Zh1, Diff.Egy., 2018.03.21. NEPTUN: :

Név: $(2+3+3+2 \text{ pont})$ Aláírás:

1a. Mi az $y'(t) = 4 + \delta(t)$, $y(3) = 4$ DE megoldasa?

$t>0:\ y'=4,\ y(3)=4 \Rightarrow y(t)=4t-8$

$y(0^+)-y(0^-)=1 \Rightarrow y(0^-)=-9$

$t<0\quad y(t)=-9+4t$

$$y(t)=\begin{cases}4t-8, & \text{ha } t>0\\ -9+4t, & \text{ha } t<0\end{cases}$$

1b. Mi az $y''(t) = 4 + \delta(t)$, $y(3) = 4$, $y'(3) = 5$ DE megoldasa?

$\left[\begin{array}{l} y''=4,\ y=2t^2+C_1t+C_2 \\ 4=2\cdot 3^2+C_1\cdot 3+C_2 \\ 5=4\cdot 3+C_1 \\ \Rightarrow C_1=-7,\ C_2=7 \end{array}\right]$

$t>0:\ y''=4,\ y(3)=4,\ y'(3)=5 \Rightarrow y(t)=2t^2-7t+7$

$y(0^+)-y(0^-)=0,\ y'(0^+)-y'(0^-)=1 \Rightarrow y(0^-)=7,\ y'(0^-)=-8$

$t<0:\quad y(t)=2t^2-8t+7$

$$y(t)=\begin{cases}2t^2-7t+7, & \text{ha } t>0\\ 2t^2-8t+7, & \text{ha } t<0\end{cases}$$

1c1. Mi az $y''(t) = -4y(t) + \delta(t)$ DE retardalt fundamentalis megoldasa?

$$\left.\begin{array}{l} t<0:\ y(t)=0 \\ t\approx 0:\ y(0^+)-y(0^-)=0,\ y'(0^+)-y'(0^-)=1 \end{array}\right\} \Rightarrow y(0^+)=0,\ y'(0^+)=1$$

$t>0:$ az $y''=-4y,\ y(0)=0,\ y'(0)=1$ DE megoldása: $y(t)=\frac{1}{2}\sin(2t)$

$$G(t)=\begin{cases}0, & \text{ha } t<0\\ \frac{1}{2}\sin(2t), & \text{ha } t>0\end{cases}$$

1a. $y_1'(t)=4$ $\quad y_1(t)=4t$

$y_2'(t)=\delta(t)$ $\quad y_2(t)=H(t)$

$y'_{hom}(t)=0$ $\quad y_{hom}(t)=C$

$y'(t)=4+\delta(t)$ $\quad y_{gen}(t)=4t+H(t)+C$

$y(3)=4 \to 4\cdot 3+H(3)+C=13+C=4$

$y_{part}(t)=4t+H(t)-9$

1c2. Mi az $y''(t) = -4y(t) + f(t)$, $y(t) = f(t) = 0$, ha $t << 0$ DE megoldasa?

$$y(t)=\int_{-\infty}^{\infty} G(t-s)f(s)\,ds=\int_{-\infty}^{t}\frac{1}{2}\sin(2(t-s))\,f(s)\,ds$$

1b. $y_1''(t)=4$ $\quad y_1(t)=2t^2$

$y_2''(t)=\delta(t)$ $\quad y_2(t)=K(t)=\begin{cases}0, & \text{ha } t<0\\ t, & \text{ha } t>0\end{cases}$

$y''_{hom}(t)=0$ $\quad y_{hom}(t)=C_1t+C_2$

$y''(t)=4+\delta(t)$ $\quad y_{gen}(t)=2t^2+K(t)+C_1t+C_2$

$y'_{gen}(t)=4t+H(t)+C_1$

$y_{part}=2t^2+K(t)-8t+7$

$y(3)=4:$

$4=2\cdot 3^2+K(3)+C_1\cdot 3+C_2=$

$=18+3+3C_1+C_2$

$y'(3)=5:$

$5=4\cdot 3+H(3)+C_1=13+C_1$

tehát $C_1=-8,\ C_2=7$

4a. (4 pont)
$y' = (y^2+4)y.$ $= y^3+4y = f(y)$
Keresd meg a DE fixpontjait!

$$(y^2+4)y = 0 \implies y_1 = 0$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{d}{dy}f(y) = 3y^2+4\,,\quad f'(y_1) = f'(0) = 3\cdot 0^2+4 = 4, \text{ tehát a}$$

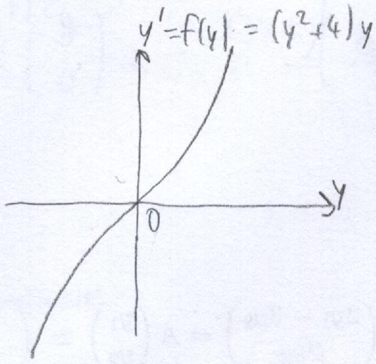
linearizált DE: $\frac{d}{dt}(y-0) = \frac{d}{dt}\Delta y = 4\Delta y.$

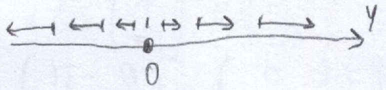
(ez amúgy triviális, mivel $y' = y^3+4y \approx 4y$, ha $y \approx y_1 = 0$)

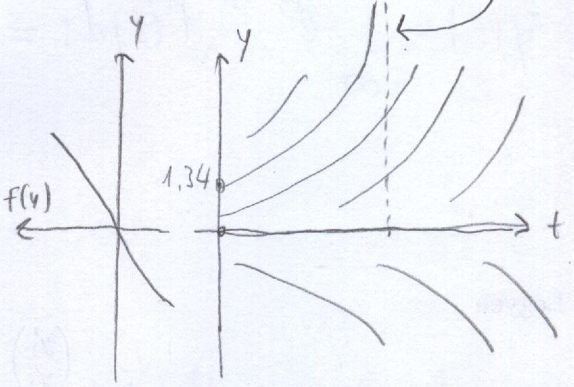
Ha $y(0) = 1.34$, mennyi
$\lim_{x\to\infty} y(x) = +\infty$ || $\lim_{x\to-\infty} y(x) = 0$

(pontosabban $\lim\limits_{x\to+\infty} y(x)$ nem létezik, mivel a DE-nek nincs globális megoldása)

Vazold a DE megoldasgorbeit!







4b. (6 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_2-2)y_1 \\ (y_1-4)(y_2+5). \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$\left.\begin{aligned} (y_2-2)y_1 &= 0 \\ (y_1-4)(y_2+5) &= 0 \end{aligned}\right\} \implies \bar{y}_A = \begin{pmatrix} 0 \\ -5 \end{pmatrix},\ \bar{y}_B = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\mathrm{Jac} = \begin{pmatrix} \partial_{y_1}[(y_2-2)y_1] & \partial_{y_2}[(y_2-2)y_1] \\ \partial_{y_1}[(y_1-4)(y_2+5)] & \partial_{y_2}[(y_1-4)(y_2+5)] \end{pmatrix} = \begin{pmatrix} y_2-2 & y_1 \\ y_2+5 & y_1-4 \end{pmatrix}$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$\bar{y}_A:\ \mathrm{Jac}(\bar{y}_A) = \begin{pmatrix} -7 & 0 \\ 0 & -4 \end{pmatrix} \qquad \bar{y}_B:\ \mathrm{Jac}(\bar{y}_B) = \begin{pmatrix} 0 & 4 \\ 7 & 0 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-0 \\ y_2-(-5) \end{pmatrix} = \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} -7 & 0 \\ 0 & -4 \end{pmatrix}\overline{\Delta y} \qquad \frac{d}{dt}\begin{pmatrix} y_1-4 \\ y_2-2 \end{pmatrix} = \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} 0 & 4 \\ 7 & 0 \end{pmatrix}\overline{\Delta y}$$

2. (4+2+4 pont) Legyen

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 - 3y_2 \\ 5y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{illetve} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix}, \quad \begin{vmatrix} 2-\lambda & -3 \\ 0 & 5-\lambda \end{vmatrix} = 0 \implies \lambda_1 = 2, \ \lambda_2 = 5$$

$$\lambda_1: \begin{pmatrix} 0 & -3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \lambda_2: \begin{pmatrix} -3 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3u - 3v = 0, \ u = -v$$

$$\overline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \overline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Ell:
$$\begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \checkmark$$
$$\begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \checkmark$$

Ird fel a DE altalanos megoldasat!

$$\overline{y}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasait!

$$\overline{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \quad c_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{matrix} c_1 + c_2 = 1 \\ -c_2 = 0 \end{matrix} \implies \begin{matrix} c_1 = 1 \\ c_2 = 0 \end{matrix}$$

$$\overline{y}^{I}_{part}(t) = e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$$

$$\overline{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \quad c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{matrix} c_1 + c_2 = 0 \\ -c_2 = 1 \end{matrix} \implies \begin{matrix} c_1 = 1 \\ c_2 = -1 \end{matrix}$$

$$\overline{y}^{II}_{part}(t) = e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{2t} - e^{5t} \\ e^{5t} \end{pmatrix}$$

3a1. Mennyi e^{tA} ? (3+2+3+2 pont)

$$e^{tA} = \begin{pmatrix} \bar{y}^{I}_{part}(t) & \bar{y}^{II}_{part}(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & e^{2t}-e^{5t} \\ 0 & e^{5t} \end{pmatrix}$$

vagy

$$e^{tA} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1}$$

3a2. Mi az $\frac{d}{dt}\bar{y}(t) = A\bar{y}(t) + \bar{f}(t)$, $\bar{y}(t) = \bar{f}(t) = 0$, ha $t << 0$, DE megoldasa?

$$\bar{y}(t) = \int_{-\infty}^{t} e^{(t-s)A} \bar{f}(s)\,ds = \int_{-\infty}^{t} \begin{pmatrix} e^{2(t-s)} & e^{2(t-s)} - e^{5(t-s)} \\ 0 & e^{5(t-s)} \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds$$

3b. Legyen

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 - 3y_2 \\ 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Mennyi e^{tA} ?

$$e^{tA} = \exp\begin{pmatrix} 2t & -3t \\ 0 & 2t \end{pmatrix} = \exp\left[\begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix} + \begin{pmatrix} 0 & -3t \\ 0 & 0 \end{pmatrix}\right] =$$

$$= \exp\begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & -3t \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -3t \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\overbrace{\begin{pmatrix} 0 & -3t \\ 0 & 0 \end{pmatrix}^2}^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}\right] =$$

$$= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -3t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & -3t\,e^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

Mi az elozo DE partikularis megoldasa az $(y_1(0), y_2(0))^T = (4, 5)$ kezdeti feltetel mellett?

$$\bar{y}_{part}(t) = \begin{pmatrix} e^{2t} & -3t\,e^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$