{dt}\overline{y_1} = A(t)\overline{y_1} + \overline{f}_1(t) \quad \text{es} \quad \frac{d}{dt}\overline{y_2} = A(t)\overline{y_2} + \overline{f}_2(t)$$

akkor

$$\frac{d}{dt}\left(\alpha_1\overline{y_1}+\alpha_2\overline{y_2}\right)=A(t)\left(\alpha_1\overline{y_1}+\alpha_2\overline{y_2}\right)+\alpha_1\overline{f_1}+\alpha_2\overline{f_2}.$$

Az (12.1) egyenlet megoldasi strategiaja: Odjuk meg a DE-t, ha $f(t)=$

$$\delta(t), \qquad e^{ipt}, \qquad e^{-st},$$

majd irjuk fel f-et ezen elemi jelek linearis kombinaciojakent (integraljakent):

$$\begin{aligned}
f(t) &\approx \sum_i f(t_i)\delta(t-t_i)\Delta t,\\
f(t) &= \frac{1}{\sqrt{2\pi}}\int \tilde{f}(p)e^{ipt}\,dp,\\
f(t) &= \frac{1}{2\pi i}\int F(s)e^{st}\,ds.
\end{aligned}$$

Az input-output relacio linearitasa alapjan ezzel megoldottuk (12.1)-t.

11.2 Disztribuciok (altalanositott fuggvenyek)

Legyen $\mathcal{D}$ a sima, egy veges intervallumon kivul nulla erteku fuggvenyek tere (tesztfuggvenyek). Ekkor minden $f\in\mathcal{D}$ definial egy linearis funkcionalt ($\mathcal{D}$-t $\mathbb{R}$-be lekepezo linearis es "folytonos" fuuggvenyt)

$$L_f:\ g\to L_f[g]=\int_{-\infty}^{\infty} f(x)g(x)\,dx=\langle f,g\rangle.$$

Viszont pl. a $\delta: g\to g(0)$ funkcionalhoz nincs olyan $f\in\mathcal{D}$ fuggveny, ami ezt tudna. (Viszont δ kozelitheto (ahogy $n\to\infty$) olyan $\delta_n(x)\in\mathcal{D}$ pozitiv fuggvenyekkel, amelyekre $\int\delta_n(x)\,dx=1$, es az a tartomany, ahol $\delta_n(x)$ nem nulla, az egyre jobban koncentralodik 0 kore.) Az osszes, $\mathcal{D}$-n ertelmezett lin.funkcionalt hivjuk a disztribuciok $\mathcal{D}'$ terenek. Ezeket lehet derivalni is. Ennek a motivacioja: Ha $f,g\in\mathcal{D}$ akkor

$$\langle f',g\rangle=-\langle f,g'\rangle$$

a parcialis integralas szabalyai szerint. Tehat definialjuk a $d\in\mathcal{D}'$ disztribucio d' derivaltjat a

$$\langle d',g\rangle=-\langle d,g'\rangle \qquad \forall g\in\mathcal{D}$$

szaballyal. Ekkor mar bebizonyithatjuk, hogy $H'=\delta$:

$$\begin{aligned}
\langle H',g\rangle=-\langle H,g'\rangle &= -\int_{-\infty}^{\infty}H(x)g'(x)\,dx=-\int_0^{\infty}g'(x)\,dx\\
&= -g(x)\ |_0^{\infty}=-(g(\infty)-g(0))=g(0)=\langle\delta,g\rangle.
\end{aligned}$$

Problema 70. *Legyen*

$$F(t)=\begin{cases}0 & ha\ t<0\\ t & ha\ 0<t.\end{cases} \tag{11.2}$$

Bizonyitsd be, hogy $F'=H$, vagyis $F''=\delta$!

Problema 71. *Legyen*

$$G(t) = \begin{cases} 0 & ha\ t < 0 \\ e^{-7t} & ha\ 0 < t. \end{cases} \tag{11.3}$$

Bizonyitsd be, hogy $G' + 7G = \delta$!

Problema 72. 1. *Magyarazd meg, hogy miert nincs ertelme a disztribuciok szorzasanak, pl. miert nincs ertelme $(\delta(x))^2$-nek!*

2. *Magyarazd meg, hogy viszont miert definialhato pl. $\delta(x_1)\delta(x_2) = \delta(\bar{x})$!*

11.3 Inhomogen egyeletek

11.3.1

Az $y' = f(t)$ DE az inhomogen linearis egyenletek legegyszerubb peldaja. Adott az $y'(t) = f(t)$ sebesseg, mennyi az $y(t)$ pozicio?

$$y'(t) = f(t),\ y(t_0) = y_0 \quad \Longleftrightarrow \quad y(t) = y_0 + \int_{t_0}^{t} f(\tau)\, d\tau.$$

Legyen $y(t) = f(t) = 0$, ha $t \ll 0$. Ekkor

$$y(t) = \int_{-\infty}^{t} f(\tau)\, d\tau = \int_{-\infty}^{\infty} H(t,\tau) f(\tau)\, d\tau = \int_{-\infty}^{\infty} H(t-\tau) f(\tau)\, d\tau, \tag{11.4}$$

ahol H a Heaviside egysegugras fuggveny:

$$H(t) = \begin{cases} 0 & \text{ha } t < 0 \\ 1 & \text{ha } 0 < t. \end{cases}$$

Milyen tulajdonsagai lennenek a (csak az altalanositott fuggvenyek (disztribuciok) kozott letezo) $H'(t) = \delta(t)$ Dirac-delta fuggvenynek?

$$\delta(t) = 0, \text{ ha } t \neq 0, \quad \text{illetve} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Dirac-delta mint hatarertek (pl.):

$$\delta_n = \begin{cases} n + n^2 t & \text{ha } -1/n < t < 0 \\ n - n^2 t & \text{ha } 0 < t < 1/n \\ 0 & \text{amugy.} \end{cases}$$

$H_n' = \delta_n$:

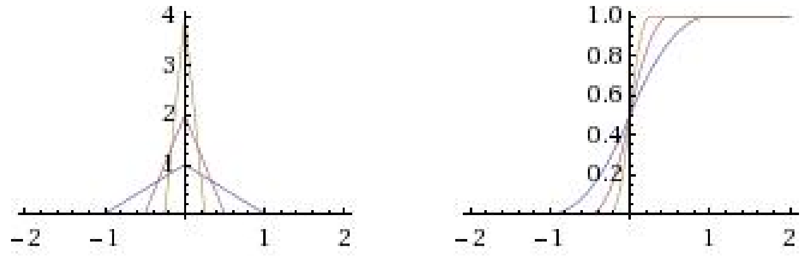


Figure 11.1: $\delta_{1,2,4} \to \delta$, $H_{1,2,4} \to H$.

Ha meg tudjuk keresni az $y' = \delta$ egyenlet $y = H$ megoldasat (fundamentalis megoldas, impulzusvalasz, retardalt Green-fuggveny), akkor az $y' = f$ egyenlet megoldasa az (12.4) kifejezes.

11.3.2

Adott az $y''(t) = f(t)$ gyorsulas, mennyi az $y(t)$ pozicio?

Pl.:

$$
\begin{gathered}
y'' = 10,\ y(1) = 2,\ y'(1) = 3. \\
y' = \int 10\, dt = 10t + C_1, \quad y = \int 10t + C_1\, dt = 5t^2 + C_1 t + C_2, \\
5 \cdot 1^2 + C_1 \cdot 1 + C_2 = 2, \quad 10 \cdot 1 + C_1 = 3 \qquad \Longrightarrow \qquad y = 5t^2 - 7t + 4.
\end{gathered}
$$

$$
y'' = f(t), \quad v = y', \quad v' = f(t) \qquad \Longrightarrow \qquad y'(t) = v(t) = v(t_0) + \int_{t_0}^{t} f(\tau)\, d\tau,
$$

$$
\begin{aligned}
y(t) &= y(t_0) + \int_{t_0}^{t} \left(v(t_0) + \int_{t_0}^{\tau} f(\tau_2)\, d\tau_2 \right) d\tau \\
&= y(t_0) + (t - t_0) v(t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{\tau} f(\tau_2)\, d\tau_2 \right) d\tau
\end{aligned}
$$

Ha $y(t) = f(t) = 0$ az $x \ll 0$ esetben:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} f(\tau_2)\, d\tau_2 \right) d\tau = \int_{-\infty}^{\tau_2} \left(\int_{\tau_2}^{t} f(\tau_2)\, d\tau \right) d\tau_2 \\
&= \int_{-\infty}^{\tau_2} (t - \tau_2) f(\tau_2)\, d\tau_2 = \int_{-\infty}^{\tau} (t - \tau) f(\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} F(t - \tau) f(\tau)\, d\tau,
\end{aligned}
$$

ahol

$$F(t) = \begin{cases} 0 & \text{ha } t < 0 \\ t & \text{ha } 0 < t. \end{cases} \tag{11.5}$$

$F(t)$ a $F'' = \delta$ egyenlet megoldasa, itt F' ugrik egyet $t = 0$-nal. ($H(t)$ a $H' = \delta$ egyenlet megoldasa, ott H ugrik egyet $t = 0$-nal.)

11.3.3

$y'(t) + 3y(t) = f(t)$, $y(t) = f(t) = 0$ ha $t \ll 0$. Szeretnenk y-t az $y(t) = \int_{-\infty}^{\infty} G(t-\tau)f(\tau)\,d\tau$ alakba felirni, ahol G a $G' + 3G = \delta$ egyenlet megoldasa. Mit tudunk G-rol?

1. $G(t) = 0$, ha $t < 0$ az $y(t) = f(t) = 0$ ha $t \ll 0$ feltetel miatt.
2. Ha $t \approx 0$ akkor G a nulla koruli jobboldali es baloldali hatarertekek szempontjabol ugy viselkedik mint a $H' = \delta$ egyenlet Heaviside fuggveny megoldasa, vagyis $G(0^+) - G(0^-) = 1 = H(0^+) - H(0^-)$.
3. Ha $t > 0$, akkor $G' + 3G = 0$, mivel $\delta(t) = 0$, ha $t \neq 0$.

Mivel az $u'(t) + 3u(t) = 0$, $u(0) = 1$ egyenlet megoldasa $u(t) = e^{-3t}$, igy

$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ e^{-3t} & \text{ha } t > 0. \end{cases}$$

Tehat

$$y(t) = \int_{-\infty}^{\infty} G(t-s)f(s)\,ds = \int_{-\infty}^{t} e^{-3(t-s)}f(s)\,ds.$$

Itt egy, a regmultban nyugalmi allapotban levo rendszert kezeltunk. Ha adott $y(0)$, akkor

$$y(t) = y(0)G(t) + \int_0^{\infty} G(t-s)f(s)\,ds,$$

mivel G kielegiti a $G(0^+) = 1$ kezdeti feltetelt (Duhamel-elv).

11.3.4

$y''(t) + 9y(t) = f(t)$, $y(t) = f(t) = 0$ ha $t \ll 0$. Szeretnenk y-t az $y(t) = \int_{-\infty}^{\infty} G(t-\tau)f(\tau)\,d\tau$ alakba felirni, ahol G a $G'' + 9G = \delta$ egyenlet megoldasa. Mit tudunk G-rol?

1. $G(t) = 0$, ha $t < 0$ az $y(t) = f(t) = 0$ ha $t \ll 0$ feltetel miatt.
2. Ha $t \approx 0$ akkor G a nulla koruli jobboldali es baloldali hatarertekek szempontjabol ugy viselkedik mint a $F'' = \delta$ (12.5) egyenlet megoldasa, vagyis $G(0^+) - G(0^-) = 0 = F(0^+) - F(0^-)$ es $G'(0^+) - G'(0^-) = 1 = F'(0^+) - F'(0^-)$.
3. Ha $t > 0$, akkor $G'' + 9G = 0$, mivel $\delta(t) = 0$, ha $t \neq 0$.

Mivel az $u''(t) + 9u(t) = 0$, $u'(0) = 1$, $u(0) = 0$ egyenlet megoldasa $u(t) = \frac{1}{3}\sin(3t)$, igy

$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ \frac{1}{3}\sin(3t) & \text{ha } t > 0. \end{cases}$$

Tehat

$$y(t) = \int_{-\infty}^{\infty} G(t-s)f(s)\,ds = \int_{-\infty}^{t} \frac{1}{3}\sin(3(t-s))f(s)\,ds.$$

Sajnos a Duhamel-elv itt nem alkalmazhato direkt modon, mivel G segitsegevel nem tudnank elerni egy $y(0) \neq 0$ kezdeti feltetelt.

11.3.5

Radioaktiv bomlas $1 \to 2 \to \emptyset$:

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Homogen egyenlet partikularis megoldasai: ha

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{illetve} \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

akkor

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} + 3e^{-2t} \end{pmatrix}, \qquad \text{illetve} \qquad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}.$$

Milyen a rendszer G_{i1} valasza egy $t = 0$-kor beerkezo 1 fele egysegnyi radioaktiv szennyezodesre:

$$\frac{d}{dt}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} + \begin{pmatrix} \delta(t) \\ 0 \end{pmatrix}.$$

Ha a masodik fele anyag erkezik impulzusszeruen, akkor

$$\frac{d}{dt}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ \delta(t) \end{pmatrix}.$$

Ez ugyanaz, mint

$$\frac{d}{dt}\begin{pmatrix} G_{12} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} G_{12} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} + \delta(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Igy

$$G(0^+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Tehat a retardalt ($G(t) = 0$, ha $t < 0$) megoldasa pozitiv t-re

$$G(t) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} e^{-3t} & 0 \\ -3e^{-3t} + 3e^{-2t} & e^{-2t} \end{pmatrix},$$

mig negativ t-re G nulla. Ekkor a

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix},$$

($\bar{f}(t) = \bar{y}(t) = 0$, ha $t \ll 0$) kezdeti feltetelu DE megoldasa

$$\bar{y}(t) = \int_{-\infty}^{\infty} G(t-s)\bar{f}(s)\,ds.$$

Formalisan

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\frac{d}{dt} - \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\right)^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

sajnos egy nem invertalhato (a homogen egyenlet megoldasai nulla sajatertekuek) operator invertalasanak az ertelmezese elegge bonyolult.

Duhamel elv: ha

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 13 \\ 31 \end{pmatrix}$$

akkor

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = G(t)\begin{pmatrix} 13 \\ 31 \end{pmatrix} + \int_0^{\infty} G(t-s)\bar{f}(s)\,ds, \qquad \text{ha } t > 0.$$

Problema 73. *Ird at az $y'' + 9y = f(t)$ DE-t mint egy elsorendu rendszert! Keresd meg a retardalt Green fuggvenyt! Ird fel a DE kezdetiertek feladatanak a megoldasat a Duhamel elv segitsegevel!*

Problema 74. *Ird at az $y'' + 4y' + 5y = f(t)$ DE-t mint egy elsorendu rendszert! Keresd meg a retardalt Green fuggvenyt! Ird fel a DE kezdetiertek feladatanak a megoldasat a Duhamel elv segitsegevel!*

Mintapeldak: ZH2

1. Veges differenciak.
   Keress numerikus egyenleteket a kovetkezo DE kozelito megoldasara:

   $$u''(x) + xu'(x) = x^2, \;\; u(0) = u(1) = 0.$$

   Approximaljuk az u fuggvenyt a kovetkezo vektorral: $\vec{u}_i = u(i\Delta x),\; i = 1,\ldots,4,\; \Delta x = 1/5$.

   - Kozelitsd $u''(x)$-t az $u(x \pm \Delta x), u(x)$ ertekek segitsegevel!
   - Hogyan kozelitened $u'(x)$-t ? Ird fel az ennek megfelelo veges differencias kozeliteset a DE-nek mint egy inhom.lin. egyenletet a $\vec{u}$ vektorra!

   Megoldas:

   -
     $$u''(x) = \frac{1}{\Delta x^2}(u(x+\Delta x) - 2u(x) + u(x+\Delta x))$$

- $$u'(x) = \frac{1}{\Delta x}(u(x+\Delta x) - u(x)).$$

$$\frac{1}{\Delta x}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$+\begin{pmatrix} 1\cdot\Delta x & & & \\ & 2\cdot\Delta x & & \\ & & 3\cdot\Delta x & \\ & & & 4\cdot\Delta x \end{pmatrix}\frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$= \begin{pmatrix} (1\cdot\Delta x)^2 \\ (2\cdot\Delta x)^2 \\ (3\cdot\Delta x)^2 \\ (4\cdot\Delta x)^2 \end{pmatrix}.$$

A gyakorlas kedveeret oldd meg a kovetkezo problemat:
$u''' + x^2u' = x$, $u'(x) \approx (u(x+\Delta x) - u(x-\Delta x))/(2\Delta x)$!

2. Euler-Lagrange egyenletek.
Let
$$L(x(t), y(t), \dot{x}(t), \dot{y}(t)) = \dot{x}^3 + \dot{y}^2 + x\dot{y} + x^5y^6.$$
Ird fel az EL egyenleteket erre a Lagrange fuggvenyre!

Megoldas:

$$EL: \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}_i} - \frac{\partial L}{\partial\phi_i} = 0, \qquad \phi_1 = x,\ \phi_2 = y.$$

$$\frac{\partial L}{\partial x} = \dot{y} + 5x^4y^6,\ \frac{\partial L}{\partial\dot{x}} = 3\dot{x}^2, \qquad \frac{\partial L}{\partial y} = x^5\cdot 6y^5,\ \frac{\partial L}{\partial\dot{y}} = 2\dot{y} + x,$$

$$\frac{d}{dt}\left(3\dot{x}^2\right) - \left(\dot{y} + 5x^4y^6\right) = 6\ddot{x}\dot{x} - \left(\dot{y} + 5x^4y^6\right) = 0,$$

$$\frac{d}{dt}(2\dot{y} + x) - x^5\cdot 6y^5 = (2\ddot{y} + \dot{x}) - \left(x^5\cdot 6y^5\right) = 0.$$

3. Veges elemek, variacios elv.
Oszd fel a $[0,1]$ intervallumot 4 reszre a kovetkezo pontokkal: $x_i = 0.2,\ 0.5,\ 0.8$.
Legyen $v(x)$ az a folytonos fuggveny, amelyik affine az alintervallumokon es az ertekei az $x = 0,\ 0.2,\ 0.5,\ 0.8,\ 1$ pontokban a kovetkezoek: $0, v_1, v_2, v_3, 0$.

- Szamitsd ki, mennyi
$$Energy[v] = \int_0^1 (v')^2 - xv\,dx$$
kozelitoleg vagy pontosan! Kozelito szamitas eseten add meg, hogy milyen kozelitest hasznaltal!
- Ird fel az EL egyenleteket az $Energy[u]$ funkcionalra!

Megoldas:

- $$\frac{\partial Energy}{\partial v} = -x,\ \frac{\partial Energy}{\partial v'} = 2v', \qquad \frac{d}{dx}(2v') - (-x) = 2v'' + x = 0.$$

- Az xv fuggveny integraljat a szubintervallumokon az x es v fuggvenyeknek a szubintervalumok kozeppontjaiban felvett ertekeinek a segitsegevel kozelitjuk.

$$
\begin{aligned}
&Energy[v]\\
\approx 0.2 &\left(\left(\frac{v_1-0}{0.2}\right)^2-\left(\frac{0+0.2}{2}\right)\left(\frac{0+v_1}{2}\right)\right)\\
+0.3 &\left(\left(\frac{v_2-v_1}{0.3}\right)^2-\left(\frac{0.2+0.5}{2}\right)\left(\frac{v_1+v_2}{2}\right)\right)\\
+0.3 &\left(\left(\frac{v_3-v_2}{0.3}\right)^2-\left(\frac{0.5+0.8}{2}\right)\left(\frac{v_2+v_3}{2}\right)\right)\\
+0.2 &\left(\left(\frac{0-v_3}{0.2}\right)^2-\left(\frac{0.8+1}{2}\right)\left(\frac{v_3+0}{2}\right)\right)
\end{aligned}
$$

4. Veges elemek, gyenge megfogalmazas.
Oszd fel a $[0,1]$ intervallumot 4 reszre a kovetkezo pontokkal: $x_i = 0.2,\ 0.5,\ 0.8$. Legyen $v(x)$ az a folytonos fuggveny, amelyik affine az alintervallumokon es az ertekei az $x = 0,\ 0.2,\ 0.5,\ 0.8,\ 1$ pontokban a kovetkezoek: $0, v_1, v_2, v_3, 0$. Legyen
$$u''(x) + xu'(x) = x^2, \quad u(0) = u(1) = 0.$$

- A DE gyenge u megoldasa milyen egyenleteket kell, hogy kielegitsen?
- Irj fel egy numerikus egyenletet a v kozelito megoldas v_i numerikus parametereire!

Solution:

- $$0 = \int_0^1 \phi\left(u'' + xu' - x^2\right)\,dx = \int_0^1 -\phi'u' + \phi\, xu' - \phi x^2\,dx$$
minden olyan ϕ-re, amire igaz, hogy $\phi(0) = \phi(1) = 0$.
- Legyen $\phi(x) = e_2(x)$ egy sator alaku fuggveny az $x = 0.5$ pont korul. Ekkor az integrandus ket darabra likalizalhato $x = 0.5$ korul. a kozeppontos numerikus modszert hasznalva az integralok kiszamitasara azt kapjuk, hogy

$$
\begin{aligned}
&0\\
\approx 0.3\cdot &\left(-\frac{1-0}{0.5-0.2}\frac{v_2-v_1}{0.5-0.2}+\frac{1}{2}\frac{0.2+0.5}{2}\frac{v_2-v_1}{0.3}-\frac{1}{2}\left(\frac{0.2+0.5}{2}\right)^2\right)\\
+0.3\cdot &\left(-\frac{0-1}{0.8-0.5}\frac{v_3-v_2}{0.8-0.5}+\frac{1}{2}\frac{0.5+0.8}{2}\frac{v_3-v_2}{0.3}-\frac{1}{2}\left(\frac{0.5+0.8}{2}\right)^2\right).
\end{aligned}
$$

Itt a szorzotenyezo $\frac{1}{2}$ nem mas, mint az e_2 fuggveny erteke az $[0.2, 0.5]$ es $[0.5, 0.8]$ intervalumok kozepen, tovabba pl. $((0.5 + 0.8)/2)^2$ az x^2 fuggveny erteke az $x = (0.5 + 0.8)/2$ pontban.

5. Oldd meg a $y' + 9y = f(t)$ DE-t!
   - Keresd meg es abrazold a G retardalt Green fuggvenyt!
   - Hasznald G-t arra, hogy kifejezd a partikularis megoldast a kovetkezo "kezdeti" feltetel mellett: $y(t) = f(t) = 0$ ha $t \ll 0$!
   - Hasznald G-t arra, hogy $t > 0$-ra kifejezd a megoldast az $y(0) = 7$ kezdeti feltetel mellett!
6. Oldd meg a $y'' + 9y = f(t)$ DE-t!
   - Keresd meg es abrazold a G retardalt Green fuggvenyt!
   - Hasznald G-t arra, hogy kifejezd a partikularis megoldast a kovetkezo "kezdeti" feltetel mellett: $y(t) = f(t) = 0$ ha $t \ll 0$!
7.
   - Ird at az elozo feladat masodrendu DE-et egy elsorendu rendszerre!
   - Milyen egyenletet elegit ki az ennek megfelelo G Green fuggveny?
   - Mennyi $G(0^+)$?
   - Ird fel G segitsegevel a masodrendu DE megoldasat az $y(0) = 7$, $y'(0) = 6$ kezdeti feltetelek mellett, ha $t > 0$!

folyt.kov.

Chapter 12

Inhomogeneous evolutionary linear DE

12.1 Input-output relation

Homogeneous evolutionary linear DE:

$$\bar{y} = A(t)\bar{y}.$$

($\bar{y}$ vector, A matrix.)

Superposition principle: If

$$\frac{d}{dt}\overline{y_1} = A(t)\overline{y_1} \quad \text{es} \quad \frac{d}{dt}\overline{y_2} = A(t)\overline{y_2}$$

then

$$\frac{d}{dt}\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right) = A(t)\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right),$$

so linear combination of solutions are solotions, too.

Inhomogeneous linear DE:

$$\bar{y} = A(t)\bar{y} + \bar{f}(t). \tag{12.1}$$

If

$$\frac{d}{dt}\overline{y}_{hom} = A(t)\overline{y}_{hom} \ \text{es} \ \frac{d}{dt}\overline{y}_{part} = A(t)\overline{y}_{part} + \bar{f}(t),$$

then

$$\frac{d}{dt}\left(\overline{y}_{hom} + \overline{y}_{part}\right) = A(t)\left(\overline{y}_{hom} + \overline{y}_{part}\right) + \bar{f}(t).$$

So the general solution of the inhom.eq. can be written as the sum of the $\overline{y}_{hom}$ gen.sol. of the hom. eq. and the $\overline{y}_{part}$ particular sol. of the inhom. eq.: $\overline{y}_{gen.inhom} = \overline{y}_{hom} + \overline{y}_{part}$.

Linear input-output relation: If

$$\frac{d}{dt}\overline{y_1} = A(t)\overline{y_1} + \overline{f}_1(t) \quad \text{es} \quad \frac{d}{dt}\overline{y_2} = A(t)\overline{y_2} + \overline{f}_2(t)$$

then

$$\frac{d}{dt}\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right) = A(t)\left(\alpha_1\overline{y_1} + \alpha_2\overline{y_2}\right) + \alpha_1\overline{f_1} + \alpha_2\overline{f_2}.$$

Solution strategy of (12.1): Solve first the DE if $f(t) =$

$$\delta(t), \qquad e^{ipt}, \qquad e^{-st},$$

then express f as their linear combination (integral):

$$f(t) \approx \sum_i f(t_i)\delta(t - t_i)\Delta t,$$
$$f(t) = \frac{1}{\sqrt{2\pi}} \int \tilde{f}(p)e^{ipt}\, dp,$$
$$f(t) = \frac{1}{2\pi i} \int F(s)e^{st}\, ds.$$

12.2 Distributions (generalized functions)

Let $\mathcal{D}$ be the vector space of the smooth, compactly supported (i.e. zero outside a finite interval) functions. Then every $f \in \mathcal{D}$ defines a linear functional $L_f : \mathcal{D}$-t $\mathbb{R}$ by

$$L_f : \ g \to L_f[g] = \int_{-\infty}^{\infty} f(x)g(x)\, dx = \langle f, g\rangle.$$

However for the $\delta : g \to g(0)$ funkcional there is no $f \in \mathcal{D}$ function, such that $L_f(g) = g(0)$ for all g. (Nevertheless δ can be approximated (as $n \to \infty$) by $\delta_n(x) \in \mathcal{D}$ positive functions such that $\int \delta_n(x)\, dx = 1$, and the domain where $\delta_n(x)$ is not zero is becoming more and more concentrated around 0.) The space of all linear functionals defined on $\mathcal{D}$ is called the $\mathcal{D}'$ space of distributions. Derivation is defined on them. A motivation for the definition: If $f, g \in \mathcal{D}$ then

$$\langle f', g\rangle = -\langle f, g'\rangle$$

according to the rule of partial integration. So for the $d \in \mathcal{D}'$ distribution let us define d' by the rule

$$\langle d', g\rangle = -\langle d, g'\rangle \qquad \forall g \in \mathcal{D}$$

Then we can prove that $H' = \delta$:

$$\langle H', g\rangle = -\langle H, g'\rangle = -\int_{-\infty}^{\infty} H(x)g'(x)\, dx = -\int_{0}^{\infty} g'(x)\, dx$$
$$= -g(x)\,|_0^{\infty} = -(g(\infty) - g(0)) = g(0) = \langle \delta, g\rangle.$$

Problema 75. *Let*

$$F(t) = \begin{cases} 0 & ha\ t < 0 \\ t & ha\ 0 < t. \end{cases} \tag{12.2}$$

Prove that $F' = H$, i.e. $F'' = \delta$!

Problema 76. *Let*

$$G(t) = \begin{cases} 0 & ha\ t < 0 \\ e^{-7t} & ha\ 0 < t. \end{cases} \tag{12.3}$$

Prove that $G' + 7G = \delta$!

Problema 77. *1. Explain why $(\delta(x))^2$ does not make sense!*

2. Is it possible to define $\delta(x_1)\delta(x_2) = \delta(\bar{x})$?

12.3 Inhomogeneous equations

12.3.1

The $y' = f(t)$ DE is the simplest example of an inhom.lin. DE. Given the $y'(t) = f(t)$ velocity, what is th $y(t)$ position?

$$y'(t) = f(t),\ y(t_0) = y_0 \quad \Longleftrightarrow \quad y(t) = y_0 + \int_{t_0}^{t} f(\tau)\, d\tau.$$

Let $y(t) = f(t) = 0$, ha $t \ll 0$. Then

$$y(t) = \int_{-\infty}^{t} f(\tau)\, d\tau = \int_{-\infty}^{\infty} H(t,\tau) f(\tau)\, d\tau = \int_{-\infty}^{\infty} H(t-\tau) f(\tau)\, d\tau, \qquad (12.4)$$

wher H is the Heaviside (theta, unit step) function:

$$H(t) = \begin{cases} 0 & \text{ha } t < 0 \\ 1 & \text{ha } 0 < t. \end{cases}$$

What are the properties of the $H'(t) = \delta(t)$ Dirac-delta generalized function?

$$\delta(t) = 0, \ \text{ ha } \ t \neq 0, \quad \text{illetve} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Dirac-delta as a limit (for example):

$$\delta_n = \begin{cases} n + n^2 t & \text{ha } -1/n < t < 0 \\ n - n^2 t & \text{ha } 0 < t < 1/n \\ 0 & \text{otherwise.} \end{cases}$$

$H_n' = \delta_n$:

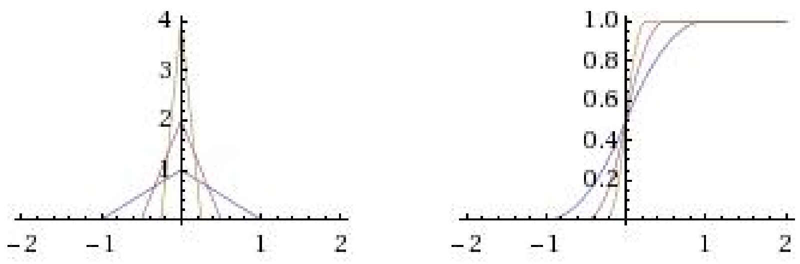


Figure 12.1: $\delta_{1,2,4} \to \delta$, $H_{1,2,4} \to H$.

Given the $y = H$ solution of $y' = \delta$ (fundamental solution, impulse response, retarded Green-function), the the solution of $y' = f$ is (12.4).

12.3.2

If the acceleration is $y''(t) = f(t)$, what is the $y(t)$ position?

Foe example:

$$y'' = 10,\ y(1) = 2,\ y'(1) = 3.$$
$$y' = \int 10\,dt = 10t + C_1, \quad y = \int 10t + C_1\,dt = 5t^2 + C_1 t + C_2,$$
$$5 \cdot 1^2 + C_1 \cdot 1 + C_2 = 2, \quad 10 \cdot 1 + C_1 = 3 \quad \Longrightarrow \quad y = 5t^2 - 7t + 4.$$

$$y'' = f(t), \quad v = y', \quad v' = f(t) \quad \Longrightarrow \quad y'(t) = v(t) = v(t_0) + \int_{t_0}^{t} f(\tau)\,d\tau,$$
$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} \left(v(t_0) + \int_{t_0}^{\tau} f(\tau_2)\,d\tau_2 \right) d\tau \\ &= y(t_0) + (t - t_0)v(t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{\tau} f(\tau_2)\,d\tau_2 \right) d\tau \end{aligned}$$

If $y(t) = f(t) = 0$ for $x \ll 0$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} f(\tau_2)\,d\tau_2 \right) d\tau = \int_{-\infty}^{\tau_2} \left(\int_{\tau_2}^{t} f(\tau_2)\,d\tau \right) d\tau_2 \\ &= \int_{-\infty}^{\tau_2} (t - \tau_2) f(\tau_2)\,d\tau_2 = \int_{-\infty}^{\tau} (t - \tau) f(\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} F(t - \tau) f(\tau)\,d\tau, \end{aligned}$$

where

$$F(t) = \begin{cases} 0 & \text{ha } t < 0 \\ t & \text{ha } 0 < t. \end{cases} \tag{12.5}$$

$F(t)$ solves $F'' = \delta$, here F' has a unit jump at $t = 0$. ($H(t)$ solves $H' = \delta$, now H has a unit jump at $t = 0$.)

12.3.3

$y'(t) + 3y(t) = f(t)$, $y(t) = f(t) = 0$ ha $t \ll 0$. We would like to express y as $y(t) = \int_{-\infty}^{\infty} G(t - \tau) f(\tau)\,d\tau$, where G solves $G' + 3G = \delta$. What do we know about G ?

1. $G(t) = 0$ if $t < 0$, since $y(t) = f(t) = 0$ if $t \ll 0$.
2. If $t \approx 0$ then G is approximately the same as the Heaviside solution of $H' = \delta$, i.e. $G(0^+) - G(0^-) = 1 = H(0^+) - H(0^-)$.
3. If $t > 0$, then $G' + 3G = 0$, since $\delta(t) = 0$, ha $t \neq 0$.

since the solution of $u'(t) + 3u(t) = 0$, $u(0) = 1$ is $u(t) = e^{-3t}$, we have

$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ e^{-3t} & \text{ha } t > 0. \end{cases}$$

Consequently

$$y(t) = \int_{-\infty}^{\infty} G(t-s)f(s)\,ds = \int_{-\infty}^{t} e^{-3(t-s)} f(s)\,ds.$$

This expression describes a system that was at rest in the distant past. If $y(0)$ is given, then

$$y(t) = y(0)G(t) + \int_{0}^{\infty} G(t-s)f(s)\,ds,$$

since G satisfies the $G(0^+) = 1$ initial condition (Duhamel-principle).

12.3.4

$y''(t) + 9y(t) = f(t)$, $y(t) = f(t) = 0$ ha $t \ll 0$. We would like to write y as $y(t) = \int_{-\infty}^{\infty} G(t-\tau)f(\tau)\,d\tau$, where G solves $G'' + 9G = \delta$. What do we know about G ?

1. $G(t) = 0$ if $t < 0$, since $y(t) = f(t) = 0$ if $t \ll 0$.
2. If $t \approx 0$, then G behaves as the (12.5) solution of $F'' = \delta$, i.e. $G(0^+) - G(0^-) = 0 = F(0^+) - F(0^-)$ and $G'(0^+) - G'(0^-) = 1 = F'(0^+) - F'(0^-)$.
3. If $t > 0$, then $G'' + 9G = 0$, since $\delta(t) = 0$, if $t \neq 0$.

As the solution of $u''(t) + 9u(t) = 0$, $u'(0) = 1$, $u(0) = 0$ is $u(t) = \frac{1}{3}\sin(3t)$, we have

$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ \frac{1}{3}\sin(3t) & \text{ha } t > 0. \end{cases}$$

So

$$y(t) = \int_{-\infty}^{\infty} G(t-s)f(s)\,ds = \int_{-\infty}^{t} \frac{1}{3}\sin(3(t-s))f(s)\,ds.$$

Unfortunately we can not apply here the Duhamel principle.

12.3.5

Radioactive decay $1 \to 2 \to \emptyset$:

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Particular solution of the hom.eq: if

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{illetve} \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} + 3e^{-2t} \end{pmatrix}, \qquad \text{illetve} \qquad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}.$$

The G_{i1} response of the system for a unit impulse of the 1 material at time zero:

$$\frac{d}{dt}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} + \begin{pmatrix} \delta(t) \\ 0 \end{pmatrix}.$$

The same for the 2 material:

$$\frac{d}{dt}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ \delta(t) \end{pmatrix}.$$

We can unify these equations:

$$\frac{d}{dt}\begin{pmatrix} G_{12} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} G_{12} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} + \delta(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So

$$G(0^+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Consequently the retarded ($G(t) = 0$, if $t < 0$) solution for $t > 0$ is

$$G(t) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} e^{-3t} & 0 \\ -3e^{-3t} + 3e^{-2t} & e^{-2t} \end{pmatrix},$$

while for negative t the value of G is zero. Then the solution of

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix},$$

($\bar{f}(t) = \bar{y}(t) = 0$, if $t \ll 0$) is

$$\bar{y}(t) = \int_{-\infty}^{\infty} G(t-s)\bar{f}(s)\,ds.$$

Formally

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\frac{d}{dt} - \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\right)^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

Unfortunately it is not easy to define the inverse of an operator with zero eigenvectors (solutions of the hom.eq.)

Duhamel principle: if

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 13 \\ 31 \end{pmatrix}$$

then

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = G(t)\begin{pmatrix} 13 \\ 31 \end{pmatrix} + \int_0^{\infty} G(t-s)\bar{f}(s)\,ds, \qquad \text{ha } t > 0.$$

Problema 78. *Express $y''+9y = f(t)$ as a first order system! Find the retarded Green function! Write down the solution of initial value problem with the help of the Duhamel principle!*

Problema 79. *Express $y'' + 4y' + 5y = f(t)$ as a first order system! Find the retarded Green function! Write down the solution of initial value problem with the help of the Duhamel principle!*

Chapter 13

Finite elements

13.1 Calculus of variations, Euler-Lagrange equation

Reference: P.Olver: *http : //www.math.umn.edu/˜olver/ln_/cv.pdf*

How can we minimize the

$$E[u] = \int_0^1 \frac{1}{2}[u'(x)]^2 - f(x)u(x)\,dx = \int_0^1 \frac{1}{2}L(x,u,u')\,dx$$

energy functional with boundary conditions $u(0) = u(1) = 0$?

At the critical point u we have

$$\begin{aligned} E[u+\delta u] - E[u] &\approx \int_0^1 \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}(\delta u)'\,dx \\ &= \frac{\partial L}{\partial u'}\bigg|_0^1 + \int_0^1 \left[\frac{\partial L}{\partial u} - \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right)\right]\delta u\,dx \end{aligned}$$

as this is true for all $\delta u : \delta u(0) = \delta u(1) = 0$ variation,

$$\frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) - \frac{\partial L}{\partial u} = 0.$$

This is the Euler-Lagrange equation.

In higher dimensions: $L = L(\bar{x}, \phi^\mu(\bar{x}), \partial_{x_i}\phi^\mu(\bar{x}))$,

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial(\partial_{x_i}\phi^\mu)}\right) - \frac{\partial L}{\partial \phi^\mu} = 0.$$

Problema 80. *Write down the EL equations for the following Lagrangians:*

1. *A point particle $\bar{x}(t)$ moving in a $V(\bar{x})$ potential field:*

$$L = \frac{1}{2}\sum_i \dot{x}_i^2 - V(\bar{x}).$$

2. *A point particle $\bar{x}(t)$ moving in a $\bar{A}(\bar{x})$ magnetic potential:*

$$L = \frac{1}{2}\sum_i \dot{x}_i^2 + \bar{A}(\bar{x}) \cdot \dot{\bar{x}}.$$

3. *A monocycle's coordinates: $x(t), y(t), \phi(t)$.*

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\phi}^2\right) + \lambda(t) \cdot (\dot{x}\sin\phi - \dot{y}\cos\phi).$$

The EL equation correponding to λ ensures that the velocity vector $(\dot{x}, \dot{y})$ has the same direction as $(\cos(\phi), \sin(\phi))$ has.

Problema 81. *Write down the Euler-Lagrange equations for the Lagrangians L and M !*

$$(y')^2 - y^2, \quad y' + 8, \quad (y')^2 + y', \quad L = (y')^4 + (y-1)^2,$$
$$M = \left((y_1')^2 + (y_2')^2\right)/2 - V(y_1, y_2),$$
$$\left((y_1')^2 + (y_2')^2\right)/2 + A_1(y_1, y_2)y_1' + A_2(y_1, y_2)y_2'$$

Megoldas 46. *Solution:*
L:

$$\frac{\partial L}{\partial y} = 2(y-1), \qquad \frac{\partial L}{\partial y'} = 4(y')^3,$$
$$\frac{d}{dx}\left(4(y')^3\right) - 2(y-1) = 0.$$

M:

$$\frac{\partial M}{\partial y_1'} = y_1', \qquad \frac{\partial M}{\partial y_2'} = y_2', \qquad \frac{\partial M}{\partial y_1} = -\frac{\partial V}{\partial y_1}, \qquad \frac{\partial M}{\partial y_2} = -\frac{\partial V}{\partial y_2},$$
$$\frac{d}{dx}y_1' - (-\frac{\partial V}{\partial y_1}) = 0, \qquad \frac{d}{dx}y_2' - (-\frac{\partial V}{\partial y_2}) = 0,$$

vagy

$$y_1'' = -\frac{\partial V}{\partial y_1}, \qquad y_2'' = -\frac{\partial V}{\partial y_2}$$

13.2 Finite differences

Try to approximate $u \in Fun([0,1])$ by its values at $x_i = i\Delta x$, $i = 1, \ldots, (N-1)$, where $N = 1/\Delta x$:

$$\vec{u} = (u(1 \cdot \Delta x, \ldots, u(i \cdot \Delta x), \ldots, u((N-1)\Delta x))^T = (u_1, \ldots, u_{N-1})^T,$$

so the components of the vector are $u_i = u(x_i) = u(i \cdot \Delta x)$.

Derivation: Numerical approximation of the derivative:

$$u'(x) \approx \frac{1}{\Delta x}\left(u(x+\Delta x) - u(x)\right) \approx \frac{1}{\Delta x}\left(u(x) - u(x-\Delta x)\right)$$
$$\approx \frac{1}{2\Delta x}\left(u(x+\Delta x) - u(x-\Delta x)\right).$$

If the vector space of the functions $u(x)$ are approximated by finite (three) dimensional vectors

$$\vec{u} = (u(0.25), u(0.5), u(0.75))^T = (u_1, u_2, u_3) \in \mathbb{R}^3$$

(here $\Delta x = 1/4$), then the approximations of u' are

$$\frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$
$$\frac{1}{\Delta x}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$
$$\frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

The last one is the most accurate, its matrix is also antisymmetric.

Derivation in the infinite dimensional space of functions $\Longrightarrow$ *linear transzformations, matrices in finite dimensional vector spaces.*

The approximation ofu'' :

$$u''(x) \approx \frac{1}{\Delta x^2}(u(x+\Delta x) - 2u(x) + u(x-\Delta x)).$$

So a numerical approximation of the equation

$$-u''(x) = f(x), \quad \text{vagy} \quad -\frac{d^2}{dx}u = f, \qquad u(0) = u(1) = 0,$$

might be:

$$\frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \text{vagy} \quad \Delta x \cdot L\,\vec{u} = \vec{f}.$$

Then

$$(\Delta x \cdot L)\,\vec{u} = \vec{f},$$
$$(\Delta x \cdot G)\,\vec{f} = \vec{u} = \Delta x\left(\Delta x^2 L\right)^{-1}\vec{f},$$

so

$$G = \left(\Delta x^2 L\right)^{-1} = \Delta x \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0.1875 & 0.125 & 0.0625 \\ 0.125 & 0.25 & 0.125 \\ 0.0625 & 0.125 & 0.1875 \end{pmatrix}.$$

13.2.1 Finite elements method (FEM)

Ref: P.Olver: $http://www.math.umn.edu/\tilde{\ }olver/num\_/lnf.pdf$

Energy minimalization: Instead of approximating u by a finite dimensional vector, let us try to approximate it by a function in a finite dimensional subspace $V \subset Fun$.

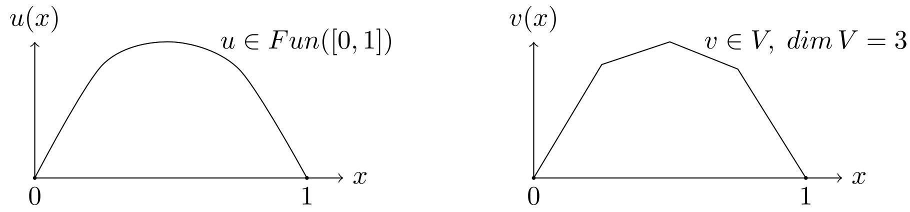


Figure 13.1: The piecewise affine and continuous functions v form a three dimensional subspace in the space of "all" functions.

A basis for V:

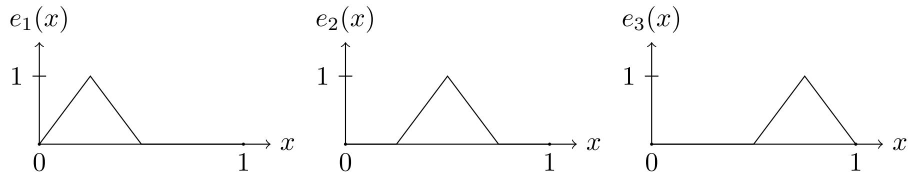


Figure 13.2: The tent shaped functions form a basis of$V$.

So the approximation of u is

$$\begin{aligned} u(x) \approx v(x) &= u(0.25)e_1(x) + u(0.5)e_2(x) + u(0.75)e_3(x) \\ &= v_1 e_1(x) + v_2 e_2(x) + v_3 e_3(x). \end{aligned}$$

This approximation does not make too much sense if it is substituted into (10.1), as v'' is zero wherever it is defined. However the energy functional (10.2)

$$\text{Energia}[v] = \int_0^1 \frac{1}{2}[v'(x)]^2 - f(x)v(x)\,dx$$

approximates Energia$[u]$ quite well, if the first derivatives of u and v are close to each other. For the computation of the integral of $f(x)v(x)$ we might use the approximation

$$\begin{aligned} f(x) &\approx f(0.25)e_1(x) + f(0.5)e_2(x) + f(0.75)e_3(x) \\ &= f_1 e_1(x) + f_2 e_2(x) + f_3 e_3(x). \end{aligned}$$

Then

$$
\begin{aligned}
E[\bar{v}] = E[(v_1, v_2, v_3)^T] = \\
+4v_1^2 - 4v_2v_1 + 4v_2^2 + 4v_3^2 - 4v_2v_3 \\
-\frac{f_1v_1}{6} - \frac{f_2v_1}{24} - \frac{f_1v_2}{24} - \frac{f_2v_2}{6} - \frac{f_3v_2}{24} - \frac{f_2v_3}{24} - \frac{f_3v_3}{6} \\
= (v_1, v_2, v_3) \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
+(f_1, f_2, f_3) \begin{pmatrix} -\frac{1}{6} & -\frac{1}{24} & 0 \\ -\frac{1}{24} & -\frac{1}{6} & -\frac{1}{24} \\ 0 & -\frac{1}{24} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
= \bar{v}^T L\bar{v} + \bar{f}^T M\bar{v} = \bar{v}^T L\bar{v} + \bar{g}^T\bar{v},
\end{aligned} \tag{13.1}
$$

where $\bar{g} = M^T\bar{f}$. This expression was generated by $N = 4$ integration on the subintervals, for example on the third $[0.5, 0.75]$ element

$$v(x) = v_2 + (v_3 - v_2)\frac{x - 0.5}{0.25}, \qquad f(x) = f_2 + (f_3 - f_2)\frac{x - 0.5}{0.25},$$

and its contribution to Energia$[v]$ is

$$\int_{0.5}^{0.75} \frac{1}{2}\left(\frac{v_3 - v_2}{0.25}\right)^2 - \left(f_2 + (f_3 - f_2)\frac{x - 0.5}{0.25}\right)\left(v_2 + (v_3 - v_2)\frac{x - 0.5}{0.25}\right) dx.$$

Of course other approximations can be used for the computation of the integrals. For example one might wish to use some sort of midpoint method:

$$\int_{0.5}^{0.75} f(x)v(x)\,dx \approx 0.25 \cdot \frac{f_2 + f_3}{2} \cdot \frac{v_2 + v_3}{2}.$$

Let us minimize (13.2.1)! At the critical point

$$0 = \frac{\partial}{\partial v_k}\left(\sum_{i,j} v_i L_{ij} v_j + \sum_i g_i v_i\right) = 2\sum_j v_i L_{ik} + g_k$$

for all k, so

$$2\bar{v}^T L = -\bar{g}^T \quad \Leftrightarrow \quad 2L^T\bar{v} = -\bar{g} \quad \Leftrightarrow \quad \bar{v} = -\frac{1}{2}(L^T)^{-1}\bar{g} = -\frac{1}{2}(L^{-1})^T M^T\bar{f}.$$

The vector $\bar{v}$ gives the approximative solution.

Weak solution: How can we generate similar equations, if the the energy functional is unknown Let u be the solution of (10.1)

$$u''(x) + f(x) = 0, \;\; u(0) = u(1) = 0.$$

Then for all $\phi(x) : \phi(0) = \phi(1) = 0$ we have

$$0 = \int_0^1 \phi(x)(u''(x) + f(x))\,dx = \int_0^1 -\phi'(x)u'(x) + \phi(x)f(x)\,dx \tag{13.2}$$

since in the partial integration the boundary term $\phi(x)u'(x)\mid_0^1$ drops out by the boundary condition for ϕ. just as in the case of the energy functional Energia$[v]$, all we need is the first derivative of u. The approximation of the solution $u(x)$

$$u(x) \approx v(x) = v_1e_1(x) + v_2e_2(x) + v_3e_3(x)$$

can be found as follows. We require that the second integral in (13.2) must be zero, if we write v instead of u-t, and if ϕ is substituted by $\phi(x) = c_1e_1(x) + c_2e_2(x) + c_2e_2(x)$.

As $\bar{c}$ is arbitrary, the next three integlas must be zero:

$$0 = \int_0^1 -e_i'(x)v'(x) + e_i(x)f(x)\,dx, \qquad i = 1, 2, 3. \tag{13.3}$$

If we use the $f \approx f_1e_1 + f_2e_2 + f_3e_3$ approximation, then we obtain the following set of equations:

$$\begin{aligned}
\frac{1}{6}f_1 + \frac{1}{24}f_2 + 0f_3 - 8v_1 + 4v_2 + 0v_3 &= 0 \\
\frac{1}{24}f_1 + \frac{1}{6}f_2 + \frac{1}{24}f_3 + 4v_1 - 8v_2 + 4v_3 &= 0 \\
0f_1 + \frac{1}{24}f_2 + \frac{1}{6}f_3 + 0v_1 + 4v_2 - 8v_3 &= 0.
\end{aligned}$$

For example the first one is obtained as

$$0 = \int_0^{0.25} -\frac{1}{0.25}\frac{v_1}{0.25} + \frac{x}{0.25}\frac{f_1x}{0.25}\,dx$$
$$+\int_{0.25}^{0.5} -\frac{-1}{0.25}\frac{v_2 - v_1}{0.25} + \left(1 - \frac{x - 0.25}{0.25}\right)\left(v_1 + (v_2 - v_1)\frac{x - 0.25}{0.25}\right)\,dx.$$

If $\bar{f}$ is given, the vector $\bar{v}$ determines the approximate solution $v(x)$.

Try to work out the following problems using both of the methods of energy minimization and weak solutions!

Problema 82. *What kind of changes are necessary if the boundary condition is* $u(0) = 2$, $u(1) = 3$ *?*

Problema 83. *Let* $S[u] = \int_0^1 (y'(x))^4 + xy(x)\,dx$ *where* u *is defined on* $[0, 1]$ *and vanishes at the endpoints. Let* V *be defined on* $[0, 1]$*, assume that it vanishes at the endpoints and is continuous. Assume also that elements of* V *are piecewise affine on the* $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$ *intervals. Let* ϕ_1 *and* ϕ_2 *be a basis of* V*, such that* $\phi_1(1/3) = \phi_2(2/3) = 1$ *and* $\phi_2(1/3) = \phi_1(2/3) = 0$*. Let* $u_h = c_1\phi_1 + c_2\phi_2$*. Compute the* $S[u_h] = s(c_1, c_2)$ *two variable function! (For the computation of the* $xy(x)$ *term in the integral use some approximate method!)*

Megoldas 47. *Solution:*

$$S[u_h]$$
$$\approx \left[(3c_1)^4 \cdot \frac{1}{3} + (3(c_2 - c_1))^4 \cdot \frac{1}{3} + (-3c_2)^4 \cdot \frac{1}{3}\right]$$
$$+\left[\frac{1}{2}\Big(0 \cdot 0 + \frac{1}{3} \cdot c_1\Big) \cdot \frac{1}{3} + \frac{1}{2}\Big(\frac{1}{3} \cdot c_1 + \frac{2}{3} \cdot c_2\Big) \cdot \frac{1}{3} + \frac{1}{2}\Big(\frac{2}{3} \cdot c_2 + 1 \cdot 0\Big) \cdot \frac{1}{3}\right]$$

Sample problems for Test 2

1. Finite differences.
   Find numerical equations for an approximate solution of the DE

   $$u''(x) + xu'(x) = x^2, \quad u(0) = u(1) = 0.$$

   Approximate the function u by the vector $\vec{u}_i = u(i\Delta x),\ i = 1, \ldots, 4,\ \Delta x = 1/5$.

   - Express $u''(x)$ by $u(x \pm \Delta x), u(x)$!
   - How do you approximate $u'(x)$? Write down the corresponding finite difference approximation of the DE as an inhom.lin. equation for $\vec{u}$!

   Solution:

   - $$u''(x) = \frac{1}{\Delta x^2}(u(x + \Delta x) - 2u(x) + u(x + \Delta x))$$
   - $$u'(x) = \frac{1}{\Delta x}(u(x + \Delta x) - u(x)).$$

   $$\begin{aligned} &\frac{1}{\Delta x}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \\ + &\begin{pmatrix} 1 \cdot \Delta x & & & \\ & 2 \cdot \Delta x & & \\ & & 3 \cdot \Delta x & \\ & & & 4 \cdot \Delta x \end{pmatrix} \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \\ &= \begin{pmatrix} (1 \cdot \Delta x)^2 \\ (2 \cdot \Delta x)^2 \\ (3 \cdot \Delta x)^2 \\ (4 \cdot \Delta x)^2 \end{pmatrix}. \end{aligned}$$

   Just for practice, solve a similar problem:
   $u''' + x^2 u' = x$, $u'(x) \approx (u(x + \Delta x) - u(x - \Delta x))/(2\Delta x)$!

2. Euler-Lagrange equations.
   Let

   $$L(x(t), y(t), \dot{x}(t), \dot{y}(t)) = \dot{x}^3 + \dot{y}^2 + x\dot{y} + x^5 y^6.$$

   Write down the EL equations for this Lagrange function!

   Solution:

   $$EL: \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_i} - \frac{\partial L}{\partial \phi_i} = 0, \qquad \phi_1 = x,\ \phi_2 = y.$$

   $$\frac{\partial L}{\partial x} = \dot{y} + 5x^4 y^6, \quad \frac{\partial L}{\partial \dot{x}} = 3\dot{x}^2, \qquad \frac{\partial L}{\partial y} = x^5 \cdot 6y^5, \quad \frac{\partial L}{\partial \dot{y}} = 2\dot{y} + x,$$

   $$\frac{d}{dt}\left(3\dot{x}^2\right) - \left(\dot{y} + 5x^4 y^6\right) = 6\ddot{x}\dot{x} - \left(\dot{y} + 5x^4 y^6\right) = 0,$$

   $$\frac{d}{dt}(2\dot{y} + x) - x^5 \cdot 6y^5 = (2\ddot{y} + \dot{x}) - \left(x^5 \cdot 6y^5\right) = 0.$$

3. Finite Elements, variational formulation.
Divide the $[0, 1]$ intervall to 4 subintervals by the points $x_i = 0.2,\ 0.5,\ 0.8$. Let $v(x)$ be the continuous function which is affine on the subintervals and its values at $x = 0,\ 0.2,\ 0.5,\ 0.8,\ 1$ are $0, v_1, v_2, v_3, 0$.

   - Write down the EL equation for the $Energy[u]$ functional, where $Energy$ is given by the next item!
   - Compute
   $$Energy[v] = \int_0^1 (v')^2 - xv\,dx$$
   approximatly or exactly!

   Solution:

   - $$\frac{\partial Energy}{\partial v} = -x, \quad \frac{\partial Energy}{\partial v'} = 2v', \qquad \frac{d}{dx}(2v') - (-x) = 2v'' + x = 0.$$
   - We compute (approximately) the integral of xv by using the midpoint values of x and v for each subinterval.
   $$\begin{aligned} &Energy[v] \\ \approx 0.2 &\left(\left(\frac{v_1 - 0}{0.2}\right)^2 - \left(\frac{0 + 0.2}{2}\right)\left(\frac{0 + v_1}{2}\right)\right) \\ +0.3 &\left(\left(\frac{v_2 - v_1}{0.3}\right)^2 - \left(\frac{0.2 + 0.5}{2}\right)\left(\frac{v_1 + v_2}{2}\right)\right) \\ +0.3 &\left(\left(\frac{v_3 - v_2}{0.3}\right)^2 - \left(\frac{0.5 + 0.8}{2}\right)\left(\frac{v_2 + v_3}{2}\right)\right) \\ +0.2 &\left(\left(\frac{0 - v_3}{0.2}\right)^2 - \left(\frac{0.8 + 1}{2}\right)\left(\frac{v_3 + 0}{2}\right)\right) \end{aligned}$$

4. Finite Elements, weak formulation.
Divide the $[0, 1]$ intervall to 4 subintervals by the points $x_i = 0.2,\ 0.5,\ 0.8$. Let $v(x)$ be the continuous function which is affine on the subintervals and its values at $x = 0,\ 0.2,\ 0.5,\ 0.8,\ 1$ are $0, v_1, v_2, v_3, 0$. Let
$$u''(x) + xu'(x) = x^2, \quad u(0) = u(1) = 0.$$

   - What equations should be satisfied by the weak solution u of the DE?
   - Write down a numerical equation which should be satisfied by the parameters v_i of the approximate solution v !

   Solution:

   - $$0 = \int_0^1 \phi\left(u'' + xu' - x^2\right)\,dx = \int_0^1 -\phi' u' + \phi\, xu' - \phi x^2\,dx$$
   for all ϕ, such that $\phi(0) = \phi(1) = 0$.

- Let $\phi(x) = e_2(x)$ be the tent-like basis function centered at $x = 0.5$. The integrand is localized to two subintervals around $x = 0.5$. The previous equation reads as now (we are using the midpoint approximation for the computations of the integrals)

$$
\begin{aligned}
&0 \\
&\approx 0.3 \cdot \left(-\frac{1-0}{0.5-0.2}\frac{v_2 - v_1}{0.5-0.2} + \frac{1}{2}\frac{0.2+0.5}{2}\frac{v_2-v_1}{0.3} - \frac{1}{2}\left(\frac{0.2+0.5}{2}\right)^2 \right) \\
&+0.3 \cdot \left(-\frac{0-1}{0.8-0.5}\frac{v_3 - v_2}{0.8-0.5} + \frac{1}{2}\frac{0.5+0.8}{2}\frac{v_3-v_2}{0.3} - \frac{1}{2}\left(\frac{0.5+0.8}{2}\right)^2 \right).
\end{aligned}
$$

  Here the factor $\frac{1}{2}$ is the value of e_2 at the midpoints of the intervals $[0.2, 0.5]$ and $[0.5, 0.8]$, while (for example) $((0.5{+}0.8)/2)^2$ is the value of x^2 at $(0.5 + 0.8)/2$.

5. Solve the $y' + 9y = f(t)$ DE!
   - Find and plot the retarded Green function G!
   - Use G to express the solution of the DE under the conditions $y(t) = f(t) = 0$ for $t \ll 0$!
   - Use G to express the solution of the DE for $t > 0$ under the initial condition $y(0) = 7$!
6. Solve the $y'' + 9y = f(t)$ DE!
   - Find and plot the retarded Green function G!
   - Use G to express the solution of the DE under the conditions $y(t) = f(t) = 0$ for $t \ll 0$!
7.
   - Rewrite the previous second order DE as a first order system!
   - What equation is satisfied by the corresponding Green function G ?
   - What are the values of the matrix elements of G at $t = 0^+$?
   - Use G to express the solution of the DE for $t > 0$ under the initial conditions $y(0) = 7$, $y'(0) = 6$!