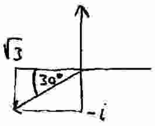
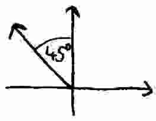
Zárthelyi feladatok megoldásai (Minta)

Komplex számok.

① $z_1 = \sqrt{3} - i$. trig. alak:  $z_1 = 2(\cos 210° + i \sin 210°)$

$\sqrt{\sqrt{3}^2 + (-1)^2}$

$z_2 = 4(\cos 135° + i \cdot \sin 135°)$. alg. alak:  $z_2 = 4 \cdot \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}(-1+i)$

$\frac{z_1}{z_2}$ trig. alak: $\frac{z_1}{z_2} = \frac{2(\cos 210° + i \sin 210°)}{4(\cos 135° + i \sin 135°)} = \frac{1}{2}(\cos 75° + i \sin 75°)$

$210° - 135°$

$z_2^3 = \left[4(\cos 135° + i \sin 135°)\right]^3 = 4^3(\cos 405° + i \sin 405°) =$ $3 \cdot 135°$

$= 64 \cdot (\cos 45° + i \sin 45°) = 64\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 32\sqrt{2}\,(1+i)$

$405° - 1 \cdot 360°$

$\sqrt{z_1} = \sqrt{2(\cos 210° + i \sin 210°)} = \sqrt{2}\left(\cos\left[\frac{210°}{2} + k\frac{360°}{2}\right] + i \sin\left[\frac{210°}{2} + k\frac{360°}{2}\right]\right),$

$k = 0, 1$

első gyök $W_1 = \sqrt{2}(\cos 105° + i \sin 105°)$

második gyök $W_2 = \sqrt{2}(\cos 105° + i \sin 105°)$

② $z_1 + (1+i)z_2 = 1$ (x)

$i z_1 - z_2 = i$ (xx)

(x) $\rightarrow$ $z_1 = 1 - (1+i)z_2$

$i(-(1+i))$

(xx) $\rightarrow$ $i[1 - (1+i)z_2] - z_2 = i \rightarrow i + [(1-i) - 1]z_2 = i \rightarrow \boxed{z_2 = 0}$

(x) $\rightarrow$ $z_1 + (1+i) \cdot 0 = 1 \rightarrow \boxed{z_1 = 1}$

Ellenőrzés: $1 + (1+i) \cdot 0 = 1$

$i \cdot 1 - 0 = i$

Vektorok

③ $\bar{v}_1=[2,1]$, $\bar{v}_2=[-2,1]$

$$[3,0]=\alpha\bar{v}_1+\beta\bar{v}_2=\alpha[2,1]+\beta[-2,1]=[2\alpha-2\beta,\alpha+\beta]$$

$$\left.\begin{array}{l}2\alpha-2\beta=3\\ \alpha+\beta=0\end{array}\right\}\longrightarrow \alpha=\frac{3}{4},\ \beta=-\frac{3}{4}$$

④ $\bar{v}_1=[2,1]$, $\bar{v}_2=[-2,1]$

$$[x,y]=\alpha\bar{v}_1+\beta\bar{v}_2=[2\alpha-2\beta,\alpha+\beta]$$

$$\left.\begin{array}{l}2\alpha-2\beta=x\\ \alpha+\beta=y\end{array}\right\}\longrightarrow \alpha=\frac{x+2y}{4},\ \beta=\frac{2y-x}{4}$$

ha adott α,β, akkor $x=2\alpha-2\beta$, $y=\alpha+\beta$

ha adott x,y akkor $\alpha=\frac{1}{4}x+\frac{1}{2}y$, $\beta=-\frac{1}{4}x+\frac{1}{2}y$

(Mátrix jelöléssel:

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2&-2\\1&1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}\qquad \begin{bmatrix}\alpha\\\beta\end{bmatrix}=\begin{bmatrix}1/4&1/2\\-1/4&1/2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}2&-2\\1&1\end{bmatrix}\begin{bmatrix}1/4&1/2\\-1/4&1/2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=E$$)

⑤ $\bar{a}=\bar{i}-\bar{k}$, $\bar{b}=\bar{j}+\bar{k}$, $\bar{c}=[1,0,2]$

$\bar{a}^0=\dfrac{[1,0,-1]}{\sqrt{2}}$ $\sqrt{2}\leftarrow\sqrt{1^2+0^2+(-1)^2}$, $\quad\bar{a}\bar{b}=[1,0,-1]\cdot[0,1,1]=1\cdot0+0\cdot1+(-1)\cdot1=-1$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\1&0&-1\\0&1&1\end{vmatrix}=\bar{i}\begin{vmatrix}0&-1\\1&1\end{vmatrix}-\bar{j}\begin{vmatrix}1&-1\\0&1\end{vmatrix}+\bar{k}\begin{vmatrix}1&0\\0&1\end{vmatrix}=\underbrace{1}_{0\cdot1-(-1)\cdot1}\bar{i}-1\bar{j}+1\bar{k}=[1,-1,1]$$

$$\bar{a}\bar{b}\bar{c}=(\bar{a}\times\bar{b})\cdot\bar{c}=[1,-1,1][1,0,2]=1\cdot1+(-1)\cdot0+1\cdot2=3$$

vagy
$$\bar{a}\bar{b}\bar{c}=\begin{vmatrix}1&0&-1\\0&1&1\\1&0&2\end{vmatrix}=1\cdot\begin{vmatrix}1&1\\0&2\end{vmatrix}-0\cdot\begin{vmatrix}0&1\\1&2\end{vmatrix}+(-1)\begin{vmatrix}0&1\\1&0\end{vmatrix}=1\cdot2-0-1\cdot(-1)=3$$

$\bar{b}$ vetülete $\bar{a}^0$-ra:
$$(\bar{b}\bar{a}^0)\cdot\bar{a}^0=\left([0,1,1]\cdot\frac{[1,0,-1]}{\sqrt{2}}\right)\cdot\frac{[1,0,-1]}{\sqrt{2}}=\frac{0\cdot1+1\cdot0+1\cdot(-1)}{\sqrt{2}}\cdot\frac{[1,0,-1]}{\sqrt{2}}=-\frac{1}{2}[1,0,-1]=\left[-\frac{1}{2},0,\frac{1}{2}\right]$$

$\bar{a}$ és $\bar{b}$ szöge: $\cos\alpha=\dfrac{\bar{a}\bar{b}}{|\bar{a}||\bar{b}|}=\dfrac{-1}{\sqrt{2}\cdot\sqrt{2}}=-\dfrac{1}{2}\longrightarrow\alpha=120^0$

az $\bar{a}$ és $\bar{b}$ által kifeszített ▱ Területe: $|\bar{a}\times\bar{b}|=|[1,-1,1]|=\sqrt{1^2+(-1)^2+1^2}=\sqrt{3}$

—"— △ —"— $\dfrac{|\bar{a}\times\bar{b}|}{2}=\dfrac{\sqrt{3}}{2}$

az $\bar{a},\bar{b}$ és $\bar{c}$ által kifeszített paralelepipedon térfogata: $|\bar{a}\bar{b}\bar{c}|=|3|=3$

⑥ Egyenes két pontja: $P_1[1,0,2]$, $P_2[0,3,1]$

$$\bar{v} = P_2 - P_1 = [-1,3,-1] \quad , \quad \bar{r}_0 = P_1 = [1,0,2]$$

$$\bar{r}(t) = \bar{r}_0 + t\bar{v} = [1,0,2] + t[-1,3,-1] = [1-t, 0+3t, 2-t]$$

Sík egyenlete:

⑦ normálvektor + egy pont:

$\bar{n} = [1,-1,1]$, $P_1[1,1,1]$

$$\bar{n}\cdot(\bar{r}-\bar{r}_0) = 0$$

$$[1,-1,1]\big([x,y,z]-[1,1,1]\big) = 0$$

$$1\cdot(x-1) - 1(y-1) + 1\cdot(z-1) = 0$$

⑧ egy pont + két síkkal párhuzamos vektor:

$P_1[1,1,1]$, $\bar{a} = [1,0,-1]$, $\bar{b} = [0,1,1]$

$n = \bar{a}\times\bar{b} = [1,-1,1]$ (⑤-ös feladatból)

$1\cdot(x-1) - 1(y-1) + 1(z-1) = 0$ (⑦-es feladat)

⑨ 3 pont:

$P_1[1,1,1]$, $P_2[2,1,0]$, $P_3[1,2,2]$

$\bar{a} = P_2 - P_1 = [1,0,-1]$, $\bar{b} = P_3 - P_1 = [0,1,1]$

folytatás: ⑧-as feladat.

⑩ Az $\bar{r}(t) = [1-t, 0+3t, 2-t]$ egyenes és a $x - y + z - 1 = 0$ sík metszéspontja:

$[x,y,z] = [1-t, 3t, 2-t]$ ha az egyenesen vagyunk

$(1-t) - (3t) + (2-t) - 1 = 0$ ha ez a pont a síkon is rajta van.

tehát $-5t + 2 = 0 \longrightarrow t = \frac{2}{5}$,

a metszéspont: $\bar{r}\left(\frac{2}{5}\right) = \left[1-\frac{2}{5}, 3\cdot\frac{2}{5}, 2-\frac{2}{5}\right] = \left[\frac{3}{5}, \frac{6}{5}, \frac{8}{5}\right]$

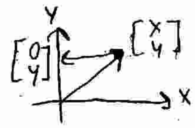
(11) lineáris tr.-ek mátrixai:

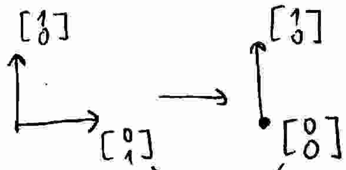
$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} y \\ x-2y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} \to [2x+3y] = [2 \;\; 3] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ 2x \\ 2x+y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(12) geometriai tr.-ek mátrixai:

a) y-tengelyre ⊥ vetítés

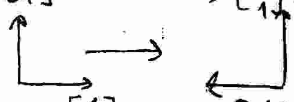




$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} \end{bmatrix} = P_{y\text{-vetítés}}$$

b) tükrözés



$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$P_{y\text{-tükrözés}} = \begin{bmatrix} \boxed{\begin{matrix} -1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} \end{bmatrix}$$

c) $x=y$ egyenesre tükrözés

(13) R^3-ban x-y-síkra vetítés

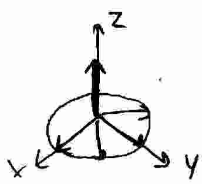
$$P_{xy} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad P_{xy} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad P_{xy} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

xy síkbeli vektorok nem változnak

xy síkra ⊥ vektorok a nullvektorba vetítődnek

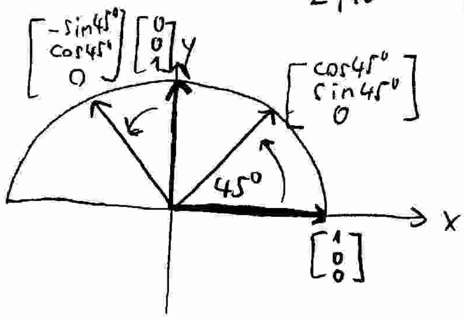
z-körüli 45° elforgatás



$$R_{z,45°} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 45° \\ \sin 45° \\ 0 \end{bmatrix} \quad R_{z,45°} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin 45° \\ \cos 45° \\ 0 \end{bmatrix}$$

$$R_{z,45°} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{z tengely a helyén marad}$$

$$R_{z,45°} = \begin{bmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



14

$\underbrace{[2\ \ 3\ \ 4]}_{3}\begin{bmatrix}1\\2\end{bmatrix}\}2$ 3 oszlop 2 sor $3 \neq 2$ NINCS értelme

$$[2\ \ 3\ \ 4]\begin{bmatrix}1\\2\\4\end{bmatrix} = [2\cdot1+3\cdot2+4\cdot4] = [24]$$

$$[2][1\ \ 2\ \ 4] = [2\ \ 4\ \ 8]$$

$$[2\ \ 3]\begin{bmatrix}1&2\\3&4\end{bmatrix} = [2\cdot1+3\cdot3\quad 2\cdot2+3\cdot4] = [11\ \ 16]$$

15

$$\begin{bmatrix}2&3\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&1\end{bmatrix} - 3\begin{bmatrix}0&1\\1&1\end{bmatrix} + 2\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}2&5\\1&2\end{bmatrix} - \begin{bmatrix}0&3\\3&3\end{bmatrix} + \begin{bmatrix}2&0\\0&2\end{bmatrix} = \begin{bmatrix}4&2\\-2&1\end{bmatrix}$$

$$B^2 - BE = B^2 - B = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&1\end{bmatrix} - \begin{bmatrix}0&1\\1&1\end{bmatrix} = \begin{bmatrix}1&1\\1&2\end{bmatrix} - \begin{bmatrix}0&1\\1&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

16

$[2][x] = 1$

$[2x] = 1$

$x = 1/2$

$[2]^{-1} = [1/2]$

$$\begin{bmatrix}2&2\\6&7\end{bmatrix}\begin{bmatrix}x&u\\y&v\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}2x+2y&2u+2v\\6x+7y&6u+7v\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$2x+2y=1 \qquad 2u+2v=0$

$6x+7y=0 \qquad 6u+7v=1$

$x=\frac{7}{2}\quad y=-3 \qquad u=-1\quad v=1$

$$\begin{bmatrix}2&2\\6&7\end{bmatrix}^{-1} = \begin{bmatrix}7/2&-1\\-3&1\end{bmatrix}$$

$$\left.\begin{array}{l}2x+2y=2\\6x+7y=4\end{array}\right\} \to \begin{bmatrix}2&2\\6&7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} \to \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2&2\\6&7\end{bmatrix}^{-1}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}7/2&-1\\-3&1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$$

Vagyis $x=3$, $y=-2$

17

$$\left|\begin{array}{ccc|c}1&-1&1&2\\2&1&-1&1\\-1&-1&2&3\end{array}\right| \xrightarrow[\text{III}\to\text{III}+\text{I}]{\text{II}\to\text{II}-2\cdot\text{I}} \left|\begin{array}{ccc|c}1&-1&1&2\\0&3&-3&-3\\0&-2&3&5\end{array}\right| \xrightarrow{\text{III}\to\text{III}+\frac{2}{3}\text{II}} \left|\begin{array}{ccc|c}1&-1&1&2\\0&3&-3&-3\\0&0&1&3\end{array}\right|$$

III sor: $1z=3 \to \boxed{z=3}$, II sor: $3y-3(\underset{z}{3})=-3 \to \boxed{y=2}$,

I sor: $1x-1\cdot(2)+1\cdot(3)=2 \to \boxed{x=1}$

Ellenőrzés:

$1-2+3=2$

$2\cdot1+2-3=1$

$-1-2+2\cdot3=3$

(18) sajátérték: $0=\det\left(\begin{bmatrix}2&0\\0&3\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=\begin{vmatrix}2-\lambda&0\\0&3-\lambda\end{vmatrix}=(2-\lambda)(3-\lambda)$

$$(2-\lambda)(3-\lambda)=0 \longrightarrow \lambda_1=2\,,\quad \lambda_2=3$$

saját vektor:

$\lambda_1=2$

$$\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=2\begin{bmatrix}x\\y\end{bmatrix}$$

$$\left.\begin{matrix}2x=2x\\3y=2y\end{matrix}\right\}\begin{matrix}x \text{ tetszőleges}\\y=0\end{matrix}$$

$$\overline{V}_1=\begin{bmatrix}1\\0\end{bmatrix}$$

$\lambda_2=3$

$$\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=3\begin{bmatrix}x\\y\end{bmatrix}$$

$$\left.\begin{matrix}2x=3x\\3y=3y\end{matrix}\right\}\begin{matrix}x=0\\y \text{ tetszőleges}\end{matrix}$$

$$\overline{V}_2=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}7\\5\end{bmatrix}=\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\alpha\\\beta\end{bmatrix}\longrightarrow \alpha=7\,,\quad\beta=5$$

$$\begin{bmatrix}2&0\\0&3\end{bmatrix}^{13}\begin{bmatrix}7\\5\end{bmatrix}=\begin{bmatrix}2&0\\0&3\end{bmatrix}^{13}\left(7\cdot\begin{bmatrix}1\\0\end{bmatrix}+5\begin{bmatrix}0\\1\end{bmatrix}\right)=7\cdot\begin{bmatrix}2&0\\0&3\end{bmatrix}^{13}\begin{bmatrix}1\\0\end{bmatrix}+5\begin{bmatrix}2&0\\0&3\end{bmatrix}^{13}\begin{bmatrix}0\\1\end{bmatrix}$$

$$=7\cdot\underset{\lambda_1}{(2)}^{13}\cdot\underset{\overline{V}_1}{\begin{bmatrix}1\\0\end{bmatrix}}+5\cdot\underset{\lambda_2}{(3)}^{13}\underset{\overline{V}_2}{\begin{bmatrix}0\\1\end{bmatrix}}=\begin{bmatrix}7\cdot2^{13}\\5\cdot3^{13}\end{bmatrix}$$

---

(19) sajátérték: $0=\det\left(\begin{bmatrix}2&1\\0&3\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=\begin{vmatrix}2-\lambda&1\\0&3-\lambda\end{vmatrix}=(2-\lambda)(3-\lambda)\rightarrow\lambda_1=2,\ \lambda_2=3$

saját vektor:

$\lambda_1=2$

$$\begin{bmatrix}2&1\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=2\begin{bmatrix}x\\y\end{bmatrix}$$

$$\left.\begin{matrix}2x+y=2x\\3y=2y\end{matrix}\right\}\begin{matrix}x \text{ tetszőleges}\\y=0\end{matrix}$$

$$\overline{V}_1=\begin{bmatrix}1\\0\end{bmatrix}$$

$\lambda_2=3$

$$\begin{bmatrix}2&1\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=3\begin{bmatrix}x\\y\end{bmatrix}$$

$$\left.\begin{matrix}2x+y=3x\\3y=3y\end{matrix}\right\}\begin{matrix}y \text{ tetszőleges}\\x=y\end{matrix}$$

$$\overline{V}_2=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}7\\5\end{bmatrix}=\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\alpha+\beta\\\beta\end{bmatrix}\longrightarrow\left.\begin{matrix}\alpha+\beta=7\\\beta=5\end{matrix}\right\}\rightarrow\alpha=2$$

vagyis $\begin{bmatrix}7\\5\end{bmatrix}=2\begin{bmatrix}1\\0\end{bmatrix}+5\begin{bmatrix}1\\1\end{bmatrix}$

$$\begin{bmatrix}2&1\\0&3\end{bmatrix}^{13}\begin{bmatrix}7\\5\end{bmatrix}=2\cdot2^{13}\begin{bmatrix}1\\0\end{bmatrix}+5\cdot3^{13}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\cdot2^{13}+5\cdot3^{13}\\5\cdot3^{13}\end{bmatrix}$$