Test1, Diff.Eq., 2017.03.23. NEPTUN: :

Name: Signature:

(2+2+(2+4) point)

1a. $y' = e^{t^2}$, $y(4) = 6$. Express $y(7)$ using definite integration

$$y(7) = 6 + \int_4^7 e^{t^2}\, dt$$

1b. Let $f(x) = \sqrt[2]{x}$. Determine the linear approximation of f around $x_0 = 8$! Find an upper bound for the error of the linear approximation, i.e. estimate $|f(8 + \Delta x) - f(8) - f'(8)\Delta x|$, if $\Delta x \in [0, 0.1]$!

$f(x) = x^{1/2}$ $\quad f(8) = \sqrt{8} = 2\sqrt{2}$ $\quad f(8+\Delta x) \approx 2\sqrt{2} + \frac{1}{4\sqrt{2}}\Delta x$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(8) = \frac{1}{2}\frac{1}{\sqrt{8}} = \frac{1}{4\cdot\sqrt{2}}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\quad f''(8) = -\frac{1}{4}\frac{1}{\sqrt{8^3}} = -\frac{1}{64\cdot\sqrt{2}}$

$$\text{hiba}(\Delta x) \le \frac{1}{2}\Delta x^2 \max_{z\in[8,8+\Delta x]} |f''(z)| = \frac{1}{2}\Delta x^2 \cdot \frac{1}{64\cdot\sqrt{2}}$$

1c.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_2 - 2)(1 - y_1) \\ (y_2 - 3)(y_1 - 4). \end{pmatrix}$$

Find the fixed points of the DE!!

$\left.\begin{matrix}(y_2-2)(1-y_1)=0 \\ (y_2-3)(y_1-4)=0\end{matrix}\right\}$ $\quad y_1 = 1,\ y_2 = 3$ $\quad$ or $\quad y_1 = 4,\ y_2 = 2$

fixed points: $\bar{z}_A = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\bar{z}_B = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Write down the linearized versions of the DE around the fixed points!

$$\text{Jac} = \begin{pmatrix} \partial_{y_1}[(y_2-2)(1-y_1)] & \partial_{y_2}[(y_2-2)(1-y_1)] \\ \partial_{y_1}[(y_2-3)(y_1-4)] & \partial_{y_2}[(y_2-3)(y_1-4)] \end{pmatrix} = \begin{pmatrix} -y_2+2 & 1-y_1 \\ y_2-3 & y_1-4 \end{pmatrix}$$

$$\text{Jac}(\bar{z}_A) = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \qquad \text{Jac}(\bar{z}_B) = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}$$

Lin. DE:

$$\frac{d}{dt}\begin{pmatrix} y_1-1 \\ y_2-3 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \quad \text{A}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-4 \\ y_2-2 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \quad \text{B}$$

3a. (1+2+1+2 point)
$y' = (y^2 - 1)(2 - y)$. $\quad = -y^3 + 2y^2 + y - 2$ $\qquad \dfrac{d(-y^3+2y^2+y-2)}{dy} = -3y^2+4y+1 = \mathrm{Jac}(y)$

Find the fixed points of the DE!

$$y_1 = -1, \quad y_2 = 1, \quad y_3 = 2$$

Write down the linearized versions of the DE around the fixed points!

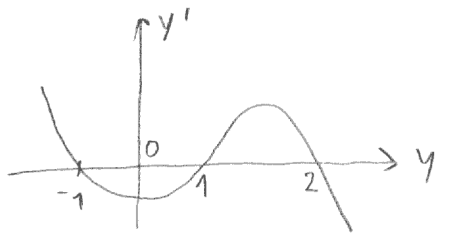
$y_1 = -1$: $\quad \mathrm{Jac}(-1) = -6 \qquad \frac{d}{dx}(y-(-1)) = \frac{d}{dx}\Delta y = -6\Delta y$

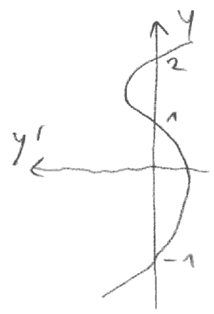
$y_2 = 1$: $\quad \mathrm{Jac}(1) = 2 \qquad \frac{d}{dx}(y-1) = \frac{d}{dx}\Delta y = 2\Delta y$

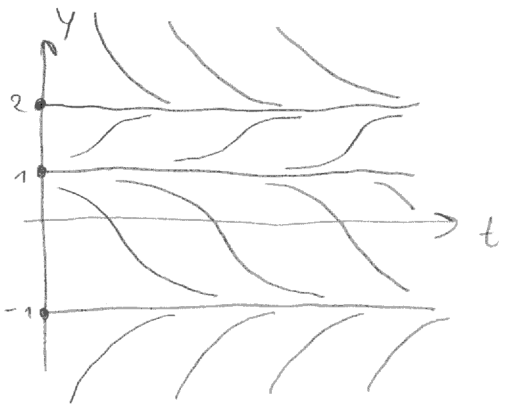
$y_3 = 2$: $\quad \mathrm{Jac}(2) = -3 \qquad \frac{d}{dx}(y-2) = \frac{d}{dx}\Delta y = -3\Delta y$

If $y(0) = 1.34$, how much are
$\lim_{x\to\infty} y(x) =$ 2 $\qquad\qquad \lim_{x\to-\infty} y(x) =$ 1

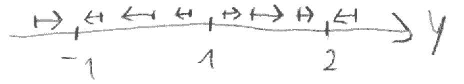
Plot the solution curves of the DE!







1 dim phase space:



3b. (4 pont) How much is

$$\exp\left[t\begin{pmatrix}5 & 6 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7\end{pmatrix}\right]$$

Block diag. so it is enough to compute $\exp\left[t\begin{pmatrix}5 & 6\\ 0 & 5\end{pmatrix}\right]$ and $\exp[t\cdot(7)]$

$$= \exp\left[\begin{pmatrix}5t & 0 & 0\\ 0 & 5t & 0\\ 0 & 0 & 7t\end{pmatrix} + \begin{pmatrix}0 & 6t & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] =$$

$$= \exp\begin{pmatrix}5t & 0 & 0\\ 0 & 5t & 0\\ 0 & 0 & 7t\end{pmatrix} \cdot \exp\begin{pmatrix}0 & 6t & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}e^{5t} & 0 & 0\\ 0 & e^{5t} & 0\\ 0 & 0 & e^{5t}\end{pmatrix}\begin{pmatrix}1 & 6t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= \left(\begin{array}{cc|c} e^{5t} & 6t\,e^{5t} & 0\\ 0 & e^{5t} & 0\\ \hline 0 & 0 & e^{7t}\end{array}\right)$$

block diag. mat.

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3y_1 \\ 4y_1 + 5y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of A !

Char. eq: $\det(A-\lambda E) = 0 = \begin{vmatrix} 3-\lambda & 0 \\ 4 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) - 4\cdot 0 = 0$

so $\lambda_1 = 3,\ \lambda_2 = 5.$ (diag. entries of the tridiag. mat.)

$(A-\lambda E)\bar{v} = 0$

$\lambda_1 = 3$: $\begin{pmatrix} 3-3 & 0 \\ 4 & 5-3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$4u + 2v = 0 \rightarrow v = -2u$

$\bar{V}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = 5$: $\begin{pmatrix} 3-5 & 0 \\ 4 & 5-5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2u = 4u = 0 \rightarrow u = 0$

$\bar{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Find the general solution of the DE!

$$\bar{Y}_{gen}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Find the particular solution of the DE!

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = C_1 e^{3\cdot 0}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{5\cdot 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$= C_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Longrightarrow C_1 = 3,\ C_2 = 7$$

$$\bar{Y}_{part}(t) = 3e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 7e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(5 × 2 pont)
What is the relation between A and the diagonal matrix D consisting of the eigenvalues?

$$D = S^{-1} A S$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \text{or}$$

$$A = S D S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$$

How much is e^{xA}?

$$e^{xA} = e^{x S D S^{-1}} = S e^{xD} S^{-1} =$$
$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} e^{3x} & 0 \\ 0 & e^{5x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}^{-1}$$

Express the paricular solution with e^{xA} !

$$\bar{y}_{part}(t) = e^{xA} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

2b) Rewrite the following DE as a first order time-independent system!

$$\frac{d^2}{dt^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' - t y_2^2 \\ y_2' - t^2 y_1' - t \end{pmatrix}$$

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \\ s \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_1 - s y_2^2 \\ v_2 - s v_1 - s \\ 1 \end{pmatrix}$$

Determine the algebraic form of $e^{-2+i\pi/3}$!

$$e^{-2+i\frac{\pi}{3}} = e^{-2}\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right) =$$
$$= e^{-2}\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$$