Test2, Diff.Eq., 2018.05.02. NEPTUN: Name:

Signature:

1. (3+3+4 pints)
Compute $F(s)$!
a) $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\exp(-2t+7))$.
$F(s)=$ $\int_0^\infty e^{-st} e^{-2t+7} dt = \int_0^\infty e^7 \cdot e^{-(s+2)t} dt = e^7 \frac{e^{-(s+2)t}}{-(s+2)} \Big|_{t=0}^{\infty}$

$= e^7 \cdot \left(\frac{0}{-(s+2)} - \frac{1}{-(s+2)} \right) = e^7 \cdot \frac{1}{s+2}$

For what values of s does the improper integral defining the laplace transform exist?

$\mathrm{Re}\, s > -2$

b) Let $f(t) = t-1$ and $g(t) = t$. What is their convolution $h = f * g$!

$(f*g)(t) = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t (t-\tau-1)\tau \, d\tau = \int_0^t t\tau - \tau^2 - \tau \, d\tau =$

$= (t-1) \cdot \left[\frac{\tau^2}{2}\right]_0^t - \left[\frac{\tau^3}{3}\right]_0^t = (t-1)\frac{t^2}{2} - \frac{t^3}{6} = \frac{1}{6}t^3 - \frac{1}{2}t^2$

c) Let

$$\frac{d}{dt}\bar{y}(t) = \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

How much is the Laplace transform $\bar{Y}(s)$ of the solution $\bar{y}(t)$?

$\begin{pmatrix} sY_1(s) - 2 \\ sY_2(s) - 3 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}$

$-\begin{pmatrix} 5-s & 6 \\ 7 & 8-s \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = -\begin{pmatrix} 5-s & 6 \\ 7 & 8-s \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

2. ((2+2)+(4+2) points)
Which function is holomorph (complex differentiable)? Explain why!

- $f(z) = f(x+iy) = -x - iy, = u(x,y) + i\,v(x,y)$ $\quad$ CR: $u'_x = v'_y\,,\quad -v'_x = u'_y$

$$(-x)'_x = (-y)'_y \qquad -(-y)'_x = (-x)'_y$$
$$-1 = -1 \checkmark \qquad 0 = 0 \checkmark \qquad \text{holomorph}$$

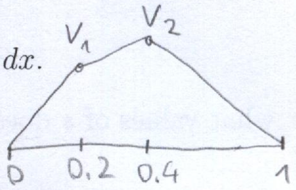
- $f(z) = f(x+iy) = x^2 + 3ixy - 4y^2$,

$$(x^2-4y^2)'_x \neq (3xy)'_y$$
$$2x \neq 3x \qquad \text{not holomorph}$$

Finite elements, variational calculus.
Divide the $x \in [0,1]$ interval by the points $x_i = 0.2, 0.4$. Let $v(x)$ be a continuous and (on the subintervals) affine function whose values at the points $(x = 0, 0.2, 0.4, 1)$ are $(0, v_1, v_2, 0)$.

- Compute

$$Energy[v] = \tilde{E}(v_1, v_2) = \int_0^1 2(v')^2 - v' - (x^2-1)v\, dx.$$



- Write down the Euler-Lagrange equation for $Energy[v]$!

$$Energy[v] = \left[2\cdot\left(\frac{v_1-0}{0.2}\right)^2 - \left(\frac{v_1-0}{0.2}\right) - \left(\left(\frac{0+0.2}{2}\right)^2-1\right)\cdot\left(\frac{0+v_1}{2}\right)\right]\cdot 0.2$$
$$+\left[2\cdot\left(\frac{v_2-v_1}{0.2}\right)^2 - \left(\frac{v_2-v_1}{0.2}\right) - \left(\left(\frac{0.2+0.4}{2}\right)^2-1\right)\cdot\left(\frac{v_1+v_2}{2}\right)\right]\cdot 0.2$$
$$+\left[2\cdot\left(\frac{0-v_2}{0.6}\right)^2 - \left(\frac{0-v_2}{0.6}\right) - \underbrace{\left(\left(\frac{0.4+1}{2}\right)^2-1\right)\cdot\left(\frac{v_2+0}{2}\right)}_{\text{midpoint values of } x^2-1 \text{ and } v(x)}\right]\cdot 0.6$$

$$\frac{d}{dx}\frac{\partial L}{\partial v'} - \frac{\partial L}{\partial v} = 0$$

$$\frac{d}{dx}\left[4v'-1\right] - \left[-(x^2-1)\right] = 0$$

$$4v''(x) + (x^2-1) = 0$$

3. (3+4+3 points)
A) Let $f(x) = \sqrt[3]{x}$. What is f the linear approximation of f around $x_0 = 8$! Give an upper bound for $|f(8 + \Delta x) - f(8) - f'(8)\Delta x|$ when $\Delta x \in [0, 0.1]$!

$$f(8+\Delta x) \approx \sqrt[3]{8} + \frac{1}{3} 8^{-2/3} \cdot \Delta x = 2 + \frac{1}{3 \cdot 4} \Delta x$$

$$|\mathrm{error}(\Delta x)| \leq \frac{1}{2} \Delta x^2 \max_{z \in [8, 8+\Delta x]} |f''(z)| = \frac{1}{2} \Delta x^2 \cdot \left| \frac{1}{3} \cdot \left(-\frac{2}{3}\right) \cdot 8^{-5/3} \right|$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2^5} \Delta x^2 = \frac{1}{9 \cdot 32} \Delta x^2 = \frac{1}{288} \Delta x^2$$

B) Let $y'(t) = (2 + y(t))(3 + t)$, $y(1) = 2$. Compute the second order Taylor polynom of $y(1 + \Delta t)$!

$$y''(t) = \left(\partial_t + (2+y)(3+t)\partial_y\right)\left[(2+y)(3+t)\right] = (2+y) + (2+y)(3+t) \cdot (3+t)$$
$$= (2+y) \cdot \left[1 + (3+t)^2\right]$$

$y(1) = 2$
$y'(1) = (2+2) \cdot (3+1) = 16$
$y''(1) = (2+2) \cdot [1 + (3+1)^2] = 4 \cdot 17 = 68$

$$y(1+\Delta t) \approx 2 + 16\Delta t + \frac{68}{2!} \Delta t^2 = 2 + 16\Delta t + 34\Delta t^2$$

C) Let $y'(t) = (y(t) + 1)(t - 1)$, $y(3) = 2$. What is the prediction of Heun's method for $y(3.001)$?

Euler: $y(3.001) \approx 2 + [\underbrace{(2+1) \cdot (3-1)}] \cdot 0.001 = 2.006$

Heun: $y(3.001) \approx 2 + \frac{1}{2}\left[6 + (2.006+1) \cdot (3.001 - 1)\right] \cdot 0.001$

4. ((1+2+2)+(2+2+1) points)
A) Let

$$\partial_t \phi(t,x) = \partial_x^2 \phi(t,x), \quad \phi(t,x+2\pi) = \phi(t,x), \quad \phi(0,x) = f(x),$$

where $f(x) = 5$ if $x \in [0,2]$, otherwise is 0.
1. Present an orthonormal basis of $L^2([-\pi,\pi],dx)$!

$$\vec{e}_n = e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z}$$

2. Express f by a linear combination of this basis!

$$\vec{f} = f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \vec{e}_n = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}, \text{ where}$$

$$\hat{f}_n = (\vec{e}_n, \vec{f}\,) = \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \cdot f(x)\,dx = \int_0^2 \frac{e^{-inx}}{\sqrt{2\pi}} \cdot 5\,dx = \frac{5}{\sqrt{2\pi}} \cdot \frac{e^{-inx}}{-in}\Big|_0^2$$

$$= \frac{5}{\sqrt{2\pi}} \cdot i \cdot \frac{1}{n}\left(e^{-2in} - 1\right)$$

3. Compute (with the help of Fourier series) $\phi(t,x)$!

$$\varphi(t,x) = \sum_{n\in\mathbb{Z}} \frac{5i}{\sqrt{2\pi}\cdot n}\left(e^{-in} - 1\right) \cdot e^{-n^2 t} \cdot \frac{e^{inx}}{\sqrt{2\pi}},$$

$$\text{since } \partial_x^2 \frac{e^{inx}}{\sqrt{2\pi}} = (in)^2 \cdot \frac{e^{inx}}{\sqrt{2\pi}} = -n^2 \cdot \frac{e^{inx}}{\sqrt{2\pi}}$$

B) $y'' + 4y' + 8y = 5(t+1)^2$, $y(0) = 2$, $y'(0) = 3$. How much is $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)

$$(s^2 Y(s) - 2s - 3) + 4(sY(s) - 2) + 8Y(s) = 5 \cdot \left(\frac{2!}{s^3} + 2 \cdot \frac{1}{s^2} + \frac{1}{s}\right)$$

$$Y(s) = \frac{1}{s^2+4s+8}\left(2s + 3 + 2 + 5 \cdot \left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right]\right)$$

What is the partial fraction decompositionn of $Y(s)$? $\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda_{1,2} = \frac{-4 \pm \sqrt{16-32}}{2}$

$$Y(s) = \frac{A}{s-(-2+2i)} + \frac{B}{s-(-2-2i)} + \frac{C}{s^3} + \frac{D}{s^2} + \frac{E}{s}$$

What is $y(t)$?

$$y(t) = A\,e^{(-2+2i)t} + B\,e^{(-2-2i)t} + C \cdot \frac{1}{2!} t^2 + Dt + E$$