A. Test 1. Econ.Anal 13.Oct.21. NEPTUN: Name: A

1. A. Compute the derivatives of the following functions!

   1. $e^x \cos(2x-1)$
   2. $e^7 \ln(2x-1)$
   3. $\frac{\ln(2x)}{\ln(x)}$

B. What is the prediction of the linear approximation of the function $f(x)$ at $x = x_0$ for the value of $f(x_0 + \Delta x)$?
$f(x) = \ln x$, $x_0 = e$, $\Delta x = 0.1$.

A. 1. $\left[e^x\cos(2x-1)\right]' = [e^x]'\cos(2x-1) + e^x[\cos(2x-1)]'$

2 $\quad = e^x\cos(2x-1) + e^x(-\sin(2x-1)\cdot 2)$

2 $\quad$ 2. $[e^7\ln(2x-1)]' = e^7\frac{1}{2x-1}\cdot 2$ (as e^7 is just a constant)

3. $\left[\frac{\ln(2x)}{\ln x}\right]' = \frac{[\ln(2x)]'\cdot\ln x - \ln(2x)\cdot[\ln x]'}{(\ln x)^2} =$

3 $\quad = \frac{\left[\frac{1}{2x}\cdot 2\right]\ln x - \ln(2x)\cdot\frac{1}{x}}{(\ln x)^2}$

B. $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x$

$f(x) = \ln x \quad , f'(x) = \frac{1}{x}$

3 $\quad f(e) = \ln e = 1 \; , \; f'(e) = \frac{1}{e}$

$f(e+0.1) \approx 1 + \frac{1}{e}\cdot 0.1$

A

2. A. Study the monotonicity, convexity and local extremal values of the following function!
$f(x) = 2x^3 - 3x^2$.
Draw its graph!

B. Study the monotonicity of the following sequence!
$\frac{3n+4}{5n+6}$.

A. $f = 2x^3 - 3x^2 = x^2(2x-3)$

(2) $f' = 6x^2 - 6x = x(6x-6)$

$f'' = 12x - 6$

MIN/MAX:

$f' = 0 = x(6x-6)$

(2) $x_1 = 0$ $\qquad$ $x_2 = 1$

$f''(0) = 12\cdot 0 - 6 = -6$ $\qquad$ $f''(1) = 12\cdot 1 - 6 = 6$

$-6 < 0$ $\qquad$ $6 > 0$

MAX $\qquad$ MIN

$x<0$	$x=0$	$0<x<1$	$x=1$	$x>1$
$\nearrow$	MAX	$\searrow$	MIN	$\nearrow$

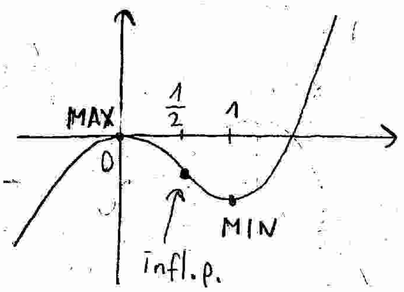
convexity

(2) inflection point: $f'' = 0$

$12x - 6 = 0$ $\qquad$ $x_{inf} = \frac{1}{2}$

$x<\frac{1}{2}$	$x=\frac{1}{2}$	$x>\frac{1}{2}$
$\frown$	INFL.	$\smile$
concave		convex

(1)



B. $a_{n+1} - a_n = \frac{3(n+1)+4}{5(n+1)+6} - \frac{3n+4}{5n+6} = \frac{(3n+7)(5n+6) - (3n+4)(5n+11)}{(5n+11)(5n+6)}$

(3) $= \frac{-2}{(5n+11)(5n+6)} < 0 \Rightarrow$ decreasing sequence

2

A

3. A. Compute the limit of the following sequence! $a_n = (1 + \frac{4}{3n})^{3n-7}$.

B. Let $\phi(x) = 3x - 9$, $x_0 = 13$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\cancel{\phi^n(1) =} x_n$? $\varphi^n(13) = x_n$?

1. Find the fixed point x_f of ϕ !
2. Introduce $\Delta x = x - x_f$ and $\tilde{\phi}(\Delta x) = \phi(x_f + \Delta x) - x_f$. Calculate $\tilde{\phi}$ and $\tilde{\phi}^n$!
3. Compute x_n !

A. $\lim_{n\to\infty} \left(1+\frac{4}{3n}\right)^{3n-7} = \lim \left[\left(1+\frac{4/3}{n}\right)^n\right]^3 \cdot \left(1+\frac{4/3}{n}\right)^{-7}$

3 $= \left[e^{4/3}\right]^3 \cdot 1^{-7} = e^4$

B. 1. fixed point: $\varphi(x_f) = x_f$

1 $3x_f - 9 = x_f \longrightarrow x_f = \frac{9}{2}$

2 $\tilde{\varphi}(\Delta x) = 3 \cdot \Delta x$

2 $\tilde{\varphi}(\Delta x) = \left[3 \cdot \left(\frac{9}{2} + \Delta x\right) - 9\right] - \frac{9}{2} = 3\Delta x$

$\tilde{\varphi}^n(\Delta x) = 3^n \Delta x$

3 $\Delta x_0 = 13 - \frac{9}{2}$

2 $\Delta x_n = 3^n\left(13 - \frac{9}{2}\right)$

$x_n = \Delta x_n + x_f = 3^n\left(13 - \frac{9}{2}\right) + \frac{9}{2}$

φ^{-1}: $\varphi(x) = 3x - 9$

$y = 3x - 9$

2 $x = \frac{y+9}{3} = \frac{1}{3}y + 3$

$\varphi^{-1}(y) = \frac{1}{3}y + 3$

$\varphi^{-1}(x) = \frac{1}{3}x + 3$

A

4. A. Let $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ x+2y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$, $\psi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x+4y \\ +y \end{pmatrix} = B\begin{pmatrix} x \\ y \end{pmatrix}$. Calculate A and B ! Let $\phi\left(\psi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) = C\begin{pmatrix} x \\ y \end{pmatrix}$. Compute C !

B. Let $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2y \\ 7x+y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$. Calculate the A^{-1} matrix of the inverse ϕ^{-1} mapping!

1. Calculate $det(A)$! Does A^{-1} exist? Why?
2. Write down the matrix equation that defines A^{-1} !
3. Write down and solve the corresponding linear system of scalar equations!
4. Use A^{-1} to find the solution of the system of equations

$$2y = 12$$
$$7x + 1y = 13.$$

A. $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ $\quad B = \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}$

3 $\quad C = A \cdot B = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0\cdot 2 + (-1)\cdot 0 & 0\cdot 4 + (-1)\cdot 1 \\ 1\cdot 2 + 2\cdot 0 & 1\cdot 4 + 2\cdot 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & -1 \\ 2 & 6 \end{pmatrix}$

B. 1 $\quad A = \begin{pmatrix} 0 & 2 \\ 7 & 1 \end{pmatrix}$ $\quad det\, A = \begin{vmatrix} 0 & 2 \\ 7 & 1 \end{vmatrix} = 0\cdot 1 - 2\cdot 7 = 14$

1 $\quad$ as $|A| = 14 \neq 0$, A^{-1} exists.

2 $\quad A A^{-1} = E$

1 $\quad \begin{pmatrix} 0 & 2 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3

$0x + 2y = 1$ $\quad\quad 0u + 2v = 0$

$7x + 1y = 0$ $\quad\quad 7u + 1v = 1$

$y = \frac{1}{2}$ $\quad x = -\frac{1}{14}$ $\quad\quad v = 0$ $\quad u = \frac{1}{7}$

3 $\quad A^{-1} = \begin{pmatrix} -\frac{1}{14} & 1/7 \\ 1/2 & 0 \end{pmatrix}$

4 $\quad \begin{matrix} 2y = 12 \\ 7x + 1y = 13 \end{matrix} \Rightarrow \begin{pmatrix} 0 & 2 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 7 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ 13 \end{pmatrix}$

2 $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/14 & 1/7 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} 12 \\ 13 \end{pmatrix} = \begin{pmatrix} -1/14 \cdot 12 + 1/7 \cdot 13 \\ 1/2 \cdot 12 + 0 \cdot 13 \end{pmatrix}$

B. Test 1. Econ.Anal 13 Oct.21. NEPTUN: Name: B

1. A. Compute the derivatives of the following functions!

   1. $\sqrt[3]{\sin(3x)}$
   2. $\sqrt[3]{x}\, tg(2x-1)$
   3. $\frac{x^7}{sin(3x)}$

B. Let $f(x) = -x^2 - 2x$. Compute $\frac{f(5+\Delta x)-f(5)}{\Delta x}$! What is the limit of this fraction as $\Delta x \to 0$? What is $f'(5)$?

2 A. 1 $\left[\sqrt[3]{\sin(3x)}\right]' = \left[(\sin(3x))^{1/3}\right]' =$

$= \frac{1}{3}(\sin(3x))^{-3/3} \cdot (\sin(3x))' = \frac{1}{3}(\sin(3x))^{-2/3} \cdot \cos(3x) \cdot 3$

2 2 $\left[x^{1/3} \cdot tg(2x-1)\right]' = (x^{1/3})' \cdot tg(2x-1) + x^{1/3} \cdot [tg(2x-1)]' =$

$= \frac{1}{3} x^{-2/3} \cdot tg(2x-1) + x^{1/3} \cdot \frac{1}{\cos^2(2x-1)} \cdot 2$

2 3 $\left[\frac{x^7}{\sin(3x)}\right]' = \frac{(x^7)' \sin(3x) - x^7 \cdot (\sin(3x))'}{(\sin(3x))^2}$

$= \frac{7x^6 \sin(3x) - x^7 \cos(3x) \cdot 3}{(\sin(3x))^2}$

4 B. $\frac{\Delta f}{\Delta x} = \frac{[-(5+\Delta x)^2 - 2\cdot(5+\Delta x)] - [-5^2 - 2\cdot 5]}{\Delta x} = \frac{(-5\cdot 2 - 2)\Delta x - \Delta x^2}{\Delta x}$

$= -5\cdot 2 - 2 + \Delta x$

$\lim_{\Delta x \to 0} -5\cdot 2 - 2 + \Delta x = -5\cdot 2 - 2 = -12$, so $f'(5) = -12$

B

2. A. Study the monotonicity, convexity and local extremal values of the following function!

$f(x) = x^2 - x^4$.

Draw its graph!

B. Study the boundedness and convergence of the following sequence!

$\frac{3n+4}{5n+6}$.

A. $f(x) = x^2 - x^4 = x^2(1-x^2)$

$f'(x) = 2x - 4x^3 = 2x(1-2x^2)$

① $f''(x) = 2 - 12x^2$

$f' = 0 = 2x(1-2x^2)$

$x_1 = 0$ $\quad x_2 = \frac{1}{\sqrt{2}}$ $\quad x_3 = -\frac{1}{\sqrt{2}}$

② $f''(0) = 2 - 12\cdot 0^2 = 2$ $\quad f''(\frac{1}{\sqrt{2}}) = 2 - 12\cdot(\frac{1}{\sqrt{2}})^2 = -4$ $\quad f''(-\frac{1}{\sqrt{2}}) = -4$

$2 > 0$ MIN $\quad -4 < 0$ MAX $\quad -4 < 0$ MAX

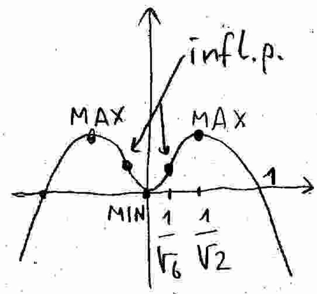
$f'' = 0 = 2 - 12x^2 \rightarrow$ inflection points

$x_1 = -\frac{1}{\sqrt{6}}$ $\quad x_2 = \frac{1}{\sqrt{6}}$

④

$x < -\frac{1}{\sqrt{2}}$	$x = -\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}} < x < 0$	$x = 0$	$0 < x < \frac{1}{\sqrt{2}}$	$x = \frac{1}{\sqrt{2}}$	$x > \frac{1}{\sqrt{2}}$
↗	MAX	↘	MIN	↗	MAX	↘

$x < -\frac{1}{\sqrt{6}}$	$x = -\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$	$x = \frac{1}{\sqrt{6}}$	$x > \frac{1}{\sqrt{6}}$
∩	INFL.P.	∪	INFL.P.	∩



Ⓑ $\lim_{n\to\infty} \frac{3n+4}{5n+6} = \lim_{n\to\infty} \frac{3+4/n}{5+6/n} = \frac{3}{5}$. Since the sequence is convergent, it is necessarily bounded.

③

B

3.A. Compute the limit of the following sequence! $a_n = \frac{2^{2n-88}}{3^{n+77}5^n}$.

B. Let $\phi(x) = 4x + 16$, $x_0 = 13$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\cancel{\phi^n(1) = x_n}$? $\varphi^n(13) = x_n$?

1. Find the fixed point x_f of ϕ !
2. Introduce $\Delta x = x - x_f$ and $\tilde{\phi}(\Delta x) = \phi(x_f + \Delta x) - x_f$. Calculate $\tilde{\phi}$ and $\tilde{\phi}^n$!
3. Compute x_n !

A. $$\lim_{n\to\infty} \frac{2^{2n-88}}{3^{n+77}\cdot 5^n} = \lim_{n\to\infty} \left(\frac{2^2}{3\cdot 5}\right)^n \cdot \frac{2^{-88}}{3^{77}} =$$

$$= \lim_{n\to\infty} \left(\frac{4}{15}\right)^n \cdot \left(\frac{2^{-88}}{3^{77}}\right) = 0$$ 3

B. 1 fixpoint: $\varphi(x_f) = x_f$, $4x_f + 16 = x_f \longrightarrow x_f = -\frac{16}{3}$ 2

2 $\Delta x = x - (-\frac{16}{3})$ $\tilde{\varphi}(\Delta x) = 4 \cdot \Delta x$

(or $\tilde{\varphi}(\Delta x) = \left[4 \cdot (-\frac{16}{3} + \Delta x) + 16\right] - (-\frac{16}{3}) = 4\Delta x$) 4

$\tilde{\varphi}^n(\Delta x) = 4^n \Delta x$

3 $x_0 = 13 \longrightarrow \Delta x_0 = 13 - (-\frac{16}{3})$

$\Delta x_n = 4^n\left(13 - (-\frac{16}{3})\right)$

$x_n = 4^n\left(13 - (-\frac{16}{3})\right) + \left(-\frac{16}{3}\right)$

[Remark: if $x_n = \varphi^n(1)$,
then $x_n = 4^n\left(1 - (-\frac{16}{3})\right) + \left(-\frac{16}{3}\right)$] 3

φ^{-1}: $\varphi(x) = 4x + 16$ 1

$y = 4x + 16$

$x = \frac{y - 16}{4} = \frac{1}{4}y - 4$

$\varphi^{-1}(y) = \frac{1}{4}y - 4$ $\varphi^{-1}(x) = \frac{1}{4}x - 4$

3.

B

4. A. Let $\bar{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$. Compute $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$!

B. Let T be a 2×2 matrix formed by the transition probabilities of a two state (labeled by 1 and 2) stochastic system, where
$T(1 \leftarrow 1) = T_{11} = 0.5$, $T(2 \leftarrow 1) = T_{21} = 0.5$, $T(1 \leftarrow 2) = T_{12} = 0.5$, $T(2 \leftarrow 2) = T_{22} = 0.5$.

1. Find an eigenvector $\bar{v}_1$ corresponding to the eigenvalue $\lambda_1 = 1$! (This is the equilibrium state.)
2. Find the eigenvalue λ_2 of T corresponding to the eigenvector $\bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$!
3. Calculate α and β in $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$!
4. Calculate $T(\alpha\bar{v}_1 + \beta\bar{v}_2)$, $T^2(\alpha\bar{v}_1 + \beta\bar{v}_2)$, etc.

A. $\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \alpha\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\alpha + 3\beta \\ 4\beta \end{pmatrix}$

$\left.\begin{matrix} 2\alpha + 3\beta = 12 \\ 4\beta = 8 \end{matrix}\right\} \Rightarrow \beta = 2 \qquad \alpha = 3$

2

B. $T = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$

2 1 $\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ $\left.\begin{matrix} 0.5x + 0.5y = 1 \cdot x \\ 0.5x + 0.5y = 1 \cdot y \end{matrix}\right\} x = y, \ \bar{V}_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

2 2 $\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 0.5 \cdot 1 + 0.5 \cdot (-1) \\ 0.5 \cdot 1 + 0.5 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow \lambda_2 = 0$

2 3 $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{matrix} 1 = \frac{\alpha}{2} + \beta \\ 0 = \frac{\alpha}{2} - \beta \end{matrix} \Rightarrow \alpha = 1, \beta = \frac{1}{2}$

2 4 $T\left(1 \cdot \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \underset{\lambda_1}{\uparrow 1} \cdot \underset{\alpha}{\uparrow 1} \cdot \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \underset{\lambda_2}{\uparrow 0} \cdot \underset{\beta}{\uparrow \frac{1}{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$T^2\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = 1^2 \cdot 1 \cdot \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + 0^2 \cdot \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

⋮