1. Let

$$\partial_t \phi(t,x) = \partial_x^2 \phi(t,x), \quad \phi(t,x+2\pi) = \phi(t,x), \quad \phi(0,x) = f(x),$$

where $f(x) = 55$, if $x \in [1,2]$, otherwise 0 on $[-\pi,\pi]$..

1. Give an orthonormal basis of $L^2([0,4],dx)$!
2. Express f with the help of this basis!
3. What is $\phi(t,x)$? use Fourier series to express ϕ !

2.
- Compute the Laplace transform $\mathcal{L}(f(t)) = F(s)$ of the following functions! $H(-4t+8),\ H(-4t-8),\ H(t-3)H(-t-5),\ \exp(-4t+8),\ \sin(-4t+8),\ \cos(-4t+8)$ (here H is the Heaviside theta function).
- Let $f(t) = t-1$ and $g(t) = t$. What is their $h = f * g$ convolution? How much is $\mathcal{L}(f(t))\mathcal{L}(g(t)) - \mathcal{L}(h(t))$?

  Variansok: Repeat the exercise for the following pairs of functions:

  (a) $1,\ 1,$
  (b) $5,\ t,$
  (c) $t,\ t,$
  (d) $t,\ t^2.$

3. Finite elements, variational principles.
Divide the interval $[0,1]$ into four subinterval with the following list of points: $x_i = 0.2,\ 0.4,\ 0.8$. Let $v(x)$ be the function which is affine on the subintervals and its values at the points $x = 0,\ 0.2,\ 0.4,\ 0.8,\ 1$ are $0, v_1, v_2, v_3, 0$.

- Compute
$$Energy[v] = \int_0^1 2(v')^2 - v' - (x^2-1)v\,dx$$
exactly or approximately (specify your method of approxiamtion.)
- Write down the Euler-Lagrange equations for $Energy[u]$!

Variansok:
Repeat the exercise for

$$Energy[v] = \int_0^1 2(v')^2 - v' - (x^2-1)v\,dx,$$
$$Energy[v] = \int_0^1 2(v')^4 - v' - (x^2-1)v^4\,dx$$

Write down the Euler-Lagrange equations for the following $L(\bar{u}, \bar{u}')$ Lagrange functions:

$$(u_1')^2 + (u_2')^2 + u_1 u_2' + u_1 u_2,$$
$$u_1 u_1' + u_2 u_2'.$$

4. $y'' + 4y' + 8y = 5t^3,\ y(0) = 2,\ y'(0) = 3$. What is $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)
What is the partial fraction decomposition of $Y(s)$?
Mennyi $y(t)$?

Variansok: Repeat the exercise for

$$y'' + 9y = 5(t-1)^2,\ y(0) = 2,\ y'(0) = 3,$$
$$y''' + 9y = 5(t-1)^2,\ y(0) = 2,\ y'(0) = 3,\ y''(0) = 2,$$
$$y' + 9y = t^2 - 1,\ y(0) = 2.$$

5. Let

$$\frac{d}{dt}\bar{y}(t) = \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Compute the $\bar{y}(t)$ solution's Laplace transform $\bar{Y}(s)$!

Variansok: Repeat the exercise for

$$\frac{d^2}{dt^2}\bar{y}(t) = \frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} y_1'(0) \\ y_2'(0) \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

6. Which functions are holomorphic?

$$f(z) = f(x+iy) = -x - iy, \quad f(z) = f(x+iy) = -x + iy, \quad f(z) = f(x+iy) = -x - iy,$$
$$f(z) = f(x+iy) = x^2 + 2ixy - 2y^2, \quad f(z) = f(x+iy) = x^2 + 2ixy + 2y^2,$$
$$f(z) = f(x+iy) = e^x(\cos y + i\sin x), \quad f(z) = f(x+iy) = e^x(\cos y + i\sin y)$$

Solution: $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ holomorph, if Cauchy-Riemann equations are satisfied

$$u_x' = v_y', \qquad u_y' = -v_x'.$$

For example $f(z) = f(x+iy) = x^2 + 2ixy + 2y^2$ is not holomorph, since

$$\left(x^2 + 2y^2\right)_x' = 4x = (2xy)_y',$$
$$\left(x^2 + 2y^2\right)_y' = 4y \neq -(2xy)_y' = 2x,$$

so the second CR equation is violated.

7. Let $f(x) = \sqrt[2]{x}$. What is the linear approximation of f around $x_0 = 4$? Give an upper bound for $|f(4+\Delta x) - f(4) - f'(4)\Delta x|$, if $\Delta x \in [0, 0.1]$!

Variansok: Repeat the exercise for

$$f(x) = \sqrt[6]{x}, \quad x_0 = 64,$$
$$f(x) = 1/x, \quad x_0 = 4,$$
$$f(x) = 1/x^2, \quad x_0 = 4,$$

8. Let $y'(t) = (2 + y^2(t))t, \;\; y(1) = 1$. Compute the third order Taylor polynomof $y(1+\Delta t)$!

Variansok: Repeat the exercise for

$$y'(t) = y(t)^2(t+1),$$
$$y'(t) = y(t) + 3.$$

9. Let $y'(t) = (y^2(t) - 1)(t-1), \;\; y(3) = 2$. What is Heun's method prediction for $y(3.001)$?

Variansok: Repeat the exercise for

$$y'(t) = y^2(t) - t, \;\; y(3) = 2,$$
$$\frac{d}{dt}\bar{y}(t) = \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(3) \\ y_2(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$