① $\bar{a}\bar{b} = (1,2,3)(4,-2,3) = 1\cdot 4 + 2\cdot(-2) + 3\cdot 3 = 9$

$\bar{b}\bar{c}$ not defined

$|\bar{d}| = |(1,2,3,4)| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$

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④ $\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\alpha + 3\beta \\ 4\beta \end{pmatrix} \iff \begin{matrix} 2\alpha + 3\beta = 12 \\ 4\beta = 8 \end{matrix} \Rightarrow \begin{matrix} \alpha = 3 \\ \beta = 2 \end{matrix}$

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⑤ $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 4 \end{pmatrix} \iff \begin{matrix} 2\alpha + 3\beta = x \\ 4\alpha = y \end{matrix} \Rightarrow \begin{matrix} \beta = x/2 - 3y/8 \\ \alpha = y/4 \end{matrix}$

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1/2 & -3/8 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix}$

$AB = BA = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, as the mappings $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \xrightarrow{B} \begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ are inverse to each other.

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⑥ $\bar{n}_1$ and $\bar{n}_2$ are orthonormal: $\bar{n}_1^2 = \bar{n}_2^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$,

$\bar{n}_1 \bar{n}_2 = \frac{1}{2}\cdot\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\cdot\frac{1}{2} = 0.$

$\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \alpha \bar{n}_1 + \beta \bar{n}_2 \Rightarrow \begin{pmatrix} 12 \\ 8 \end{pmatrix} \cdot \bar{n}_1 = (\alpha \bar{n}_1 + \beta \bar{n}_2)\bar{n}_1 = \alpha \cdot 1 + \beta \cdot 0 = \alpha$

$\begin{pmatrix} 12 \\ 8 \end{pmatrix} \cdot \bar{n}_2 = (\alpha n_1 + \beta \bar{n}_2)\bar{n}_2 = \alpha \cdot 0 + \beta \cdot 1 = \beta,$

so $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 12\cdot\frac{1}{2} + 8\cdot\frac{\sqrt{3}}{2} \\ 12\cdot\left(-\frac{\sqrt{3}}{2}\right) + 8\cdot\frac{1}{2} \end{pmatrix}.$

---

⑦ $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x - y \\ x + 2y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$, $\psi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 4y \\ 3x + y \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \end{pmatrix}$

$\varphi \circ \psi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \varphi\left(\psi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) = AB\begin{pmatrix} x \\ y \end{pmatrix}$, where

$AB = \begin{pmatrix} \boxed{1 \quad -1} \\ \boxed{1 \quad 2} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 4 \\ 1 \end{matrix}} \end{pmatrix} = \begin{pmatrix} 1\cdot 2 - 1\cdot 3 & 1\cdot 4 - 1\cdot 1 \\ 1\cdot 2 + 2\cdot 3 & 1\cdot 4 + 2\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 8 & 6 \end{pmatrix}.$

So $\varphi\left(\psi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) = \begin{pmatrix} -1x + 3y \\ 8x + 6y \end{pmatrix}.$

(8) $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = (8x - 9y) = (8 \quad -9)\begin{pmatrix} x \\ y \end{pmatrix}$

$$\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x+4y \\ 3x+1y \\ 5x+7y \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 1 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

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(9) $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 7x+2y \\ y \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$

(1) As $\det\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix} = \begin{vmatrix} 7 & 2 \\ 0 & 1 \end{vmatrix} = 7\cdot 1 - 2\cdot 0 = 7 \neq 0$, A^{-1} exists.

(2)
$$A A^{-1} = E$$
$$\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(3)
$$\left.\begin{array}{ll} 7x+2y=1 & 7u+2v=0 \\ 0x+1y=0 & 0x+1v=1 \end{array}\right\} \Longrightarrow \begin{array}{ll} x=1/7 & u=-2/7 \\ y=0 & v=1 \end{array}$$

$$A^{-1} = \begin{pmatrix} 1/7 & -2/7 \\ 0 & 1 \end{pmatrix}$$

(4)
$$\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/7 & -2/7 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7\cdot\frac{1}{7}+2\cdot 0 & 7\cdot\left(-\frac{2}{7}\right)+2\cdot 1 \\ 0\cdot\frac{1}{7}+1\cdot 0 & 0\cdot\left(-\frac{2}{7}\right)+1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(5)
$$A\overline{v} = \overline{d}$$
$$A^{-1}A\overline{v} = A^{-1}\overline{d}$$
$$E\overline{v} = A^{-1}\overline{d}$$
$$\overline{v} = A^{-1}\overline{d},$$

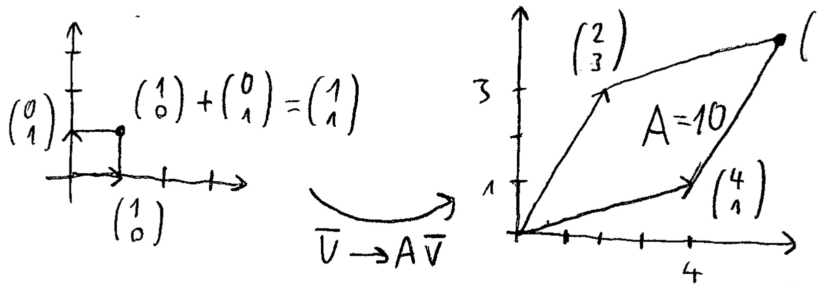
Consequently

$$\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ 13 \end{pmatrix} =$$
$$= \begin{pmatrix} 1/7 & -2/7 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ 13 \end{pmatrix},$$

so $x = \frac{1}{7}\cdot 12 - \frac{2}{7}\cdot 13$, $\quad y = 0\cdot 12 + 1\cdot 13$

(11) $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$



$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

$\det A = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 4\cdot 3 - 2\cdot 1 = 10$

10 is the area of the parallelogramm

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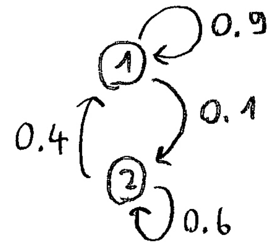
(14)



Evolution law:

$T: \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \to \begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} T_{11}p_1 + T_{12}p_2 \\ T_{21}p_1 + T_{22}p_2 \end{pmatrix}$

Eigenvector-eigemvalue eq.: $T\overline{v}_i = \lambda_i \overline{v}_i$

(1) $\lambda_1 = 1$

$\begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\cdot\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{matrix} 0.9x + 0.4y = 1\cdot x \\ 0.1x + 0.6y = 1\cdot y \end{matrix} \Rightarrow x = 4y$

Choose a nonzero $\overline{v}_1$ vector from $\left\{\begin{pmatrix} 4y \\ y \end{pmatrix}\right\}$: $\overline{v}_1 = \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix}$

(2) $\overline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} = 0.5\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix},$

so $\lambda_2 = 0.5$.

(3) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{5}{21}\cdot\begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix} + \frac{1}{21}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

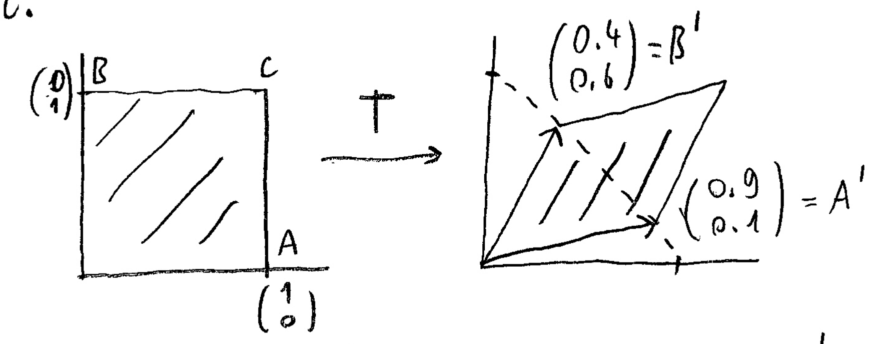
(4) $T(\alpha\overline{v}_1 + \beta\overline{v}_2) = \alpha T\overline{v}_1 + \beta T\overline{v}_2 = \lambda_1\cdot\alpha\overline{v}_1 + \lambda_2\beta\overline{v}_2$

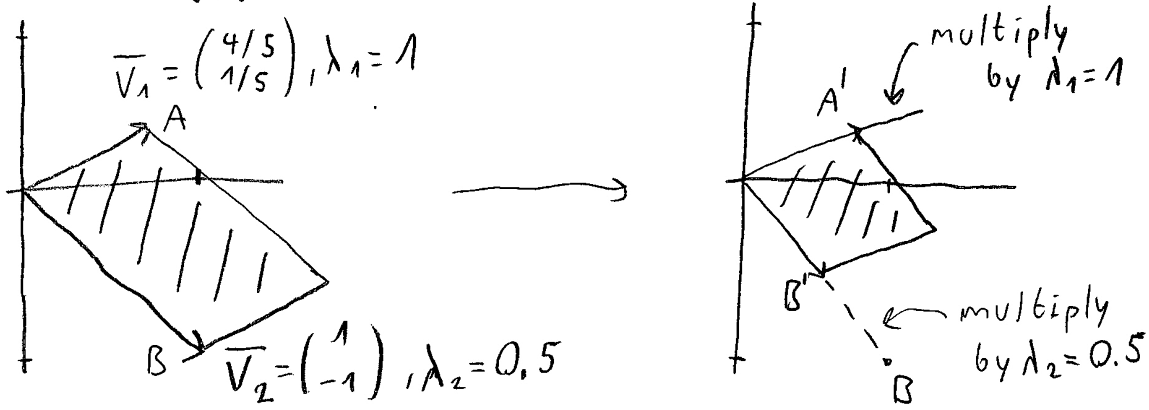
$T^2(\alpha\overline{v}_1 + \beta\overline{v}_2) = \qquad = \lambda_1^2\cdot\alpha\overline{v}_1 + \lambda_2^2\beta\overline{v}_2$

(5) $T^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T^n\left[\frac{5}{21}\begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix} + \frac{1}{21}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{5}{21}\cdot 1^n\cdot\begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix} + \frac{1}{21}0.5^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(14) cont.

(6)





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(16) $\varphi(x) = 2x+1 \qquad \varphi^{-1}:$

$$y = 2x+1$$

$$x = \frac{y-1}{2}$$

$$\varphi^{-1}(y) = \frac{y-1}{2} \quad \text{or} \quad \varphi^{-1}(x) = \frac{x-1}{2}$$

Fixpoint: $\varphi(x_f) = x_f$

$$2x_f + 1 = x_f$$

$$x_f = -1$$

dynamics around the fixedpoint x_f:

$$\Delta x = x - (-1)$$

$$\tilde{\varphi}(\Delta x) = \varphi(x_f + \Delta x) - x_f$$

$$\tilde{\varphi}(\Delta x) = \left(2 \cdot ((-1) + \Delta x) + 1\right) - (-1) = 2\Delta x$$

$$\tilde{\varphi}^n(\Delta x) = 2^n \Delta x$$

as $x = \Delta x + (-1)$

$$\varphi^n(x) = \tilde{\varphi}^n(\underbrace{x-(-1)}_{\Delta x}) + (-1)$$

$$= 2^n(x-(-1)) + (-1) = 2^n(x+1) - 1$$

If $x_0 = 1$, $\varphi^n(1) = 2^n(1+1) - 1 = 2^{n+1} - 1$.

(19) $a_n = \dfrac{2n+2}{3n+6}$.

Monotonicity: $a_{n+1} - a_n = \dfrac{2(n+1)+2}{3(n+1)+6} - \dfrac{2n+2}{3n+6} =$

$$= \frac{(2n+4)(3n+6)-(2n+2)(3n+9)}{(3n+9)(3n+6)} = \frac{6}{(3n+9)(3n+6)} > 0,$$

if $n = 0, 1, 2, \dots$, so a_n is increasing.

$$\lim_{n\to\infty} \frac{2n+2}{3n+6} = \lim_{n\to\infty} \frac{2+2/n}{3+6/n} = \frac{2}{3}.$$

Since a_n is convergent, it is necessarily bounded.

---

(20) $$\lim_{n\to\infty} \frac{2n^3+2n}{3n^4+6} = \lim_{n\to\infty} \frac{2/n + 2/n^3}{3 + 6/n^4} = \frac{0+0}{3+0} = 0$$

$$\lim_{n\to\infty} \left(1 - \frac{3}{2n}\right)^{4n+3} = \lim_{n\to\infty} \left[\left(1 + \frac{-3/2}{n}\right)^n\right]^4 \left(1 + \frac{-3/2}{n}\right)^3 =$$

$$= \left[e^{-3/2}\right]^4 \cdot 1^3 = e^{-12/2} = e^{-6}$$

$$\lim_{n\to\infty} \frac{2^{3n}}{3^n \cdot 5^{n+6}} = \lim_{n\to\infty} \left(\frac{2^3}{3\cdot 5}\right)^n \cdot \frac{1}{5^6} = \lim_{n\to\infty} \Big(\underbrace{\frac{8}{15}}_{\text{abs. val.} < 1}\Big)^n \cdot \frac{1}{5^6} = 0$$

---

(22) $$\frac{f(5+\Delta x) - f(5)}{\Delta x} = \frac{[3\cdot(5+\Delta x)^2 - 2(5+\Delta x)] - [3\cdot 5^2 - 2\cdot 5]}{\Delta x} =$$

$$= \frac{3\cdot 2\cdot 5\cdot \Delta x + 3\Delta x^2 - 2\Delta x}{\Delta x} = 3\cdot 2\cdot 5 + 3\Delta x - 2$$

$$\lim_{\Delta x\to 0} 3\cdot 2\cdot 5 + 3\Delta x - 2 = 3\cdot 2\cdot 5 - 2 = 28,$$

consequently $f'(5) = 28$. Indeed, $f' = 3\cdot 2x - 2$, $f'(5) = 3\cdot 2\cdot 5 - 2$.

---

(23). $f(x) = \sqrt{x}$, $f'(x) = (x^{1/2})' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f(9) = 3$, $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

So $f(9.1) = f(9+0.1) \approx f(9) + f'(9)\cdot 0.1 =$

$= 3 + \frac{1}{6}\cdot 0.1$

(24) $\left(\sqrt[3]{x} + \frac{1}{x^6} + \cos(3x) + \ln(2x)\right)' = \left(x^{1/3} + x^{-6} + \cos(3x) + \ln(2x)\right)'$

$= \frac{1}{3}x^{-2/3} + (-6)\cdot x^{-7} + (-\sin(3x))\cdot 3 + \frac{1}{2x}\cdot 2$

$\left(\sin\sqrt[3]{x}\right)' = \left(\sin x^{1/3}\right)' = \sin'(x^{1/3})\cdot(x^{1/3})' =$

$= \cos(x^{1/3})\cdot\frac{1}{3}x^{-2/3}$ We used $[f(g(x))]' = f'(g(x))\,g'(x)$.

$\left[x^7\cos(2x-1)\right]' = (x^7)'\cos(2x-1) + x^7\cdot(\cos(2x-1))' =$

$= 7x^6\cdot\cos(2x-1) + x^7(-\sin(2x-1)\cdot 2)$

We used $(f\cdot g)' = f'\cdot g + f\cdot g'$.

(26.) (1)

$f(x) = x^3 - x$

$f'(x) = 3x^2 - 1$

$f''(x) = 6x$

$f' = 0 = 3x^2 - 1$

$x_1 = -\frac{1}{\sqrt{3}}$ $\qquad x_2 = \frac{1}{\sqrt{3}}$

$f''\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}}\cdot 6 < 0$ $\qquad f''\left(\frac{1}{\sqrt{3}}\right) = 6\cdot\frac{1}{\sqrt{3}} > 0$

(local) MAX $\qquad$ (local) MIN

infl. point: $f'' = 6x = 0$, $x_{inf} = 0$.

convexity: $f'' = 6x > 0 \rightarrow x > 0$ convex

$f'' = 6x < 0 \rightarrow x < 0$ concave

$x < -\frac{1}{\sqrt{3}}$	$x = -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$x = \frac{1}{\sqrt{3}}$	$x > \frac{1}{\sqrt{3}}$
$\nearrow$	MAX	$\searrow$	MIN	$\nearrow$

$x < 0$	$x_{inf} = 0$	$0 < x$
$\frown$	INFLECTION POINT	$\smile$