Zh2, Lin.Alg., 2021.11.30. NEPTUN: Aláírás:

Név:

1. (3+4+3 pont) Legyen

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

Keresd meg A sajatertekeit!

$$\det(A-\lambda E) = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 2^2 = \lambda^2 - 8\lambda + 12 = 0$$

$$\lambda_{1,2} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\cdot1\cdot12}}{2\cdot1} \text{, vagy } \lambda_1\lambda_2 = 12,\ \lambda_1+\lambda_2 = 8 \implies \lambda_1 = 6,\ \lambda_2 = 2$$

Keresd meg A sajatvektorait!

$$(A-\lambda E)\vec{V} = \vec{0}$$

$\lambda_1 = 6$:

$$\begin{pmatrix} 4-6 & 2 \\ 2 & 4-6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -2x+2y=0 \\ 2x-2y=0 \end{array}\right\} \longrightarrow x = y$$

sajátvektorok: $\begin{bmatrix} x \\ x \end{bmatrix}$, $x \neq 0$

$$\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} 4-2 & 2 \\ 2 & 4-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 2x+2y=0 \\ 2x+2y=0 \end{array}\right\} \longrightarrow -x = y$$

sajátvektorok: $\begin{bmatrix} x \\ -x \end{bmatrix}$, $x \neq 0$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Mennyi $A^{13}\,(6,8)^T$?

$$\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \alpha_1\vec{V}_1 + \alpha_2\vec{V}_2 = \alpha_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \overset{\text{vagy}}{=} \begin{bmatrix} \boxed{\begin{matrix}1\\1\end{matrix}} & \boxed{\begin{matrix}1\\-1\end{matrix}} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\left.\begin{array}{l} \alpha_1+\alpha_2=6 \\ \alpha_1-\alpha_2=8 \end{array}\right\} \alpha_1 = \frac{6+8}{2} = 7,\ \alpha_2 = \frac{6-8}{2} = -1$$

vagy $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$$A^{13}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = A^{13}(7\vec{V}_1 - 1\vec{V}_2) = 7A^{13}\vec{V}_1 - 1\cdot A^{13}\vec{V}_2 =$$

$$= 7\cdot 6^{13}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\cdot 2^{13}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

vagy

$$= \begin{bmatrix} \underset{\vec{V}_1}{\boxed{\begin{matrix}1\\1\end{matrix}}} & \underset{V_2}{\boxed{\begin{matrix}1\\-1\end{matrix}}} \end{bmatrix}\begin{bmatrix} \underset{\lambda_1}{6^{13}} & 0 \\ 0 & \underset{\lambda_2}{2^{13}} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

2. (7+3 pont) Legyen

$$A = \begin{pmatrix} 5 & 2 & 4 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \end{pmatrix}.$$

Szamold ki pivotalassal A inverzet! Ellenorizd az eredmenyed!

	a_1	a_2	a_3	e_1	e_2	e_3
e_1	5	2	4	1	0	0
e_2	3	2	$\boxed{1}$	0	1	0
e_3	5	3	2	0	0	1
e_1	-7	-6	0	1	-4	0
a_3	3	2	1	0	1	0
e_3	$\boxed{-1}$	-1	0	0	-2	1
e_1	0	$\boxed{1}$	0	1	10	-7
a_3	0	-1	1	0	-5	3
a_1	1	1	0	0	2	-1
a_2	0	1	0	1	10	-7
a_3	0	0	1	1	5	-4
a_1	1	0	0	-1	-8	6

Ellenőrzés:

$$\begin{bmatrix} 5 & 2 & 4 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & -8 & 6 \\ 1 & 10 & -7 \\ 1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Legyen

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}.$$

Mennyi $(A^{-1})_{32}$, ha az indexálás 1-nel kezdodik?

$$\det A = -0 \cdot \begin{vmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 0 \\ 6 & 0 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 6 & 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 5 & 0 & 1 \\ 6 & 0 & 0 \end{vmatrix}$$

$$= 1 \cdot \left(1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 5 & 0 \\ 6 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 5 & 1 \\ 6 & 0 \end{vmatrix} \right) = 1 \cdot \left(1 \cdot 1 - 0 \cdot 5 + 3 \cdot (-6) \right) = -17$$

$$(A^{-1})_{32} = \frac{1}{-17} \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 6 & 0 & 1 \end{vmatrix} = \frac{1}{-17} \cdot (-1)^{3+2} \cdot \left[-5 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 3 \\ 6 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix} \right]$$

$$= \frac{1}{-17} \cdot (-1) \cdot (-5) \cdot 2 = -\frac{10}{17}$$

23-as elem!
NEM a 32-es!

$$\begin{bmatrix} 1 & 2 & \boxed{0} & 3 \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} \\ 5 & 0 & \boxed{1} & 0 \\ 6 & 0 & \boxed{0} & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 6 & 0 & 1 \end{bmatrix}$$

3. (5+5 pont) Legyen

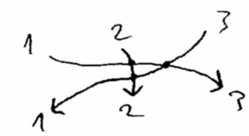
$$A = \begin{pmatrix} 4 & 2 & 5 \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix}.$$

Szamold ki pivotalassal $\det(A)$-t!

$$\begin{array}{c|ccc|} & a_1 & a_2 & a_3 \\ \hline e_1 & 4 & 2 & 5 \\ e_2 & 2 & 3 & 5 \\ e_3 & \boxed{1} & 2 & 3 \\ \hline e_1 & 0 & -6 & -7 \\ e_2 & 0 & \boxed{-1} & -1 \\ a_1 & 1 & 2 & 3 \\ \hline e_1 & 0 & 0 & \boxed{-1} \\ a_2 & 0 & 1 & 1 \\ a_1 & 1 & 0 & 1 \\ \hline a_3 & 0 & 0 & 1 \\ a_2 & 0 & 1 & 0 \\ a_1 & 1 & 0 & 0 \\ \hline \end{array}$$

$$\det A = \boxed{1} \cdot \boxed{-1} \cdot \boxed{-1} \cdot \operatorname{sgn} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}_{\text{permutáció:}}$$

$$= 1 \cdot (-1) \cdot (-1) \cdot (-1) = -1$$



3 (páratlan) számú kereszteződés: páratlan permutáció

$\operatorname{sgn} = -1$

Legyen

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 3 & 5 \end{pmatrix}.$$

Ird fel ezen matrix $A = LU$ felbontasat!

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 3 & 5 \end{bmatrix} \xrightarrow[\text{III.} \to \text{III.} - 3 \cdot \text{I.}]{\text{I.} \to \text{I.},\ \text{II.} \to \text{II.} - 0 \cdot \text{I.}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow[\text{III.} \to \text{III.} - (-3) \text{II.}]{\text{I.} \to \text{I.},\ \text{II.} \to \text{II.}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \boxed{0} & 1 & 0 \\ \boxed{-3} & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{3} & 1 \end{bmatrix}$$

előjelváltás

$L = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{0} & 1 & 0 \\ \boxed{3} & \boxed{-3} & 1 \end{bmatrix}$ $\qquad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 11 \end{bmatrix}$

4.A (5+(2+3) pont) Oldd meg pivotálással a kovetkezo egyenletrenszert! Adj meg egy partikularis megoldast! Adj meg egy bazisát a homogen egyenlet megoldasainak! Ird fel az altalanos megoldast!

$$9x_1 + 3x_2 - 6x_3 = 39$$
$$6x_1 + 2x_2 - 4x_3 = 26$$
$$3x_1 + 1x_2 - 2x_3 = 13.$$

$$\begin{array}{c|cccc|} & a_1 & a_2 & a_3 & b \\ \hline e_1 & 9 & 3 & -6 & 39 \\ e_2 & 6 & 2 & -4 & 26 \\ e_3 & 3 & \boxed{1} & -2 & 13 \\ \hline \end{array} \longrightarrow \begin{array}{c|cccc|} & a_1 & a_2 & a_3 & b \\ \hline e_1 & 0 & 0 & 0 & 0 \\ e_2 & 0 & 0 & 0 & 0 \\ a_2 & 3 & 1 & -2 & 13 \\ \hline \end{array}$$

part. megold: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ 0 \end{bmatrix}$

$$\begin{array}{c|c} & a_1 \\ \hline e_1 & 0 \\ e_2 & 0 \\ a_2 & 3 \end{array} \longrightarrow \vec{a}_1 = 3\vec{a}_2$$

$$-1\cdot\vec{a}_1 + 3\vec{a}_2 + 0\vec{a}_3 = 0$$

$$\vec{X}_{hom_1} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

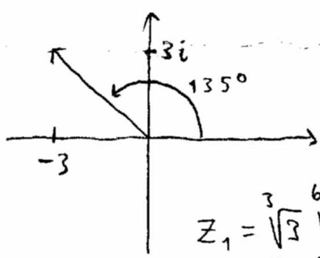
$$\begin{array}{c|c} & a_3 \\ \hline e_1 & 0 \\ e_2 & 0 \\ a_2 & -2 \end{array} \longrightarrow \vec{a}_3 = -2\vec{a}_2$$

$$0\vec{a}_1 - 2\vec{a}_2 - 1\cdot\vec{a}_3 = 0$$

$$\vec{X}_{hom_2} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

$$\vec{X}_{ált} = \begin{bmatrix} 0 \\ 13 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

4.B.1 Legyen $z = -3 + 3i$. Sorold fel $\sqrt[3]{z}$ trigonometrikus alakjait!



$$z = \underbrace{\sqrt{(-3)^2+3^2}}_{3\sqrt{2}} \cdot (\cos 135° + i\cdot\sin 135°)$$

$$\sqrt[3]{z} = \sqrt[3]{3\sqrt{2}} \cdot \left(\cos\left[\frac{135°}{3} + k\cdot\frac{360°}{3}\right] + i\cdot\sin\left[\frac{135°}{3} + k\cdot\frac{360°}{3}\right]\right),$$

$k = 0, 1, 2$

$$z_1 = \sqrt[3]{3}\,\sqrt[6]{2}\cdot(\cos 45° + i\cdot\sin 45°)$$
$$z_2 = \sqrt[3]{3}\,\sqrt[6]{2}\cdot(\cos 165° + i\cdot\sin 165°)$$
$$z_3 = \sqrt[3]{3}\cdot\sqrt[6]{2}\cdot(\cos 285° + i\cdot\sin 285°)$$

(vagy: $\pi/4 = \frac{3}{12}\pi$
$11/12\cdot\pi$
$19/12\cdot\pi$)

4.B.2 Legyen

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{K}(a,b) = aE + bI.$$

Ha $\mathcal{K}(2,3)\,(\mathcal{K}(1,2))^{-1} = \mathcal{K}(a,b)$, akkor mennyi a és b ?

① $I^2 = -E$, így $K(a,b)$ úgy viselkedik, mint $a+bi \in \mathbb{C}$, ha $a, b \in \mathbb{R}$

$$2+3i\cdot\frac{1}{1+2i} = \frac{2+3i}{1+2i}\cdot\frac{1-2i}{1-2i} = \frac{2-6i^2-i}{1^2+2^2} = \frac{8-i}{5} = \frac{8}{5} + \frac{-1}{5}i \sim K\left(\frac{8}{5}, -\frac{1}{5}\right)$$

tehát $a = \frac{8}{5}$; $b = -\frac{1}{5}$

② $$K(2,3)(K(1,2))^{-1} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\cdot\frac{1}{5}\cdot\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 & 1 \\ -1 & 8 \end{bmatrix} \longrightarrow \begin{matrix} a = 8/5 \\ b = -1/5 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1}: \quad \begin{matrix} 1 & -2 \\ 2 & 1 \end{matrix} \xrightarrow[\text{aldet}]{\text{kieg.}} \begin{matrix} 1 & 2 \\ -2 & 1 \end{matrix} \xrightarrow{\text{transzpoz.}} \begin{matrix} 1 & -2 \\ 2 & 1 \end{matrix} \xrightarrow{\pm 1} \begin{matrix} 1 & 2 \\ -2 & 1 \end{matrix} \xrightarrow{\text{inverz:}} \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1^2 + 2^2 = 5$$