Problems for Test 1.

1. Let $\bar{a} = (1, 2, 3),\ \bar{b} = (4, -2, 3),\ \bar{c} = (1, 2),\ \bar{d} = (1, 2, 3, 4)$. Compute $\bar{a}\bar{b},\ \bar{b}\bar{c},\ |\bar{a}|,\ |\bar{d}|$!

2$*$. The price of $(apple, orange) = (2, 3) = \overline{price}$. What is the total price of $\overline{quantity} = (4, 5)$ of apples and oranges? Express the result as a scalar product of the vectors $\overline{price}$ and $\overline{quantity}$!

3*. Let $\bar{a} = (2, 0),\ \bar{b} = (3, 4)$. Draw a figure! Convince yourself that $\bar{a}\bar{b} = |\bar{a}||\bar{b}|\cos\alpha$, where α is the angle between the vectors $\bar{a}$ and $\bar{b}$.

4. Let $\bar{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$. Compute $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$!

5. Let $\bar{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$. Compute the matrix A in the equation $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$! Compute B if $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$! What is AB and BA ?

6. Let $\bar{n}_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$, $\bar{n}_2 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix}$. Do $\bar{n}_1$ and $\bar{n}_2$ form and othonormed basis? Why? Let $\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \alpha\bar{n}_1 + \beta\bar{n}_2$. Compute $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$!

7. Let $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x - y \\ x + 2y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$, $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 4y \\ 3x + y \end{pmatrix} = B\begin{pmatrix} x \\ y \end{pmatrix}$. Calculate A and B ! Let $\phi\left(\psi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) = C\begin{pmatrix} x \\ y \end{pmatrix}$. Compute C !

8. Let $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 8x - 9y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$, $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 4y \\ 3x + y \\ 5x + 7y \end{pmatrix} = B\begin{pmatrix} x \\ y \end{pmatrix}$. Calculate A and B !

9. Let $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 7x + 2y \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$. Calculate the A^{-1} matrix of the inverse ϕ^{-1} mapping!

- Calculate $det(A)$! Does A^{-1} exist? Why?
- Write down the matrix equation that defines A^{-1} !
- Write down and solve the corresponding linear system of scalar equations!
- Check your result!
- Use A^{-1} to find the solution of the system of equations

$$\begin{aligned} 7x + 2y &= 12 \\ 0x + 1y &= 13. \end{aligned}$$

10. Repeat the previous exercise for the mapping $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 7x + 2y \\ 3x + y \end{pmatrix}$!

11. Let $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 4x + 2y \\ 1x + 3y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.

- Calculate $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$!
- Draw the image $\phi(S)$ of the unit square $S = \left\{\begin{pmatrix} x \\ y \end{pmatrix};\ x, y \in [0, 1]\right\}$!

- Calculate $det(A)$!

12.* Repeat the previous exercise for the mapping $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 4x+2y \\ 3y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$! What is the relation between the area of $\phi(S)$ and $det(A)$? Answer these questions for $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x+4y \\ 3x \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$!

13.* Draw $\phi_i(S)$ for the matrices $A_1 = \begin{pmatrix} 2 & 5 \\ 2 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 2-2\cdot 5 & 5 \\ 2-2\cdot 1 & 1 \end{pmatrix}$! Explain why the areas of $\phi_i(S)$ are equal!

14. Let T be a 2×2 matrix formed by the transition probabilities of a two state (labeled by 1 and 2) stochastic system, where $T(1\leftarrow 1) = T_{11} = 0.9$, $T(2\leftarrow 1) = T_{21} = 0.1$, $T(1\leftarrow 2) = T_{12} = 0.4$, $T(2\leftarrow 2) = T_{22} = 0.6$.

- Find an eigenvector $\bar{v}_1$ corresponding to the eigenvalue $\lambda_1 = 1$! (This is the equilibrium state.)
- Find the eigenvalue λ_2 of T corresponding to the eigenvector $\bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$!
- Calculate α and β in $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$!
- Calculate $T(\alpha\bar{v}_1 + \beta\bar{v}_2)$, $T^2(\alpha\bar{v}_1 + \beta\bar{v}_2)$, etc.
- Calculate $T^n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$!
- Draw the images of $S = \left\{\begin{pmatrix} x \\ y \end{pmatrix};\ x,y\in[0,1]\right\}$ and
  $S' = \{\alpha\bar{v}_1 + \beta\bar{v}_2;\ \alpha,\beta\in[0,1]\}$ by the mapping $\bar{v}\to T\bar{v}$!

15. Let $\phi(x) = 2x$, $x_0 = 1$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\phi^n(1) = x_n$?

16. Let $\phi(x) = 2x+1$, $x_0 = 1$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\phi^n(1) = x_n$?

- Find the fixed point x_f of ϕ !
- Introduce $\Delta x = x - x_f$ and $\tilde{\phi}(\Delta x) = \phi(x_f + \Delta x) - x_f$. Calculate $\tilde{\phi}$ and $\tilde{\phi}^n$!
- Compute x_n !

17. Repeat the previous exercise for $\phi(x) = 1.1x - 20$!

18.** Let $\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = T\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, where T is the matrix of Ex.14. Compute $\phi^n\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

19. Study the monotonicity, boundedness and convergence of the following sequences!
2^n, $(-2)^n$, 0.99^n, $(-0.99)^n$, $\frac{2n+2}{3n+6}$, $(-1)^n\frac{2n+2}{3n+6}$, $(-1)^n\frac{2}{3n+6}$,

20. Compute the limits of the following sequences!
$\frac{2n^3+2n}{3n^3+6}$, $\frac{2n^3+2n}{3n^4+6}$, $\frac{2n^4+2n}{3n^3+6}$, $\left(1-\frac{3}{2n}\right)^{4n+3}$, $\left(2-\frac{3}{2n}\right)^{4n+3}$, $\left(0.9-\frac{3}{2n}\right)^{4n+3}$, $\frac{2^{3n}}{3^n 5^{n+6}}$.

21. One receives 1%/year interest for a 1 EUR deposit. What will be the balance of the account after 100 years? What is the relation of the result with the number e? Repeat the exercise for 2%/year interest rate!

22. Let $f(x) = 3x^2 + 2x$. Compute $\frac{f(5+\Delta x) - f(5)}{\Delta x}$! What is the limit of this fraction as $\Delta x\to 0$? What is $f'(5)$?

23. What is the prediction of the linear approximation of the function $f(x)$ at $x = x_0$ for the value of $f(x_0+\Delta x)$?

- $f(x) = 1/x,\ x_0 = 3,\ \Delta x = 0.1,\ 0.01,\ 0.001.$
- $f(x) = \sqrt{x},\ x_0 = 9,\ \Delta x = 0.1,\ 0.01,\ 0.001.$

24. Compute the derivatives of the following functions!

- $\sqrt[3]{x^4} + \frac{1}{x^6} + \cos(3x) + \ln(2x)$
- $\sin\left(\sqrt[3]{x}\right)$
- $x^7 \cos(2x-1)$
- $\frac{\sin(2x)}{x^2+1}$
- $\frac{\ln(2x^2+8)}{\ln(x)}$

25.* What is the price elasticity of the demand function $q(p) = 1/p^2$.

Draw the graphs of the functions $\sqrt{x},\ 1/x^2,\ 5 \cdot 10^{2x}$ in the $x-y,\ x-\log y,\ \log x - \log y$ coordinate systems!

26. Study the monotonicity, convexity and local extremal values of the following functions!

$x^2 - 2x,\ x^3 - x,\ x^2 - x^4,\ x\ln(x),\ xe^{-3x}$.

Draw their graphs!

27. Calculate the following derivatives!

$\frac{d(x^4a^3b^2)}{dx},\ \frac{d(x^4a^3b^2)}{da},\ \frac{d(x^4a^3b^2)}{db},$

$\frac{\partial(x^4y^3z^2)}{\partial x},\ \frac{\partial(x^4y^3z^2)}{\partial y},\ \frac{\partial(x^4y^3z^2)}{\partial z},$