1. Legyen $a = i + 2j - 3k$, $b = 2j - 3k$, $c = 2i + j + k$. (1 + 2 + 2 + 5 · 1 pont)

(a) Mennyi ab ?

$(1,2,-3)\cdot(0,2,-3) = 1\cdot 0 + 2\cdot 2 + (-3)\cdot(-3) = 13$

(b) Mennyi $a \times b$?

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2 & -3 \\ 0 & 2 & -3 \end{vmatrix} = \bar{i}\begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix} - \bar{j}\begin{vmatrix} 1 & -3 \\ 0 & -3 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = \underset{\uparrow\, 2\cdot(-3)-2\cdot(3)}{0}\,\bar{i} + 3\bar{j} + \underset{\uparrow\, 1\cdot 2 - 2\cdot 0}{2}\,\bar{k} = (0,3,2)$$

(c) Mennyi abc ?

$(a \times b)\cdot c = (0,3,2)(2,1,1) = 0\cdot 2 + 3\cdot 1 + 2\cdot 1 = 5,$

Vagy
$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 1\cdot\underbrace{\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}}_{5} - 2\cdot\underbrace{\begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix}}_{6} + (-3)\underbrace{\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix}}_{-4} = 5$$

(d) Mennyi az a, b vektorok kozotti szog koszinusza?

$$\cos\alpha = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|} = \frac{13}{\sqrt{1^2+2^2+(-3)^2}\,\sqrt{0^2+2^2+(-3)^2}} = \frac{13}{\sqrt{14}\cdot\sqrt{13}} = \sqrt{\frac{13}{14}}$$

(e) Mennyi az a, b vektorok altal kifeszitett paraleogramma terulete?

$$|\bar{a}\times\bar{b}| = |(0,3,2)| = \sqrt{0^2+3^2+2^2} = \sqrt{13}$$

(f) Mennyi az a, b vektorok altal kifeszitett haromszog terulete?

$$T_{\triangle} = \frac{T_{\square}}{2} = \frac{\sqrt{13}}{2}$$

(g) Mennyi az a, b, c vektorok altal kifeszitett ferde teglatest (paralepipedon) terfogata?

$$|\bar{a}\bar{b}\bar{c}| = |5| = 5$$

(h) Mennyi az a, b, c vektorok altal kifeszitett tetraeder terfogata?

$$V = \frac{1}{6}\cdot 5$$

2. Nev: Neptun:

Legyen (7 + 3 pont)

$$
\begin{aligned}
3x_2 + x_4 &= 10 \\
x_1 + x_2 + 2x_3 &= 11 \\
-x_1 + x_3 &= -2
\end{aligned}
$$

(a) Keresd meg ennek az egyenletnek az osszes megoldasat pivotalas segitsegevel!

pivottábla:

$$
\begin{array}{c|cccc|c}
 & a_1 & a_2 & a_3 & a_4 & b \\
\hline
e_1 & 0 & 3 & 0 & 1 & 10 \\
e_2 & 1 & 1 & 2 & 0 & 11 \\
e_3 & -1 & 0 & 1 & 0 & -2
\end{array}
\qquad \xrightarrow{\text{pivotálás: } (2,1),\ (3,2),\ (1,4) \text{ elemek körül}}
$$

$$
\begin{array}{c|cccc|c}
 & a_1 & a_2 & a_3 & a_4 & b \\
\hline
a_4 & 0 & 0 & -9 & 1 & -17 \\
a_1 & 1 & 0 & -1 & 0 & 2 \\
a_2 & 0 & 1 & 3 & 0 & 9
\end{array}
$$

4 pont

$\bar{b} = x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 + x_4\bar{a}_4$

vagyis $\bar{b} = 2\cdot\bar{a}_1 + 9\bar{a}_2 - 17\bar{a}_4$

tehát egy megoldás: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 0 \\ -17 \end{pmatrix}$

a_3 oszlopa: $-1\bar{a}_1 + 3\bar{a}_2 - 9\bar{a}_4 = \bar{a}_3$,

vagyis $-1\,\bar{a}_1 + 3\bar{a}_2 - 1\cdot\bar{a}_3 - 9\bar{a}_4 = \bar{0}$.

tehát $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \\ -9 \end{pmatrix}$ megoldja a homogén (zero jobboldal) egyenletet.

Megoldás: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 0 \\ -17 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ -1 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 0 \\ -17 \end{pmatrix} - x_3\begin{pmatrix} -1 \\ 3 \\ -1 \\ 9 \end{pmatrix}$

3 pont

(b) Ellenorizd az eredmenyedet!

$$
\begin{pmatrix} 0 & 3 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 9 \\ 0 \\ -17 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 9 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & 3 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -1 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

vagy

$$
\begin{aligned}
3\cdot(9+3t) + (-17+9t) &= 10 \\
(2-t) + (9+3t) + 2(-t) &= 11 \\
-(2-t) + (-t) &= -2
\end{aligned}
$$

~~Megjegyzés:~~

Más pivotálást használva az eredmény más alakban lesz,

$\begin{pmatrix} 2 \\ 9 \\ 0 \\ -17 \end{pmatrix} + t\cdot\begin{pmatrix} -1 \\ 3 \\ -1 \\ 9 \end{pmatrix}$ helyett, lehet pl.

$\begin{pmatrix} 2+7\cdot(-1) \\ 9+7\cdot 3 \\ 0+7\cdot(-1) \\ -17+7\cdot 9 \end{pmatrix} + t\cdot\begin{pmatrix} 8\cdot(-1) \\ 8\cdot 3 \\ 8\cdot(-1) \\ 8\cdot 9 \end{pmatrix}$

(a 7, 8 lehet akármi)

3. Adott harom pont: $A = (1,0,0)$, $B = (0,0,1)$, $C = (0,1,0)$. (4 + 3 + 3 pont)

(a) Ird fel az oket tartalmazo siknak egy, a sikot meghatarozo egyenletet!

$\bar{u} = B - A = (-1,0,1)$

$\bar{v} = C - A = (0,-1,1)$

$$\bar{n} = \bar{u} \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \bar{i} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - \bar{j} \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + \bar{k} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$= \bar{i} + \bar{j} + \bar{k} = (1,1,1)$$

sík egyenlete: $\bar{n} \cdot (\bar{r} - A) = 0$, vagy $\bar{n}\bar{r} = \bar{n} \cdot A$

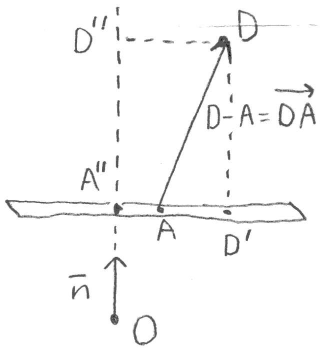
$(1,1,1) \cdot (x,y,z) = (1,1,1) \cdot (1,0,0)$

$x + y + z = 1$

$\bar{n}A = (1,1,1) \cdot (1,0,0)$

(b) Mennyi a $D = (1,0,5)$ pont tavolsaga ettol a siktol?

$$d = \left| \frac{\bar{n}}{|\bar{n}|} \cdot (D - A) \right| = \frac{1}{|\bar{n}|} \cdot \left| \bar{n} \cdot \bar{D} - \bar{n}A \right| = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} \cdot \left| (1,1,1) \cdot (1,0,5) - 1 \right| = \frac{5}{\sqrt{3}}$$



$$d = |\overrightarrow{D'D}| = |\overrightarrow{A''D''}| = \left| \frac{\bar{n}}{|\bar{n}|} \cdot \overrightarrow{AD} \right| = \left| \frac{\bar{n}}{|\bar{n}|} (D - A) \right|$$

sík-egyenes távolság

D'D vetülete az $\bar{n}$ normálvektorra

egy $\bar{b}$ vektor vetülete az $\bar{m}$ vektorra: $\frac{(\bar{n} \cdot \bar{b})}{|\bar{n}|^2} \cdot \bar{n}$,

ennek hossza: $\left| \frac{\bar{n} \cdot \bar{b}}{|\bar{n}|} \right|$

(c) Keresd meg az $E = (0,1,0)$ es a $F = (3,4,5)$ pontokon athalado egyenes es a sik metszespontjat!

$E = C$, tehát C metszéspont, mégpedig az egyetlen, mert $\overline{EF}$ nem $\perp$ $\bar{n}$-re.

Vagy: $\bar{v} = F - E = (3,3,5)$, $\bar{r}(t) = E + t\bar{v} = (0,1,0) + t(3,3,5) = (3t, 3t+1, 5t)$

$\bar{r}(t)$ a síkon van, ha $\underbrace{(3t)}_{x} + \underbrace{(3t+1)}_{y} + \underbrace{(5t)}_{z} = 1 \longrightarrow t = 0$,

a sík egyenlete: $x + y + z = 1$

a metszéspont: $\bar{r}(0) = (3 \cdot 0, 3 \cdot 0 + 1, 5 \cdot 0) = (0,1,0)$

4. Legyen $v = \sqrt{3}i - 1$, $w = 1 + i$. (2 + 2 + 3 + 3 pont)

(a) Mennyi $vw + \bar{v}$?

$$VW = (-1+\sqrt{3}\,i)(1+i) = (-1+\sqrt{3}\,i^2) + (\sqrt{3}-1)\,i = (-1-\sqrt{3}) + (\sqrt{3}-1)\,i$$

$$\bar{V} = -1-\sqrt{3}\,i$$

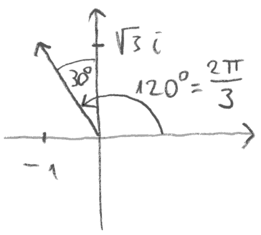
$$VW+\bar{V} = [(-1-\sqrt{3}) + (\sqrt{3}-1)\,i] + [-1-\sqrt{3}\,i] = (-2-\sqrt{3}) - i$$

(b) Mennyi w/v algebrai alakja?

$$\frac{1+i}{-1+\sqrt{3}\,i} = \frac{1+i}{-1+\sqrt{3}\,i} \cdot \frac{-1-\sqrt{3}\,i}{-1-\sqrt{3}\,i} = \frac{(-1-\sqrt{3}\,i^2) + (-1-\sqrt{3})\,i}{(-1)^2 + (\sqrt{3})^2} =$$

$$= \frac{(-1+\sqrt{3})}{4} + \frac{(-1-\sqrt{3})}{4}\,i$$

(c) Mennyi v^5 ? (Add meg az eredmenyt algebrai alakban is!)



$$-1+\sqrt{3}\,i = 2\cdot\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right)$$

$\uparrow$ $\sqrt{(-1)^2+(\sqrt{3})^2}$

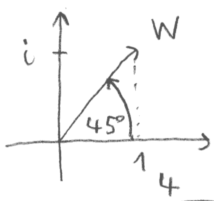
$$\frac{10\pi}{3} = 2\pi + \frac{4}{3}\pi$$

$$V^5 = \left[2\cdot\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right)\right]^5 = 2^5\cdot\left(\cos\left[5\cdot\frac{2\pi}{3}\right] + i\cdot\sin\left[5\cdot\frac{2\pi}{3}\right]\right) =$$

$$= 32\cdot\left(\cos\frac{4\pi}{3} + i\cdot\sin\frac{4\pi}{3}\right) = 32\cdot\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = -16 - 16\cdot\sqrt{3}\cdot i$$

$240^\circ \sim -120^\circ$

(d) Mennyi $\sqrt[4]{w}$?



$$W = \sqrt{2}\cdot\left(\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}\right)$$

$\uparrow$ $\sqrt{1^2+1^2}$

$$\sqrt[4]{W} = \sqrt[4]{\sqrt{2}\left(\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}\right)} = \sqrt[4]{\sqrt{2}}\left(\cos\left[\frac{\pi/4}{4} + \frac{2k\pi}{4}\right] + i\cdot\sin\left[\frac{\pi/4}{4} + \frac{2k\pi}{4}\right]\right)$$

ahol $k = 0, 1, 2, 3$

$4\cdot 2 \to 8$

$$= \sqrt[8]{2}\left(\cos\left[\frac{\pi}{16} + k\cdot\frac{\pi}{2}\right] + i\cdot\sin\left[\frac{\pi}{16} + k\cdot\frac{\pi}{2}\right]\right)$$

a szögek: $11{,}25^\circ$, $101{,}25^\circ$, 191.25°, 281.25°