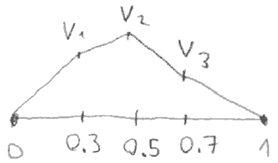
(1)

- $\frac{d}{dx}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u} = 0 = \frac{d}{dx}\frac{\partial(u'^2+(1-x^2)u)}{\partial u'} - \frac{\partial(u'^2+(1-x^2)u)}{\partial u} = 2u'' - (1-x^2) = 0$

- 

$$\text{Energy}[v] \approx 0.3\cdot\left(\frac{v_1-0}{0.3}\right)^2 + 0.2\cdot\left(\frac{v_2-v_1}{0.2}\right)^2 + 0.2\cdot\left(\frac{v_3-v_2}{0.2}\right)^2 + 0.3\cdot\left(\frac{0-v_3}{0.2}\right)^2$$
$$+ 0.3\cdot\left(\frac{(1-0^2)+(1-0.3^2)}{2}\cdot\frac{0+v_1}{2}\right) + 0.2\cdot\left(\frac{(1-0.3^2)+(1-0.5^2)}{2}\cdot\frac{v_1+v_2}{2}\right)$$
$$+ 0.2\cdot\left(\frac{(1-0.5^2)+(1-0.7)^2}{2}\cdot\frac{v_2+v_3}{2}\right) + 0.3\left(\frac{(1-0.7^2)+(1-1^2)}{2}\cdot\frac{v_3+0}{2}\right)$$

- $0 = \int_0^1 \varphi\cdot\left(2u''-(1-x^2)\right)dx = \int_0^1 -2\varphi' u' - \varphi(1-x^2)\,dx \quad , \text{ if } \varphi(0)=\varphi(1)=0$

(2)

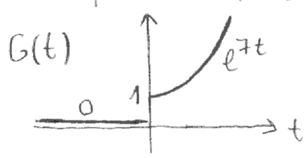
- $u''(x) \approx \frac{1}{\Delta x^2}\left(u(x+\Delta x) - 2u(x) + u(x-\Delta x)\right)$

- $$\frac{1}{\Delta x^2}\begin{bmatrix}-2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} + \begin{bmatrix}1\Delta x & & & \\ & 2\Delta x & & \\ & & 3\Delta x & \\ & & & 4\Delta x\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}1\Delta x\cdot(1-1\Delta x)\\2\Delta x\cdot(1-2\Delta x)\\3\Delta x\cdot(1-3\Delta x)\\4\Delta x\cdot(1-4\Delta x)\end{bmatrix}$$

(3)

(1) $y' - 7y = f(t)$, $\quad G'(t) - 7G(t) = \delta(t)$

- Properties of G: $\quad \left.\begin{array}{l} G(t)=0 \text{ if } t<0 \\ G'(t)-7G(t)=0 \text{ if } t>0 \\ G(0^-)=0,\ G(0^+)=1 \end{array}\right\} \Rightarrow G(t) = \begin{cases} 0 & \text{if } t<0 \\ e^{7t} & \text{if } t>0 \end{cases}$

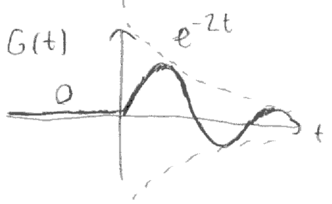


- $y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\,d\tau = \int_{-\infty}^{t} e^{7(t-\tau)} f(\tau)\,d\tau$

- $y(t) = 8\cdot G(t) + \int_0^{\infty} G(t-\tau) f(\tau)\,d\tau = 8\cdot G(t) + \int_0^t e^{7(t-\tau)} f(\tau)\,d\tau$

(2) $y'' + 4y' + 8y = f(t)$, $\quad G'' + 4G' + 8G = \delta(t)$

- Properties of G: $\quad \left.\begin{array}{l} G(t)=0 \text{ if } t<0 \\ G''+4G'+8G=0 \text{ if } t>0 \\ G(0^-)=G'(0^-)=0 \\ G'(0^+)=1,\ G(0^+)=0 \end{array}\right\} \Rightarrow G(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{1}{2}e^{-2t}\sin(2t) & \text{if } t>0 \end{cases}$



- $y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\,d\tau = \int_{-\infty}^{t} \frac{1}{2} e^{-2(t-\tau)} \sin(2(t-\tau)) f(\tau)\,d\tau$

④ ① $\mathcal{L}\{e^{-5t+7}\} = \int_0^\infty e^{-st} \cdot e^{-5t+7} dt = e^7 \cdot \frac{e^{-(s+5)t}}{-(s+5)} \Big|_0^\infty = \frac{e^7}{s+5}$

② $\mathcal{L}\{H(-5-t) \cdot e^{-5t}\} = \int_0^\infty \underbrace{H(-5-t)}_{=0 \text{ if } t>0 \text{ (as } -5-t<0)} \cdot e^{-5t} \cdot e^{st} dt = 0$

③ $(f*g)(t) = \int_0^t 4(t-\tau)^2 \cdot 5 \, d\tau = (g*f)(t) = \int_0^t 5 \cdot 4\tau^2 d\tau = 20\frac{t^3}{3}$

④ $y'' - 4y = t^2 - 2t + 1$

- $(s^2 Y(s) - 5s - 7) - 4Y(s) = \frac{2}{s^3} - 2 \cdot \frac{1}{s^2} + \frac{1}{s}$

  $Y(s) = \frac{1}{s^2-4}\left(5s + 7 + \frac{2}{s^3} - 2 \cdot \frac{1}{s^2} + \frac{1}{s}\right)$

- $Y(s) = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s^3} + \frac{D}{s^2} + \frac{E}{s}$

- $y(t) = Ae^{2t} + Be^{-2t} + \frac{C}{2}t^2 + Dt + E$

⑤ $\begin{pmatrix} sY_1(s) - 5 \\ sY_2(s) - 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s} + \frac{1}{s^2} \\ \frac{1}{s^2} + \frac{2}{s^3} \end{pmatrix}$

$\bar{Y}(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} s+1 & 3 \\ -3 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 5 + \frac{1}{s} + \frac{1}{s^2} \\ 4 + \frac{1}{s^2} + \frac{2}{s^2} \end{pmatrix}$