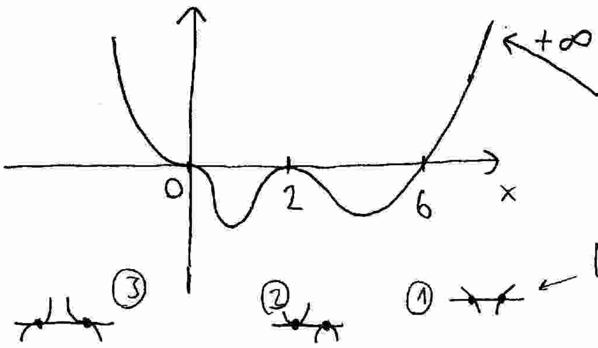
A 2 a) $f(x) = (x-2)^3 \cdot x^2 \cdot (x-6)$

gyökök: $x_1 = 2$ $\quad x_2 = 0$ $\quad x_3 = 6$

multiplicitás: 3 $\quad$ 2 $\quad$ 1

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3 \cdot x^2 \cdot x = \lim_{x\to\infty} x^6 = \infty$$



lehetséges viselkedés 1 multiplicitásnál

b) $$\lim_{x\to\infty} \frac{3x+2}{4x^2+5} = \lim_{x\to\infty} \frac{3/x + 2/x^2}{4 + 5/x^2} = \frac{0+0}{4+0} = 0$$

---

3 $f(x) = x^2 + 6x$

$$\frac{f(2+\Delta x) - f(2)}{\Delta x} = \frac{[(2+\Delta x)^2 + 6\cdot(2+\Delta x)] - [2^2 + 6\cdot 2]}{\Delta x} =$$

$$= \frac{[2^2 + 2\cdot 2\cdot \Delta x + \Delta x^2 + 6\cdot 2 + 6\Delta x] - [2^2 + 6\cdot 2]}{\Delta x} = \frac{2\cdot 2\cdot \Delta x + \Delta x^2}{\Delta x} =$$

$$= 2\cdot 2 + \Delta x$$

$$\lim_{\Delta x\to 0} 2\cdot 2 + \Delta x = 4$$

$$\lim_{x\to 0} \frac{1 - e^{-x}}{\sin(2x)} = \frac{0}{0} = \lim_{x\to 0} \frac{-e^{-x}\cdot(-1)}{\cos(2x)\cdot 2} = \frac{1}{2}$$

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

(4) $$\Big(3x^2+(4x)^3+\underbrace{\frac{1}{x^{3/2}}}_{x^{-3/2}}+e^{-x}\Big)' = 3\cdot 2x+3\cdot(4x)^2\cdot 4-\frac{3}{2}x^{-5/2}+e^{-x}(-1)$$

$$\left(\sqrt{\sin x}\right)' = \left[(\sin x)^{1/2}\right]' = \frac{1}{2}(\sin x)^{-\frac{1}{2}}\cdot\cos x = \frac{1}{2}\,\frac{\cos x}{\sqrt{\sin x}}$$

$$\big(\cos(3x)\cdot\ln(-x)\big)' = (\cos(3x))'\cdot\ln(-x)+\cos(3x)\cdot(\ln(-x))' =$$
$$= 3\cdot(-\sin(3x))\cdot\ln(-x)+\cos(3x)\cdot\frac{1}{-x}\cdot(-1)$$

$$\left(\frac{2+x^2}{3+x}\right)' = \frac{(2+x^2)'(3+x)-(2+x^2)\cdot(3+x)'}{(3+x)^2} = \frac{2x(3+x)-(2+x^2)\cdot 1}{(3+x)^2}$$

(B) (2a) $a_n=\frac{(-1)^n}{2n+1}$. $a_1=\frac{-1}{3}=-\frac{1}{3}$ $\quad a_2=\frac{1}{5}$ $\quad a_3=-\frac{1}{7}$

Mivel $a_1<a_2$, de $a_2>a_3$ $\Rightarrow$ a sorozat nem monoton.

$\lim\limits_{n\to\infty} a_n=0$, (± 1 osztva az egyre nagyobb $2n+1$ számmal)

[megjegyzés: pl. $\lim\limits_{n\to\infty}(-1)^n\frac{n-3}{2n+1}=\lim\limits_{n\to\infty}(-1)^n\frac{1-3/n}{2+1/n}=$
$=\lim\limits_{n\to\infty}(-1)^n\cdot\frac{1}{2}$ nem létezne.

(2b) $f(x)=2e^{x+4}$

$y=2\cdot e^{x+4}$

$\frac{y}{2}=e^{x+4}$

$\ln\frac{y}{2}=\ln(e^{x+4})=x+4$

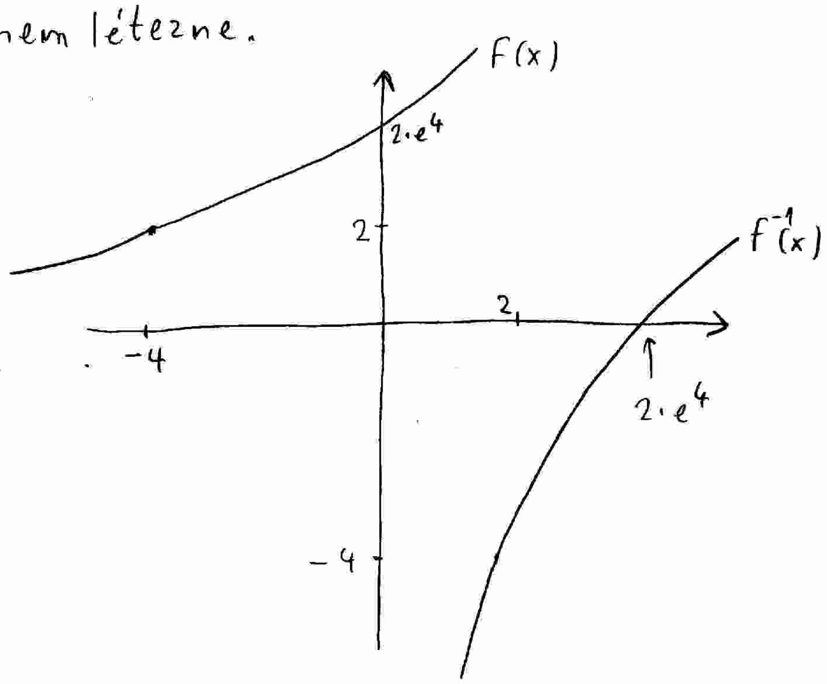
$x=\left(\ln\frac{y}{2}\right)-4$

$f^{-1}(y)=\left(\ln\frac{y}{2}\right)-4$

$f^{-1}(x)=\left(\ln\frac{x}{2}\right)-4$

$D_f=\mathbb{R}$ $\quad R_f=(0,\infty)$

$D_{f^{-1}}=(0,\infty)$ $\quad R_{f^{-1}}=\mathbb{R}$



3 a) $f(x) = x^3 + x$, $x_0 = 2$, $f(x_0) = 2^3 + 2 = 10$, $f'(x) = 3x^2 + 1$, $f'(x_0) = 3 \cdot 2^2 + 1 = 13$

Érintő $y(x) := f(x_0) + f'(x_0) \cdot (x - x_0) = 10 + 13(x-2) = 13x - 16$

$$f(2+\Delta x) - y(2+\Delta x) = \left[(2+\Delta x)^3 + (2+\Delta x)\right] - \left[13(2+\Delta x) - 16\right] =$$

$$= \underline{2^3 + 3 \cdot 2^2 \Delta x + 3 \cdot 2 \cdot \Delta x^2 + \Delta x^3 + 2 + \Delta x - 26 - 13\Delta x + 16} = 6\Delta x^2 + \Delta x^3$$

b) $f(x) = -3x^2 + 2x^3 = x^2(-3 + 2x) = 2 \cdot x^2 \cdot \left(x - \frac{3}{2}\right)$

gyökök	$x_1 = 0$	$x_2 = 3/2$
mult.	2	1

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 2x^3 = \infty$$

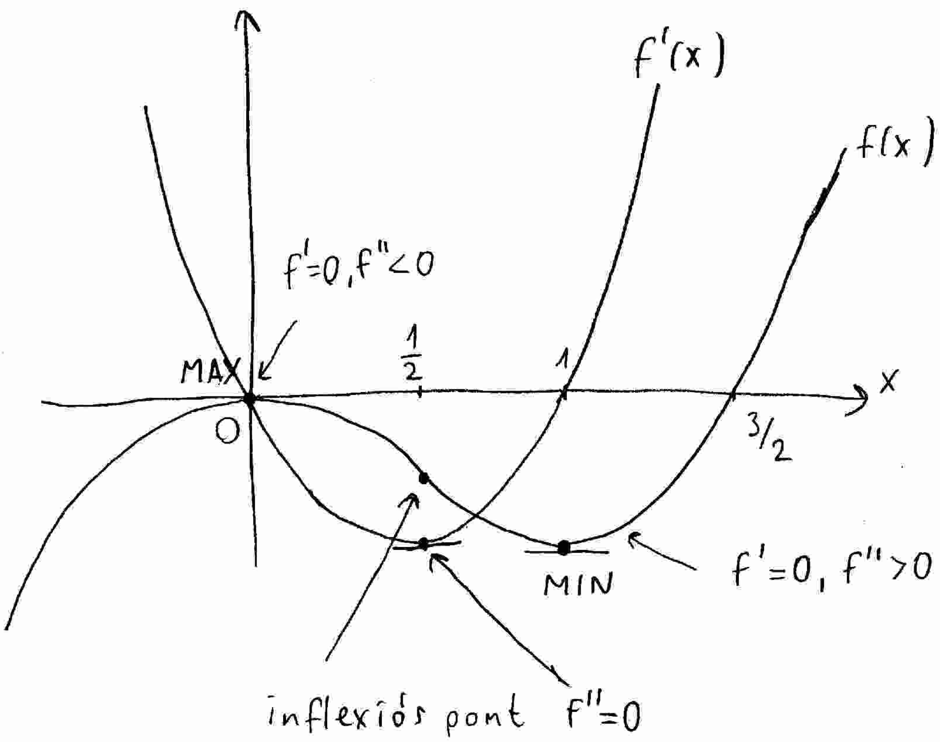
Szélsőértékek: $f'(x) = -6x + 6x^2 = 6x(x-1) \rightarrow x_1 = 0 \qquad x_2 = 1$

$f''(x) = -6 + 12x \qquad f''(0) = -6 < 0 \qquad f''(0) = 6 > 0$

lokális MAX MIN

inflexiós pont: $f''(x_{inf}) = -6 + 12x \overset{!}{=} 0$

$x_{inf} = +\frac{1}{2}$ (Ez csak a lehetséges helyeket adja meg, Mivel $f''' = -12$, tehát $f'''(+\frac{1}{2}) = -12 \neq 0$, így $+\frac{1}{2}$ valóban infl. pont)



5 folyt.

$$\lim_{n\to\infty}\left(1-\frac{4}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{-4}{n}\right)^n=e^{-4}$$

$$\lim_{n\to\infty}\left(2-\frac{4}{n}\right)^n=\lim_{n\to\infty}2^n=\infty$$

$$\lim_{n\to\infty}\frac{2^{2n+1}}{5^n}=\lim_{n\to\infty}\left(\frac{2^2}{5}\right)^n\cdot 2^1=0 \quad , \text{mivel } \left|\frac{2^2}{5}\right|<1$$

$$\lim_{n\to\infty}\frac{6^n}{3^{2n-5}}=\lim_{n\to\infty}\left(\frac{6}{3^2}\right)^n\cdot\frac{1}{3^{-5}}=0 \quad , \text{mivel } \left|\frac{6}{3^2}\right|<1$$

6

$$\lim_{x\to\infty}\left(\frac{1}{2}\right)^x=0, \quad \lim_{x\to\infty}2^x=\infty, \quad \lim_{x\to-\infty}2^x=0,$$

$$\lim_{x\to\infty}\frac{x}{x^2+1}=\lim_{x\to\infty}\frac{1/x}{1+1/x^2}=\frac{0}{1}=0$$

$$\lim_{x\to\infty}\frac{3x^2}{x^2+1}=\lim_{x\to\infty}\frac{3}{1+1/x^2}=3$$

$$\lim_{x\to\infty}\frac{x^3}{x^2+1}=\lim_{x\to\infty}\frac{x}{1+1/x^2}=\infty$$

7

$$(fg)'=f'g+fg' \qquad \left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$$

$$(f(g(x)))'=f'(g(x))\cdot g'(x) \qquad \left(\frac{1}{f}\right)'=\frac{1'\cdot f-1\cdot f'}{f^2}=-\frac{f'}{f^2}$$

8

$$(\sqrt[3]{x})'=(x^{1/3})'=\frac{1}{3}x^{-2/3} \qquad \left(\frac{1}{x^6}\right)'=(x^{-6})'=-6\cdot x^{-7}$$

$$\left(\sqrt[4]{x^3}\right)'=(x^{4/3})'=\frac{4}{3}x^{1/3} \qquad \left(e^{3x+2}\right)'=e^{3x+2}\cdot 3$$

$$(\sin(3x-1))'=\cos(3x-1)\cdot 3 \qquad \left(\frac{1}{3x-1}\right)'=\left((3x-1)^{-1}\right)'=-1\cdot(3x-1)^{-2}\cdot 3$$

$$(\ln(4x-9))'=\frac{1}{4x-9}\cdot 4 \qquad (e^{-x})'=e^{-x}\cdot(-1)=-e^{-x}$$

③ a) $f(x)=x-3x^2$

$$\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{[(2+\Delta x)-3(2+\Delta x)^2]-[2-3\cdot 2^2]}{\Delta x}=$$

$$=\frac{-3\cdot 2\cdot 2\Delta x-3\Delta x^2}{\Delta x}=\underbrace{-3\cdot 2\cdot 2}_{-12}-3\Delta x. \qquad \lim_{\Delta x\to 0} -12-3\Delta x=-12$$

b) $f(x)=(1-2x)^{-1}$

$f'(x)=-1\cdot(1-2x)^{-2}\cdot(-2)$
$=1\cdot 2\cdot(1-2x)^{-2}$

$f''(x)=1\cdot 2\cdot(-2)\cdot(1-2x)^{-3}\cdot(-2)$
$=1\cdot 2\cdot 2^2\cdot(1-2x)^{-3}$

$f'''(x)=1\cdot 2\cdot 2^2\cdot(-3)\cdot(1-2x)^{-4}\cdot(-2)$
$=1\cdot 2\cdot 3\cdot 2^3\cdot(1-2x)^{-4}$

$f(0)=1$

$f'(0)=2$

$f''(0)=8$

$f'''(0)=48$

ha $x\simeq 0$, akkor

$f(x)\approx T_4(x)=$

$=1+2\cdot x+\frac{8}{2!}x^2+\frac{48}{3!}x^3$

$=1+2x+4x^2+8x^3$

$$\left[\begin{array}{l} f^{(n)}(x)=n!\cdot 2^n\cdot(1-2x)^{-n+1}, \quad f^{(n)}(0)=n!\cdot 2^n, \\ f(x)=\sum_{n=0}^{\infty}\frac{n!\cdot 2^n}{n!}x^n=\sum_{n=0}^{\infty}(2x)^n \quad (\text{ha } |2x|<1) \end{array}\right]$$ ← Megjegyzés

④ $$\left(\sqrt[3]{7x}+\frac{1}{(2x)^3}+tg(4x)+ctg(4x)\right)'=\frac{1}{3}(7x)^{-2/3}+(-3)\frac{1}{(2x)^3}\cdot 2+$$
$$+\frac{1}{\cos^2(4x)}\cdot 4-\frac{1}{\sin^2(4x)}\cdot 4$$

$$\ln\left(\frac{1+x}{2-x}\right)=\ln'\left(\frac{1+x}{2-x}\right)\cdot\left(\frac{1+x}{2-x}\right)'=\frac{1}{\frac{1+x}{2-x}}\cdot\frac{(1+x)'(2-x)-(1+x)(2-x)'}{(2-x)^2}=$$

$$=\frac{2-x}{1+x}\,\frac{(2-x)-(1+x)\cdot(-1)}{(2-x)^2}$$

$$(x\cdot \arcsin x)'=x'\cdot\arcsin x+x(\arcsin x)'=\arcsin x+x\frac{1}{\sqrt{1-x^2}}$$

$$\left(\frac{\ln(4x)}{\frac{1}{x}+1}\right)'=\frac{(\ln(4x))'\cdot(\frac{1}{x}+1)-\ln(4x)\cdot(\frac{1}{x}+1)'}{(\frac{1}{x}+1)^2}=\frac{\frac{1}{4x}(\frac{1}{x}+1)-\ln(4x)\cdot(-\frac{1}{x^2})}{(\frac{1}{x}+1)^2}$$

D

2a) $a_n = \dfrac{2n-1}{3n+2}$. $\lim\limits_{n\to\infty} \dfrac{2n-1}{3n+2} = \lim\limits_{n\to\infty} \dfrac{2-1/n}{3+2/n} = \dfrac{2}{3}$

$$a_{n+1} - a_n = \frac{2(n+1)-1}{3(n+1)+2} - \frac{2n-1}{3n+2} = \frac{2n+1}{3n+5} - \frac{2n-1}{3n+2} =$$

$$= \frac{(2n+1)(3n+2)-(2n-1)(3n+5)}{(3n+5)(3n+2)} = \frac{7}{(3n+5)(3n+2)} > 0 \text{ (ha } n=1,2,3..),$$

tehát mivel $a_{n+1} > a_n$, a sorozat szigorúan növekvő.

2b) $f(x) = 2\sqrt{x-3}+1$

$y = 2\sqrt{x-3}+1$

$\dfrac{y-1}{2} = \sqrt{x-3}$

$x = \left(\dfrac{y-1}{2}\right)^2 + 3$

$f^{-1}(y) = \left(\dfrac{y-1}{2}\right)^2 + 3$

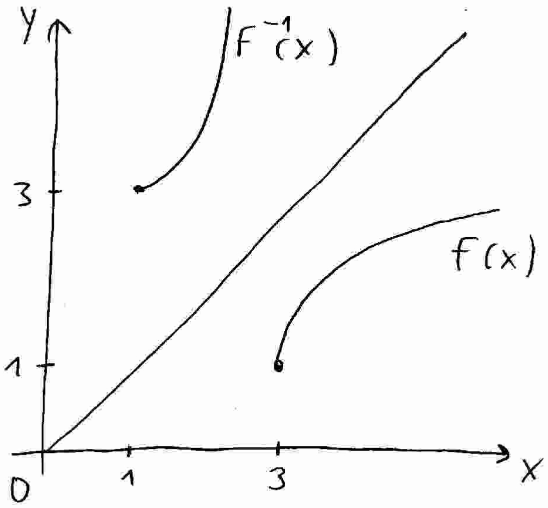
$f^{-1}(x) = \left(\dfrac{x-1}{2}\right)^2 + 3$

$D_f = [3,\infty)$

$R_f = [1,\infty)$

$D_{f^{-1}} = [1,\infty)$

$R_{f^{-1}} = [3,\infty)$



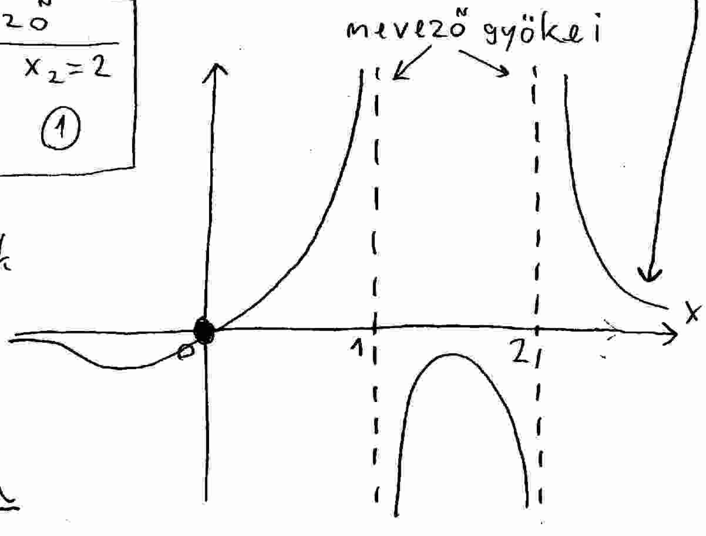
3a) $f(x) = x^2 - 3x$, $x_0 = 5$, $f(x_0) = 5^2 - 3\cdot 5 = 10$

$f'(x) = 2x-3$, $f'(x_0) = 2\cdot 5 - 3 = 7$

érintő: $y(x) = f(x_0) + f'(x_0)(x-x_0) = 10 + 7\cdot(x-5) = 7x-25$

b) $f(x) = \dfrac{x}{(x-1)(x-2)}$. Ha $|x| >> 1$, $f(x) \approx \dfrac{x}{x\cdot x} = \dfrac{1}{x}$

	számláló	nevező	
gyökök:	$x_1 = 0$	$x_1 = 1$	$x_2 = 2$
mult.:	1	1	1



lehetséges viselkedés a gyökök körül:

$x=0$, 1 $x=1$, 1 $x=2$, 1

(3c) $f(x) = x\ln(x)$

$f'(x) = x'\ln(x) + x\underbrace{(\ln x)'}_{1/x} = \ln x + 1$

$f''(x) = \frac{1}{x}$

szélsőért. lehetséges helye

$\ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1} = \frac{1}{e}$

típusa:

$f''\left(\frac{1}{e}\right) = \frac{1}{1/e} = e > 0$

MINIMUM

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \frac{\infty}{\infty} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0$$

(4) $$\Big(\underbrace{\frac{1}{\sqrt{x}}}_{x^{-1/2}} + \sqrt{2x} + \operatorname{arctg}(3x) + \arcsin(3x)\Big)' =$$

$$= -\frac{1}{2}x^{-3/2} + \frac{1}{2}(2x)^{-1/2} + \frac{1}{1+(3x)^2}\cdot 3 + \frac{1}{\sqrt{1-(3x)^2}}\cdot 3$$

$$\left[\sin(\ln[2x])\right]' = \sin'(\ln[2x])\cdot(\ln[2x])' =$$

$$= \cos(\ln[2x])\cdot\frac{1}{2x}\cdot 2$$

$$(2^x e^{-x})' = (2^x)'\cdot e^{-x} + 2^x(e^{-x})' = \ln 2\cdot 2^x\cdot e^{-x} + 2^x\cdot e^{-x}\cdot(-1)$$

$$\left(\frac{\sqrt{4x}}{\cos(3x)}\right)' = \frac{(\sqrt{4x})'\cos(3x) - \sqrt{4x}\,(\cos(3x))'}{[\cos(3x)]^2} =$$

$$= \frac{\frac{1}{2}(4x)^{-1/2}\cdot 4\cdot\cos(3x) - \sqrt{4x}\,(-\sin(3x))\cdot 3}{[\cos(3x)]^2}$$