Problems for Test 2.

1 Calculus

1. Compute the $f'_x, f'_y, f'_{xx}, f'_{xy}, f'_{yx}, f'_{yy}$ partial derivatives of the following functions!

$$x, \tag{1}$$
$$y, \tag{2}$$
$$xy, \tag{3}$$
$$x^2y^5, \tag{4}$$
$$\sin(x+y^2), \tag{5}$$
$$e^{\cos(x)+xy}, \tag{6}$$
$$x/y. \tag{7}$$

2. Find the extremal values and determine their types of the following functions!

$$5x+6y, \tag{8}$$
$$x^2+y^2, \tag{9}$$
$$xy, \tag{10}$$
$$x^2-y^2, \tag{11}$$
$$x^2-xy+4y^2-2x+4y, \tag{12}$$
$$x^2-4xy+y^2-2x+4y. \tag{13}$$

3. Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!

$$5, \tag{14}$$
$$x, \tag{15}$$
$$x^2-2, \tag{16}$$
$$x^2-y, \tag{17}$$
$$\sqrt[3]{2x^7}+\sqrt[3]{(2x)^7}+\frac{7}{x^7}, \tag{18}$$
$$e^x+\sin(x), \tag{19}$$
$$e^{3x}+\sin(3x), \tag{20}$$
$$\frac{3}{1+9x^2}. \tag{21}$$

4. Solve the following differential equations!

$$y'=1, \quad y(2)=3, \tag{22}$$
$$y'=x+1, \quad y(2)=3, \tag{23}$$
$$y'=\sin x, \quad y(2)=3, \tag{24}$$
$$y'=1/x, \quad y(2)=3, \tag{25}$$
$$y'=1/(x+1), \quad y(2)=3, \tag{26}$$
$$\tag{27}$$

5. Solve the following differential equations!

$$y'=y, \quad y(2)=3, \tag{28}$$
$$y'=-2y, \quad y(2)=3, \tag{29}$$
$$y'=0, \quad y(2)=3, \tag{30}$$
$$y'=y+5, \quad y(2)=3, \tag{31}$$
$$y'=2y+5, \quad y(2)=3, \tag{32}$$
$$\tag{33}$$

6. Plot the solutions of the following differential equations! Find their equilibrium positions and determine their stability!

$$y' = 0, \tag{34}$$
$$y' = y, \tag{35}$$
$$y' = -2y, \tag{36}$$
$$y' = -2y + 3, \tag{37}$$
$$y' = y^2 - 1, \tag{38}$$
$$y' = -y^2 + 1, \tag{39}$$
$$y' = y^2 - y, \tag{40}$$
$$y' = y^3 - y, \tag{41}$$
$$y' = y^3 - y^2. \tag{42}$$

7. Compute the following definite integrals!

$$\int_1^2 x + 1 \, dx, \tag{43}$$
$$\int_3^4 x^2 - e^x \, dx, \tag{44}$$
$$\int_5^6 1/(x+5) \, dx, \tag{45}$$
$$\int_7^8 1/(2x+5) \, dx, \tag{46}$$
$$\int_0^\pi \sin x \, dx, \tag{47}$$
$$\int_0^\pi \cos(2x) \, dx, \tag{48}$$
$$\tag{49}$$

8. Compute the following improper integrals!

$$\int_1^\infty 1/(x+1) \, dx, \tag{50}$$
$$\int_0^\infty e^{-3x} \, dx, \tag{51}$$
$$\int_1^\infty 1/(x+1)^2 \, dx. \tag{52}$$
$$\tag{53}$$

2 Probability theory

1. 3 cards are drawn from a standard deck of 52 cards. How many different 3-card hands can possibly be drawn? `http://www.statlect.com/combinations_exercise_set_1.htm` (ex.1.1)

2. What is the chance of winning the lottery grand prize (i.e. correctly guessing 5 numbers out of 90) ?

3. There are 10 black and 5 white balls in a box. Suppose that we DO NOT put back the balls after the drawings. What is the chance of drawing firstly 3 white and then 2 black balls? What is the chance of drawing 3 white and then 2 black balls if the order is irrelevant?

4. There are 10 black and 5 white balls in a box. Suppose that we DO put back the balls after the drawings. What is the chance of drawing firstly 3 white and then 2 black balls? What is the chance of drawing 3 white and then 2 black balls if the order is irrelevant?

5. Consider a sample space Ω comprising four possible outcomes $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Suppose the four possible outcomes are assigned the following probabilities:

$$P(\omega_1) = 1/10, \quad P(\omega_2) = 2/10, \quad P(\omega_3) = 3/10, \quad P(\omega_4) = 4/10.$$

Define two events: $E = \{\omega_1, \omega_2\}$, $E = \{\omega_2, \omega_3\}$ Compute $P(E|F)$, the conditional probability of E given F. Compute $P(F|E)$, too.

`http://www.statlect.com/conditional_probability_exercise_set_1.htm` (ex.1.2)

6. Suppose that we toss a dice. Six numbers (from 1 to 6) can appear face up, but we do not yet know which one of them will appear. The sample space is: $\Omega = \{1, 2, 3, 4, 5, 6\}$. Define the events E and F as follows: $E = \{1, 3, 4\}$, $F = \{3, 4, 5, 6\}$. Prove that E and F are independent.

`http://www.statlect.com/independent_events_exercise_set_1.htm` (ex.1.1)

7. A firm undertakes two projects, A and B. The probabilities of having a successful outcome are $\frac{3}{4}$ for project A and $\frac{1}{2}$ for project B. The probability that both projects will have a successful outcome is $\frac{7}{16}$. Are the two outcomes independent?

`http://www.statlect.com/independent_events_exercise_set_1.htm` (ex.1.2)

8. A class consists of 100 students, 60% of them are girls, 40% have brown hair. If "brown" is independent of "girl", how many brown haired girls are in the class?

9. There are two urns containing colored balls. The first urn contains 50 red balls and 50 blue balls. The second urn contains 30 red balls and 70 blue balls. One of the two urns is randomly chosen (both urns have probability 50% of being chosen) and then a ball is drawn at random from one of the two urns. If a red ball is drawn, what is the probability that it comes from the first urn?

`http://www.statlect.com/bayes_rule_exercise_set_1.htm` (ex.1.1)

10. An HIV test gives a positive result with probability 98% when the patient is indeed affected by HIV, while it gives a negative result with 99% probability when the patient is not affected by HIV. If a patient is drawn at random from a population in which 0,1% of individuals are affected by HIV and he is found positive, what is the probability that he is indeed affected by HIV?

`http://www.statlect.com/bayes_rule.htm` (example)

11. Roll a dice. What is the expected value of the outcome?

12. Roll a dice. What is the variance of the outcome?

13. Let X be a discrete random variable. Let its support R_X be: $R_X = \{0, 1, 2, 3\}$. Let its probability mass function be:

$$p(x) = \begin{cases} x/6 & \text{if } x \in R_X \\ 0 & \text{if } x \ni R_X. \end{cases}$$

Compute the expected value of X.

`http://www.statlect.com/expected_value_exercise_set_1.htm` (ex.1.1)

14. Let X be a discrete random variable. Let its support R_X be: $R_X = \{0, 1, 2, 3\}$. Let its probability mass function be:

$$p(x) = \begin{cases} 1/4 & \text{if } x \in R_X \\ 0 & \text{if } x \ni R_X. \end{cases}$$

Compute the variance of X.

`http://www.statlect.com/variance_exercise_set_1.htm` (ex.1.1)

15. Toss a coin two times and let X_1 (X_2) be the number of heads for the first (second) trial. Compute the probability mass functions of X_1, X_2, and $X_1 + X_2$! Check that the random variables X_1 and X_2 are independent! Compute their variances and compare $Var[X_1] + Var[X_2]$ to $Var[X_1 + X_2]$!

16. Let us suppose that 50% of the voters vote for the Huge Party, 10% vote for the Small Party and the rest is undecided. Compute the expected value and the variance of the number of votes by a single voter in the cases of the Huge and Small Parties. Do the same in the case when a polling firm asks 1000 randomly chosen voters.

What will be the typical error of measurement of the popularity of these parties?

17. a) Suppose that we toss a dice. Six numbers (from 1 to 6) can appear face up (with equal probabilities). We describe the outcome by the random variable X. Write down and plot the cumulative distribution function $F_X(x) = P(X \leq x)$!

b) Let the probability density function of the random variable X be $p(x) = 1/2$ if $x \in [1, 3]$, zero otherwise. Compute and plot the cumulative distribution function $F_X(x) = P(X \leq x)$!

c) Let the probability density function of the random variable X be $p(x) = e^{-x}$ if $x \geq 0$, zero otherwise. Compute and plot the cumulative distribution function $F_X(x) = P(X \leq x)$!

`http://statlect.com/glossary/distribution_function.htm`

`http://statlect.com/glossary/absolutely_continuous_random_variable.htm`