(1) $$\int \frac{1}{\sqrt{x}} + \sqrt[7]{x^2} + \frac{4}{x} + e^x\,dx = \int x^{-1/2} + x^{2/7} + 4\cdot\frac{1}{x} + e^x\,dx =$$
$$= \frac{x^{1/2}}{1/2} + \frac{x^{9/7}}{9/7} + 4\cdot\ln|x| + e^x + C$$

$$\int \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{2x}} + \sqrt[7]{(3x+1)^2} + \frac{4}{3-2x} + e^{-3x}\,dx =$$
$$= \frac{1}{2}\,\frac{x^{1/2}}{1/2} + \frac{\frac{(2x)^{1/2}}{1/2}}{2} + \frac{\frac{(3x+1)^{9/7}}{9/7}}{3} + 4\,\frac{\ln|3-2x|}{-2} + \frac{e^{-3x}}{-3} + C$$

$$\int \cos(3x+1) + 2^x + \frac{1}{1+(2x)^2} + \frac{1}{1+2x^2}\,dx =$$
$$= \frac{\sin(3x+1)}{3} + \frac{2^x}{\ln 2} + \frac{\operatorname{arctg}(2x)}{2} + \frac{\operatorname{arctg}(\sqrt{2}\,x)}{\sqrt{2}} + C$$

(2) $$\int x e^{-3x}\,dx = \left|\begin{matrix} f' = e^{-3x} & g = x \\ f = \frac{e^{-3x}}{-3} & g' = 1 \end{matrix}\right| = \frac{e^{-3x}}{-3}\cdot x - \int \frac{e^{-3x}}{-3}\cdot 1\,dx$$
$$= -\frac{1}{3}e^{-3x}x - \frac{\frac{e^{-3x}}{-3}}{-3} + C = -\frac{1}{3}e^{-3x}x - \frac{1}{9}e^{-3x}$$

$$\int x^6 \ln(3x)\,dx = \left|\begin{matrix} f' = x^6 & g = \ln(3x) \\ f = \frac{x^7}{7} & g' = \frac{1}{3x}\cdot 3 = \frac{1}{x} \end{matrix}\right| = \frac{x^7}{7}\cdot\ln(3x) - \int \underbrace{\frac{x^7}{7}\cdot\frac{1}{x}}_{x^6/7}\,dx$$
$$= \frac{x^7}{7}\ln(3x) - \frac{1}{7}\cdot\frac{x^7}{7} + C$$

$$\int x\cos(4x)\,dx = \left|\begin{matrix} f' = \cos(4x) & g = x \\ f = \frac{\sin(4x)}{4} & g' = 1 \end{matrix}\right| = \frac{\sin(4x)}{4}x - \int \frac{\sin(4x)}{4}\cdot 1\,dx$$
$$= \frac{\sin(4x)}{4}x - \left(-\frac{\cos(4x)}{4\cdot 4}\right) + C$$

(3) $\int f(g(x))\cdot g'(x)\,dx = F(g(x))$, ahol $\int f = F$.

$$\int x\sqrt{1+3x^2}\,dx\,dx = \frac{1}{6}\int (1+3x^2)^{1/2}\cdot\underbrace{(6x)}_{(1+3x^2)'}\,dx = \frac{1}{6}\cdot\frac{(1+3x^2)^{3/2}}{3/2} + C$$

$$\int e^{1+x^2}x\,dx = \frac{1}{2}\int e^{1+x^2}\cdot\underbrace{(2x)}_{(1+x^2)'}\,dx = \frac{1}{2}\,e^{1+x^2} + C$$

$$\int x^2\sin(x^3+1)\,dx = \frac{1}{3}\int \sin(x^3+1)\cdot\underbrace{3x^2}_{(x^3+1)'}\,dx = \frac{1}{3}\left(-\cos(x^3+1)\right) + C$$

④ $\int_1^7 8\,dx = [8x]_1^7 = 8\cdot 7 - 8\cdot 1 = 8\cdot(7-1)$

$$\int_2^3 \sqrt{x} + \frac{1}{x-1} + e^{2x}\,dx = \left[\frac{x^{3/2}}{3/2} + \ln|x-1| + \frac{e^{2x}}{2}\right]_2^3 = \left(\frac{3^{3/2}}{3/2} + \ln 2 + \frac{e^6}{2}\right) - \left(\frac{2^{3/2}}{3/2} + \underbrace{\ln 1}_{0} + \frac{e^4}{2}\right)$$

$$\int_0^1 x e^x\,dx = \left\| \begin{matrix} f' = e^x, & g = x \\ f = e^x, & g' = 1 \end{matrix} \right\|_0^1 = \left[e^x x - \int e^x \cdot 1\,dx \right]_0^1 =$$

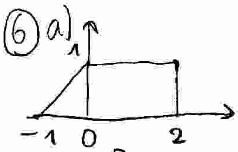
$$= \left[e^x x - e^x\right]_0^1 = (e^1\cdot 1 - e^1) - (e^0 \cdot 0 - e^0)$$

⑤ $\int_0^3 \sqrt{1+2^2}\,dx = \left[\sqrt{5}\,x\right]_0^3 = \sqrt{5}\cdot 3$ ívhossz

$\pi\int_0^3 (1+2x)^2\,dx = \pi\left[\frac{(1+2x)^3}{3}\right]_0^3 = \pi\left(\frac{7^3}{3} - \frac{1^3}{3}\right)$ térfogat

$2\pi\int_0^3 (1+2x)\sqrt{1+2^2}\,dx = 2\pi\sqrt{5}\left[\frac{(1+2x)^2}{2}\right]_0^3 = 2\pi\sqrt{5}\left(\frac{7^2}{2} - \frac{1^2}{2}\right)$ felszín

⑥ a)

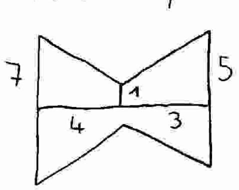


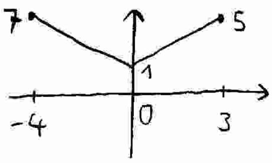
$$f(x) = \begin{cases} 1+x, & \text{ha } -1 \le x \le 0 \\ 1, & \text{ha } 0 \le x \le 2 \end{cases}$$

$$V = \pi \int_{-1}^{2} f^2(x)\,dx = \pi\left\{ \int_{-1}^{0} (1+x)^2\,dx + \int_0^2 1^2\,dx \right\} = \pi\left\{ \left[\frac{(1+x)^3}{3}\right]_{-1}^{0} + [x]_0^2 \right\} =$$

$$= \pi\left\{ \left(\frac{1^3}{3} - \frac{0^3}{3}\right) + (2-0) \right\} = \pi \cdot 2\tfrac{1}{3}$$

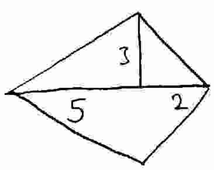
Néhány alakzat leírása:

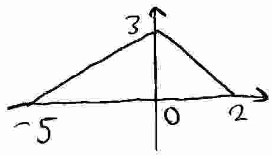




$$f(x) = \begin{cases} 1 - \frac{6}{4}x, & \text{ha } -4 \le x \le 0 \\ 1 + \frac{4}{3}x, & \text{ha } 0 \le x \le 3 \end{cases}$$

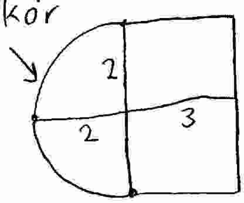
(itt $6 = 7-1$, $4 = 5-1$)

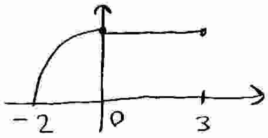




$$f(x) = \begin{cases} 3 + \frac{3}{5}x, & \text{ha } -5 \le x \le 0 \\ 3 - \frac{3}{2}x, & \text{ha } 0 \le x \le 2 \end{cases}$$

félkör





$$f(x) = \begin{cases} \sqrt{2^2 - x^2}, & \text{ha } -2 \le x \le 0 \\ 2, & \text{ha } 0 \le x \le 3 \end{cases}$$

⑥ b) $\text{Felszín} = 2\pi \int_{-1}^{2} f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi \left\{ \int_{-1}^{0} (1+x)\sqrt{1+1^2}\,dx + \int_{0}^{2} 1\cdot\sqrt{1+0^2}\,dx \right\}$

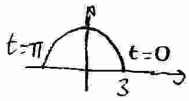
$= 2\pi \left\{ \left[\sqrt{2}\left(x+\frac{x^2}{2}\right) \right]_{-1}^{0} + \left[x\right]_0^2 \right\} = 2\pi \left\{ \sqrt{2}\left(\left[0+\frac{0^2}{2}\right] - \left[(-1)+\frac{(-1)^2}{2}\right] + [2-0] \right) \right\}$

⑦ a) ívhossz: $f(x) = \frac{e^x+e^{-x}}{2}, \quad 0 \le x \le 1$

$\int_0^1 \sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\,dx = \int_0^1 \sqrt{\frac{e^{2x}}{4}+\frac{e^{-2x}}{4}+\frac{1}{2}} = \int_0^1 \sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2}\,dx = \int_0^1 \frac{e^x+e^{-x}}{2}\,dx$

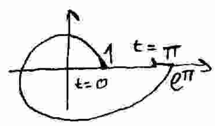
$= \left[\frac{e^x-e^{-x}}{2}\right]_0^1 = \frac{e^1-e^{-1}}{2} - \frac{e^0-e^{-0}}{2} = \frac{1}{2}\left(e-\frac{1}{e}\right)$

b) $x(t) = \cos(3t)$
$y(t) = \sin(3t)$ $\quad 0 \le t \le \pi$ (félkör)



$\int_0^\pi \sqrt{\dot{x}^2+\dot{y}^2}\,dt = \int_0^\pi \sqrt{(-3\sin(3t))^2+(3\cos(3t))^2}\,dt = \int_0^\pi \sqrt{3^2}\,dt = \int_0^\pi 3\,dt = 3\pi$

c) $x(t) = e^t\cos t$
$y(t) = e^t \sin t$ $\quad 0 \le t \le 2\pi$ (logaritmikus spirál)



$\int_0^{2\pi} \sqrt{\dot{x}^2+\dot{y}^2}\,dt =$

$\dot{x} = e^t(-\sin t) + e^t\cos t \qquad \dot{x}^2 = e^{2t}(\sin^2 t - 2\sin t\cos t + \cos^2 t)$

$\dot{y} = e^t(\cos t) + e^t \sin t \qquad +\dot{y}^2 = e^{2t}(\cos^2 t + 2\sin t\cos t + \sin^2 t)$

$= 2e^{2t}$

$= \int_0^{2\pi} \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt =$

$= \sqrt{2}\left[e^t\right]_0^\pi = \sqrt{2}\left(e^\pi - e^0\right) = \sqrt{2}\left(e^\pi - 1\right)$

⑧ $D = \{(x,y);\ 1 \le x \le 2,\ 0 \le y \le 3\}$

$\iint_D 1+x\,dx\,dy = \int_{x=1}^{2}\left(\int_{y=0}^{3} 1+x\,dy\right)dx = \int_{x=1}^{2} \left[(1+x)y\right]_{y=0}^{3} dx =$

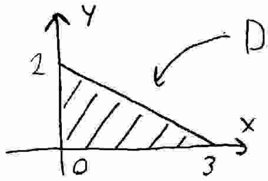
$= \int_{x=1}^{2} (1+x)\cdot 3 - (1+x)\cdot 0\,dx = \int_{x=1}^{2} 3+3x\,dx = \left[3x+3\frac{x^2}{2}\right]_{x=1}^{2} =$

$= \left(3\cdot 2 + 3\cdot\frac{2^2}{2}\right) - \left(3\cdot 1 + 3\cdot \frac{1^2}{2}\right)$

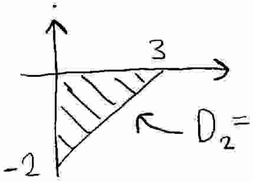
$\iint_D x\,y\,dx\,dy = \int_{y=0}^{3}\left(\int_{x=1}^{2} xy\,dx\right)dy = \int_{y=0}^{3} \left[\frac{x^2}{2}y\right]_{x=1}^{2} dy =$

$= \int_{y=0}^{3} \frac{2^2}{2}y - \frac{1^2}{2}y\,dy = \int_{y=0}^{3} \frac{3}{2}y\,dy = \left[\frac{3}{2}\frac{y^2}{2}\right]_{y=0}^{3} = \frac{3}{2}\cdot\frac{3^2}{2} - \frac{3}{2}\cdot\frac{0^2}{2}$

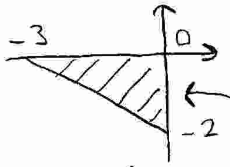
⑨ Néhány tartomány:



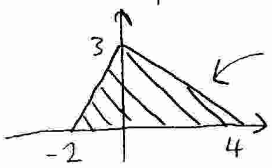
$D_1 = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\}$



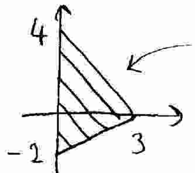
$D_2 = \{(x,y) \mid x \ge 0,\ y \le 0,\ y \ge -2 + \frac{2}{3}x\}$



$D_3 = \{(x,y) \mid x \le 0,\ y \le 0,\ y \ge -2 - \frac{2}{3}x\}$



$D_4 = \{(x,y) \mid y \ge 0,\ y \le 3,\ -2 + \frac{2}{3}y \le x \le 4 - \frac{4}{3}y\}$



$D_5 = \{(x,y) \mid 3 \ge x \ge 0,\ -2 + \frac{2}{3}x \le y \le 4 - \frac{4}{3}x\}$

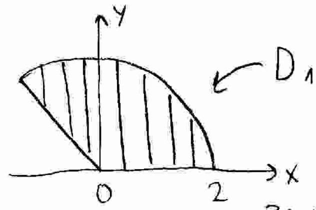
$$\iint_{D_1} x\,dx\,dy = \int_{x=0}^{3}\left(\int_{y=0}^{2-\frac{2}{3}x} x\,dy\right)dx = \int_{x=0}^{3} [xy]_{y=0}^{2-\frac{2}{3}x}\,dx = \int_{x=0}^{3} x\cdot\left(2-\frac{2}{3}x\right) - x\cdot 0\,dx$$

$$= \int_{x=0}^{3} 2x - \frac{2}{3}x^2\,dx = \left[2\frac{x^2}{2} - \frac{2}{3}\frac{x^3}{3}\right]_{x=0}^{3} = \left(2\cdot\frac{3^2}{2} - \frac{2}{3}\cdot\frac{3^3}{3}\right) - \left(2\cdot\frac{0^2}{2} - \frac{2}{3}\cdot\frac{0^3}{3}\right) = 3.$$

$$\iint_{D_2} xy\,dx\,dy = \int_{x=0}^{3}\left(\int_{y=-2+\frac{2}{3}x}^{0} xy\,dy\right)dx = \int_{x=0}^{3}\left[x\frac{y^2}{2}\right]_{y=-2+\frac{2}{3}x}^{0} dx$$

$$= \int_{x=0}^{3} x\frac{0^2}{2} - x\frac{(-2+\frac{2}{3}x)^2}{2}\,dx = \int_{x=0}^{3} -x + \frac{2}{3}x^2 - \frac{1}{9}x^3\,dx = \left[-\frac{x^2}{2} + \frac{2}{3}\frac{x^3}{3} - \frac{1}{9}\frac{x^4}{4}\right]_{x=0}^{3}.$$

⑩



$D_1 = \{(r,\varphi) \mid 0 \le r \le 2,\ 0 \le \varphi \le \frac{3}{4}\pi\}$

$$\iint_{D_1} r\,dA = \int_{\varphi=0}^{3/4\pi}\int_{r=0}^{2} r\cdot\overbrace{r\,dr\,d\varphi}^{dA} = \int_{\varphi=0}^{\frac{3}{4}\pi}\left[\frac{r^3}{3}\right]_{r=0}^{2} d\varphi = \int_{\varphi=0}^{\frac{3}{4}\pi} \frac{2^3}{3} - \frac{0^3}{3}\,d\varphi$$

$$= \frac{3}{4}\pi\cdot\frac{2^3}{3}$$

$$\iint_{D_1} x\,dA = \int_{\varphi=0}^{\frac{3}{4}\pi}\int_{r=0}^{2} \overbrace{r\cos\varphi}^{x}\ \overbrace{r\,dr\,d\varphi}^{dA} = \int_{\varphi=0}^{\frac{3}{4}\pi}\left[\frac{r^3}{3}\cos\varphi\right]_{r=0}^{2} d\varphi =$$

$$= \int_{\varphi=0}^{\frac{3}{4}\pi} \frac{2^3}{3}\cos\varphi\,d\varphi = \frac{8}{3}\left[-\sin\varphi\right]_{\varphi=0}^{\frac{3}{4}\pi} = \frac{8}{3}\left(-\sin 135^\circ + \sin 0^\circ\right) = \frac{8}{3}\left(-\left(-\frac{1}{\sqrt{2}}\right)\right) =$$

$$= \frac{4\sqrt{2}}{3}$$