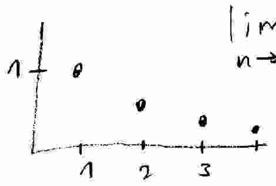
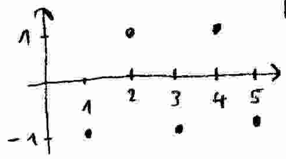
Megoldások

1. $a_n=\frac{1}{n}$



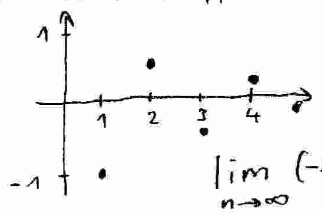
$\lim\limits_{n\to\infty}\frac{1}{n}=0$

$a_n=(-1)^n$



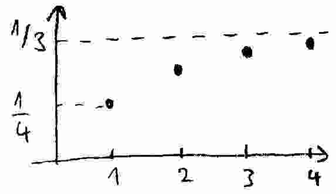
$\lim(-1)^n$ nem létezik

$a_n=(-1)^n\cdot\frac{1}{n}$



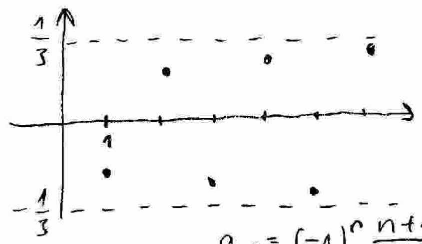
$\lim\limits_{n\to\infty}(-1)^n\cdot\frac{1}{n}=0$

$a_n=\frac{n+1}{3n+5}$



$$\lim_{n\to\infty}\frac{n+1}{3n+5}=\lim_{n\to\infty}\frac{1+1/n}{3+5/n}=\frac{1}{3}$$

$\lim\limits_{n\to\infty}(-1)^n\frac{n+1}{3n+5}$ nem létezik



$a_n=(-1)^n\frac{n+1}{3n+5}$

---

2. Monotonitás

$$a_n=\frac{n+1}{3n+1},\quad a_{n+1}-a_n=\frac{(n+1)+1}{3(n+1)+1}-\frac{n+1}{3n+1}=\frac{n+2}{3n+4}-\frac{n+1}{3n+1}=$$

$$=\frac{(n+2)(3n+1)-(n+1)(3n+4)}{(3n+4)(3n+1)}=\frac{-2}{(3n+4)(3n+1)}<0,\ \text{ha } n=1,2,3,\dots$$

tehát a_n szigorúan monoton csökkenő.

$$a_n=(-1)^n\frac{n+1}{3n+5},\quad a_1=-\frac{2}{8},\ a_2=\frac{3}{11},\ a_3=-\frac{4}{17}.$$

$a_2>a_1$, de $a_3<a_2$, a sorozat se nem növekvő, se nem csökkenő.

---

3. Határértékek

$$\lim_{n\to\infty}\frac{2n}{n^2+1}=\lim_{n\to\infty}\frac{2/n}{1+1/n^2}=\frac{0}{1+0}=0$$

$$\lim_{n\to\infty}\frac{2n^2-3}{n^2+1}=\lim_{n\to\infty}\frac{2-3/n^2}{1+1/n^2}=\frac{2-0}{1+0}=2$$

$$\lim_{n\to\infty}\frac{2n^3+3}{n^2+1}=\lim_{n\to\infty}\frac{2n+3}{1+1/n^2}=\frac{\infty}{1}=\infty$$

Elosztjuk a számlálót és a nevezőt a nevező legmagasabb fokú tagjával, az n^2-tel.

$$\lim_{n\to\infty}\frac{0.2^{n+1}}{0.5^n}=\lim_{n\to\infty}\left(\frac{0.2}{0.5}\right)^n\cdot 0.2=\lim_{n\to\infty}\underbrace{0.4^n}_{\to 0,\ \text{mert } |0.4|<1}\cdot 0.2=0\cdot 0.2=0$$

$$\lim_{n\to\infty}\frac{3^{2n}2^{3n}}{5^{3n}}=\lim_{n\to\infty}\left(\frac{3^2\cdot 2^3}{5^3}\right)^n=\lim_{n\to\infty}\left(\frac{72}{125}\right)^n=0$$

$$\lim_{n\to\infty}\left(\frac{n-3}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{-3}{n}\right)^n=e^{-3}\qquad\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x\right)$$

$$\lim_{n\to\infty}\left(\frac{n-3}{n}\right)^{2n}=\lim_{n\to\infty}\left[\left(1+\frac{-3}{n}\right)^n\right]^2=\left[e^{-3}\right]^2=e^{-6}$$

$$\lim_{n\to\infty}\left(\frac{n-3}{n}\right)^{2n+4}=\lim_{n\to\infty}\left[\left(1+\frac{-3}{n}\right)^n\right]^2\cdot\underbrace{\left(1+\frac{-3}{n}\right)^4}_{(1+0)^4=1}=e^{-6}$$

4. $\lim\limits_{x\to\infty} \frac{2x}{x^2+1} = \lim\limits_{x\to\infty} \frac{2/x}{1+1/x^2} = \frac{0}{1+0} = 0$, $\lim\limits_{x\to-\infty} \frac{2x}{x^2+1} = \frac{0}{1+0} = 0$

$\lim\limits_{x\to\infty} x^2+1 = \infty$, $\lim\limits_{x\to-\infty} x^2+1 = \infty$ — polinomok végtelenbeli viselkedése csak a legmagasabb

$\lim\limits_{x\to\infty} x^2-x^3 = \lim\limits_{x\to\infty} -x^3 = -\infty$, $\lim\limits_{x\to-\infty} x^2-x^3 = \lim\limits_{x\to-\infty} -x^3 = +\infty$ — fokú tagtól függ.

---

5. $f(x) = x^3+2x$

$$\frac{f(3+\Delta x)-f(3)}{\Delta x} = \frac{[(3+\Delta x)^3 - 2(3+\Delta x)] - [3^3+2\cdot 3]}{\Delta x} =$$

$$= \frac{[3^3+3\cdot 3^2\cdot\Delta x+3\cdot 3(\Delta x)^2+(\Delta x)^3 - 2\cdot(3+\Delta x)] - [3^3+2\cdot 3]}{\Delta x} = 3\cdot 3^2 + 3\cdot 3\,\Delta x + (\Delta x)^2 - 2$$

$$\lim_{\Delta x\to 0} \frac{f(3+\Delta x)-f(3)}{\Delta x} = \lim_{\Delta x\to 0} 3\cdot 3^2+3\cdot 3\Delta x+(\Delta x)^2-2 = 3\cdot 3^2-2 = 25 = f'(3)$$

Hasonló számítások alapján

$$\lim_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{\Delta x\to 0} 3\cdot x_0^2 + 3\cdot x_0\cdot\Delta x + (\Delta x)^2 - 2 = 3x_0^2-2 = f'(x_0)$$

---

6. $f(x) = x^3+2x$ — Érintő egyenlete $x_0=2$-ben:

$f'(x) = 3x^2+2$

$f(x_0) = f(2) = 12$

$f'(x_0) = f'(2) = 3\cdot 2^2+2 = 14$

$$f'(x_0) = 14 = \frac{y(x)-f(x_0)}{x-x_0} = \frac{y(x)-12}{x-x_0}$$

$$y(x) - 12 = 14(x-2)$$

vagy $y(x_0+\Delta x) = f(x_0) + f'(x_0)\Delta x$

$$y(2+\Delta x) = 12 + 14\Delta x$$

Az érintővel történő közelítés hibája:

Δx	hiba
0.1	0.061
0.01	0.000601
0.001	0.000006001

$f(x) = \sin x$ — Érintő egyenlete:

$f'(x) = \cos x$

$x_0 = \pi/4$

$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$\left(y(x) - \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

$$y\left(\frac{\pi}{4}+\Delta x\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Delta x$$

Δx	hiba
0.1	$-0.00365..$
0.01	$-0.0000354..$
0.001	$-3.536..\times 10^{-7}$

7. Deriválás

$$\left(2x^2+(2x)^2+\sqrt[3]{x^2}+x^{-2/3}+\ln(2x)\right)'=2\cdot 2x+2\cdot(2x)\cdot 2+\frac{2}{3}x^{-1/3}+\left(-\frac{2}{3}\right)x^{-5/3}+\frac{1}{2x}\cdot 2$$

$$\left(\sqrt[2]{(2x)^3}+\frac{1}{3\sqrt{x}}+\frac{1}{\sqrt{3x}}+\ln(x^2)\right)'=\frac{3}{2}(2x)^{1/2}\cdot 2+\frac{1}{3}\cdot\left(-\frac{1}{2}\right)x^{-3/2}+\left(-\frac{1}{2}\right)(3x)^{-3/2}\cdot 3+\frac{1}{x^2}\cdot 2x$$

$$(xe^x)'=x'e^x+x(e^x)'=1\cdot e^x+xe^x$$

$$(x^5e^{-x})'=5x^4e^{-x}+x^5e^{-x}\cdot(-1)$$

$$(\ln x\cdot\cos(2x))'=\frac{1}{x}\cos(2x)+\ln x\cdot(-\sin(2x))\cdot 2$$

$$\left(\sqrt{\cos x}\right)'=\frac{1}{2}(\cos x)^{-1/2}\cdot(-\sin x)$$

$$(\cos\sqrt{x})'=-\sin\sqrt{x}\cdot\frac{1}{2}x^{-1/2}$$

$$\left(\frac{\cos x}{x^2}\right)'=\frac{-\sin x\cdot x^2-\cos x\cdot 2x}{(x^2)^2}$$

$$\left(\frac{e^x+x}{x^2}\right)'=\frac{(e^x+1)x^2-(e^x+x)2x}{(x^2)^2}$$

$$\left(\frac{tg(3x)}{x}\right)'=\frac{\left(\frac{1}{\cos^2(3x)}\cdot 3\right)\cdot x-tg(3x)\cdot 1}{x^2}$$

$$(\mathrm{arctg}(-x)\cdot x)'=\frac{1}{1+(-x)^2}\cdot x+\mathrm{arctg}(-x)\cdot 1$$

$$(\mathrm{arctg}(x^2-1)\cdot 2^x)'=\left(\frac{1}{1+(x^2-1)^2}\cdot(2x)\right)\cdot 2^x+\mathrm{arctg}(x^2-1)\cdot(2^x\cdot\ln 2)$$

8.

f	x	x^3	x^5	$\cos x$	e^{3x}	$\frac{1}{1-x}$	$\ln(x-1)$
f'	1	$3x^2$	$5x^4$	$-\sin x$	$e^{3x}\cdot 3$	$-1(1-x)^{-2}\cdot(-1)$	$\frac{1}{x-1}$
f''	0	$3\cdot 2x$	$5\cdot 4\cdot x^3$	$-\cos x$	$e^{3x}\cdot 3^2$	$(1-x)^{-3}\cdot 2$	$-(x-1)^{-2}$
f'''	0	$3\cdot 2\cdot 1$	$5\cdot 4\cdot 3x^2$	$\sin x$	$e^{3x}\cdot 3^3$	$(1-x)^{-4}\cdot 2\cdot 3$	$(x-1)^{-3}\cdot 2$
$f^{(4)}$	0	0	$5\cdot 4\cdot 3\cdot 2\cdot x$	$\cos x$	$e^{3x}\cdot 3^4$	$(1-x)^{-5}\cdot 2\cdot 3\cdot 4$	$(x-1)^{-4}\cdot 3\cdot 2$

9. L'Hospital szabály

$$\lim_{x\to 0} \frac{tg(x)}{x} = \frac{0}{0} = \lim_{x\to 0} \frac{\frac{1}{\cos^2 x}}{1} = \frac{\frac{1}{\cos^2 0}}{1} = 1$$

$$\lim_{x\to 0} \frac{\sin(3x)}{1-e^{4x}} = \frac{0}{1-1} = \frac{0}{0} = \lim_{x\to 0} \frac{\cos(3x)\cdot 3}{0-4e^{4x}} = \frac{\cos(3\cdot 0)\cdot 3}{0-4e^{4\cdot 0}} = \frac{1\cdot 3}{0-4\cdot 1} = \frac{3}{4}$$

$$\lim_{x\to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\lim_{x\to \infty} \frac{x^2}{e^{4x}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{2x}{4e^{4x}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{2}{4\cdot 4\cdot e^{4x}} = 0$$

---

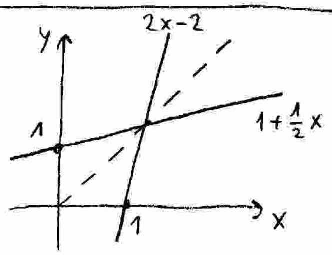
10. $f(x) = 1+\frac{1}{2}x$

$y = 1+\frac{1}{2}x$

$x = 2(y-1) = 2y-2$

$f^{-1}(y) = 2y-2$

$f^{-1}(x) = 2x-2$



$D_f = (-\infty,\infty) = R_{f^{-1}} = (-\infty,\infty)$

$R_f = (-\infty,\infty) = D_{f^{-1}} = (-\infty,\infty)$

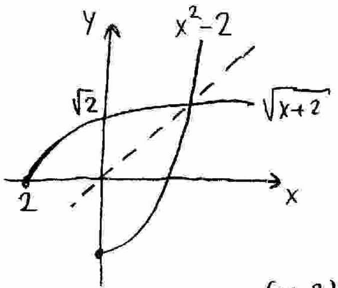
$f(x) = \sqrt{x+2}\,,\quad x \geq -2$

$y = \sqrt{x+2}\,,\quad y \geq 0$

$x = y^2-2$

$f^{-1}(y) = y^2-2\,,\quad y\geq 0$

$f^{-1}(x) = x^2-2\,,\quad x \geq 0$



$D_f = [-2,\infty) = R_{f^{-1}} = [-2,\infty)$

$R_f = [0,\infty) = D_{f^{-1}} = [0,\infty)$

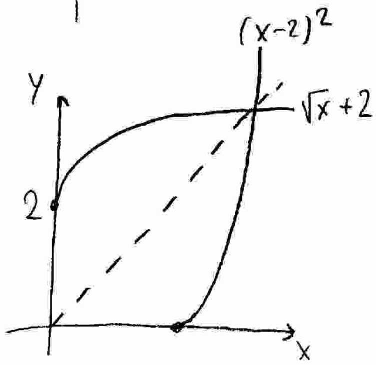
$f(x) = \sqrt{x}+2\,,\quad x\geq 0$

$y = \sqrt{x}+2 \qquad y \geq 2$

$x = (y-2)^2$

$f^{-1}(y) = (y-2)^2 \qquad y\geq 2$

$f^{-1}(x) = (x-2)^2 \qquad x \geq 2$



$D_f = [0,\infty) = R_{f^{-1}}$

$R_f = [2,\infty) = D_{f^{-1}}$

$f(x) = e^{x+2}$

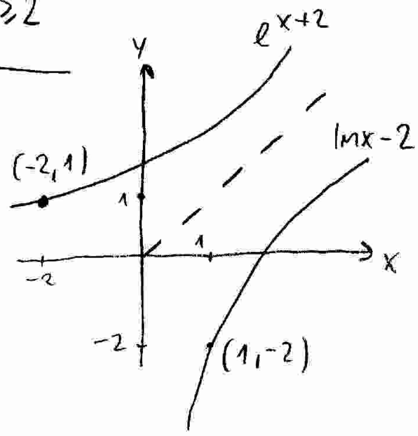
$y = e^{x+2}$

$\ln y = x+2$

$x = \ln y - 2$

$f^{-1}(y) = \ln y - 2$

$f^{-1}(x) = \ln x - 2$



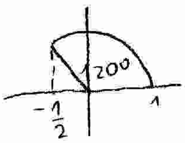
$D_f = (-\infty,\infty) = R_{f^{-1}}$

$R_f = (0,\infty) = D_f$

11. $\ln e^3 = 3$, $\ln \frac{1}{\sqrt{e}} = \ln e^{-1/2} = -1/2$,

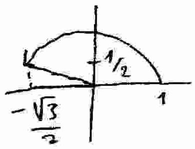
$\lg 1000 = \lg 10^3 = 3$, $\lg 0.01 = \lg \frac{1}{10^2} = \lg 10^{-2} = -2$

$\lg \sqrt[3]{100} = \lg 10^{2/3} = 2/3$.

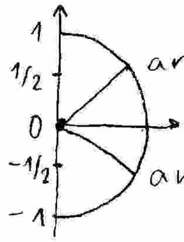


$\cos 120° = -\frac{1}{2}$
$\sin 120° = \sqrt{3}/2$
$\text{tg}\, 120° = -\sqrt{3}$

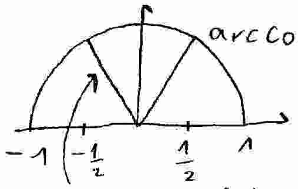
$\frac{5\pi}{6} = 150°$



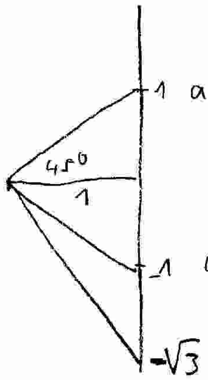
$\cos 150° = -\frac{\sqrt{3}}{2}$
$\sin 150° = 1/2$
$\text{tg}\, 150° = -\frac{1}{\sqrt{3}}$



$\arcsin 1/2 = 30°$
$\arcsin(-\frac{1}{2}) = -30°$

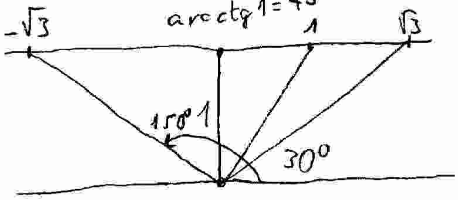


$\arccos \frac{1}{2} = 60°$
$\arccos(-\frac{1}{2}) = 120°$



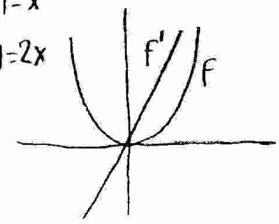
$\text{arctg}\, 1 = 45°$
$\text{arctg}(-1) = -45°$
$\text{arctg}(-\sqrt{3}) = -60°$

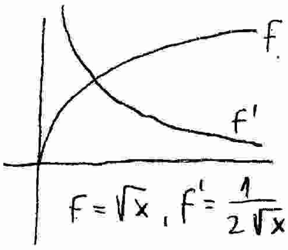
$\text{arcctg}(-\sqrt{3}) = 150°$ $\text{arcctg}\, 1 = 45°$ $\text{arcctg}\sqrt{3} = 30°$



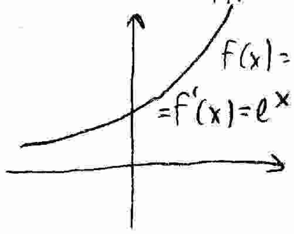
12. Rajzold le f(x)-et és f'(x)-et!

$f(x) = x^2$
$f'(x) = 2x$



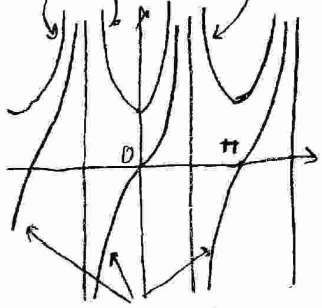


$f = \sqrt{x}$, $f' = \frac{1}{2\sqrt{x}}$



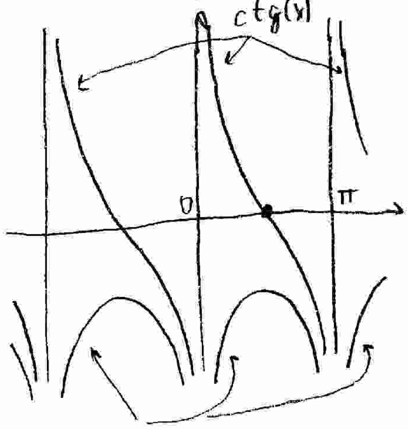
$f(x) = f'(x) = e^x$

$f'(x) = \frac{1}{\cos^2 x}$



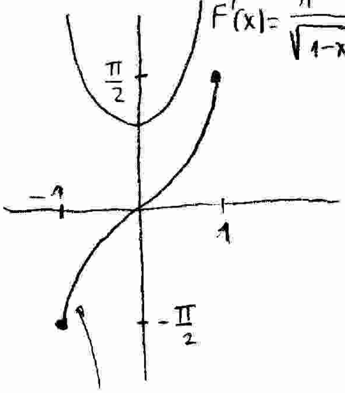
$f(x) = \text{tg}\, x$

$f(x) = \text{ctg}(x)$

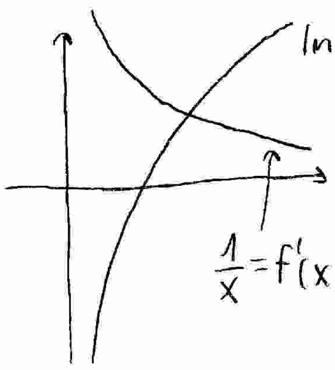


$f' = -\frac{1}{\sin^2 x}$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$



$f = \arcsin x$



$\ln x = f(x)$

$\frac{1}{x} = f'(x)$

13) a) $f(x) = -x^3 - x$

$f'(x) = -3x^2 - 1$

$f''(x) = -6x$

$-x^3 - x = -x(x^2+1)$

$x_1 = 0$ gyök, mult. = 1

$\lim_{x\to\infty} -x^3-x = \lim_{x\to\infty} -x^3 = -\infty$

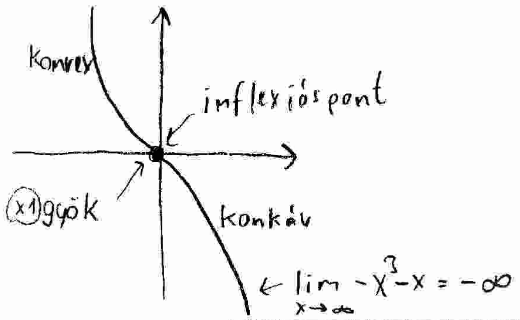
Szélsőérték helye: $f'(x_{sz}) = 0 = -3x^2 - 1$. nincs valós gyök

$f'(x) = -3x^2 - 1 < 0$ bármely x-re, nincs lokális szélsőérték.

f szig. mon. csökk. ↘

Konvexitás: $f''(x_{inf}) = -6x_{inf} = 0 \rightarrow x_{inf} = 0$

$x < 0$	$x_{inf} = 0$	$0 < x$
$f'' > 0$	$f'' = 0$	$f'' < 0$
∪ Konvex		∩ konkáv



$D_f = (-\infty, \infty)$

$R_f = (-\infty, \infty)$

b) $f(x) = x^2 - x^4 = x^2(1-x^2) = x^2(1-x)(1+x)$

gyökök: $x_1 = 0$, $x_2 = 1$, $x_3 = -1$

multiplicitás: 2, 1, 1

$f'(x) = 2x - 4x^3$

$f''(x) = 2 - 12x^2$

Szélsőérték helye: $f'(x_{sz}) = 0 = 2x - 4x^3 = 2x(1-2x^2)$

$x_1 = 0$ $\quad x_2 = -\frac{1}{\sqrt{2}}$ $\quad x_3 = \frac{1}{\sqrt{2}}$

Széls. ért. típusa:

$f''(x_1) = 2 - 12\cdot 0^2 = 2 > 0$ MIN

$f''(-\frac{1}{\sqrt{2}}) = -4 < 0$ MAX

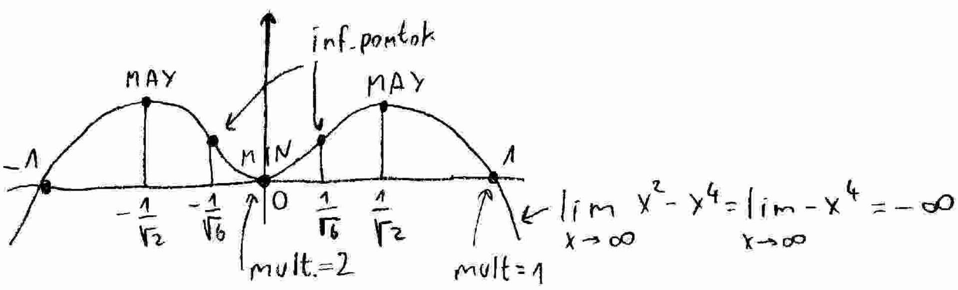
$f''(\frac{1}{\sqrt{2}}) = -4 < 0$ MAX

Inflexiós pont:

$f''(x_{inf}) = 0 = 2 - 12x^2 \rightarrow x_1 = \frac{1}{\sqrt{6}}$, $x_2 = -\frac{1}{\sqrt{6}}$

$x < -\frac{1}{\sqrt{2}}$	$x = -\frac{1}{\sqrt{2}}$	$0 > x > -\frac{1}{\sqrt{2}}$	$x = 0$	$0 < x < \frac{1}{\sqrt{2}}$	$x = \frac{1}{\sqrt{2}}$	$x > \frac{1}{\sqrt{2}}$
$f' > 0$	$f' = 0$	$f' < 0$	$f' = 0$	$f' > 0$	$f' = 0$	$f' < 0$
↗	MAX	↘	MIN	↗	MAX	↘

$x < -\frac{1}{\sqrt{6}}$	$x = -\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$	$x = \frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}} < x$
∩ $f'' < 0$, Konkáv	INFLEXIÓS PONT	∪ $f'' > 0$, konvex	INF. PONT	∩ $f'' < 0$, konkáv



13 c) $f(x)=e^{-2x}\cdot x$ gyök: $x_1=0$

$$f'(x)=-2e^{-2x}\cdot x+e^{-2x}\cdot 1=(-2x+1)e^{-2x}$$

$$f''(x)=\left((-2x+1)e^{-2x}\right)'=-2\cdot e^{-2x}+(-2x+1)(-2e^{-2x})=(-4+4x)e^{-2x}$$

Szélsőértik helye: $f'(x_{sz})=0=(-2x+1)e^{-2x} \longrightarrow x_1=\frac{1}{2}$

típusa: $f''\left(\frac{1}{2}\right)=\left(-4+4\cdot\frac{1}{2}\right)e^{-2\cdot\frac{1}{2}}=-2e<0 \longrightarrow$ MAXIMUM

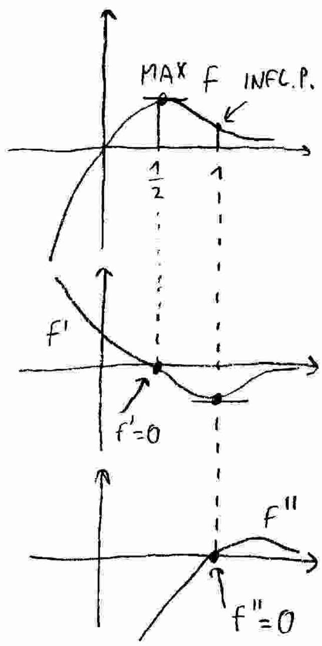
Inflexiós pont: $f''(x_{inf})=0=(-4+4x)e^{-2x} \longrightarrow x_{inf}=1$

Végtelenbeli viselkedés:

$$\lim_{x\to\infty} e^{-2x}\cdot x=\lim_{x\to\infty}\frac{x}{e^{2x}}=\frac{\infty}{\infty}=\lim_{x\to\infty}\frac{1}{2e^{2x}}=0\ ,\quad \lim_{x\to-\infty} \underbrace{e^{-2x}\cdot x}_{(+\infty)\cdot(-\infty)}=-\infty$$

$\frac{1}{2}<x$	$x=\frac{1}{2}$	$\frac{1}{2}<x$
$f'>0$ ↗	$f'=0$ MAX	$f'<0$ ↘

$x<1$	$x=1$	$1<x$
$f''<0$	$f''=0$ INFL. P.	$f''>0$



d) $f(x)=x\ln x$ $D_f=(0,\infty)$ gyök: $x_1=1$ (mivel $\ln 1=0$)

$$f'(x)=1\cdot\ln x+x\frac{1}{x}=\ln x+1$$

$$f''(x)=\frac{1}{x}$$

Szélsőért: $f'(x)=\ln x+1=0 \longrightarrow \ln x=-1 \longrightarrow x=e^{-1}=\frac{1}{e}$

típusa: $f''\left(\frac{1}{e}\right)=\frac{1}{1/e}=e>0 \longrightarrow$ MINIMUM

Infl. pont: $f''(x)=\frac{1}{x}=0 \longrightarrow$ NINCS Inflexiós pont.

$f''(x)=\frac{1}{x}>0$, ha $x\in D_f \rightarrow f$

$$\lim_{x\to\infty} x\ln x=\infty\cdot\infty=\infty$$

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\frac{\infty}{\infty}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}-x=0$$