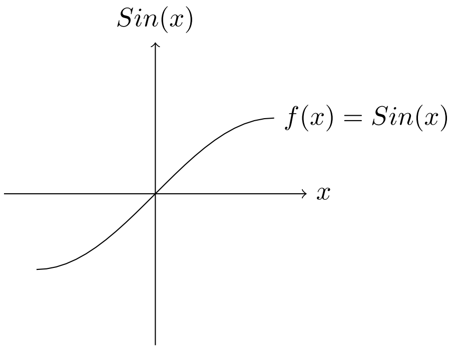
Mathematics for economic analysis, Homework set 2.

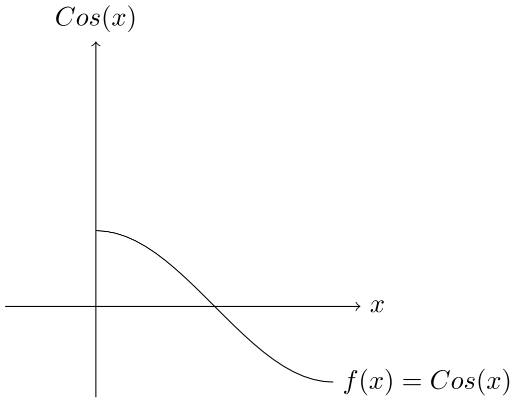
See also the HW2.ods or HW2.xlsx files. Their sheets are: ”error(Dx) of f’(x)”, ”f,f’ for 2x^2 $- x^4$”.

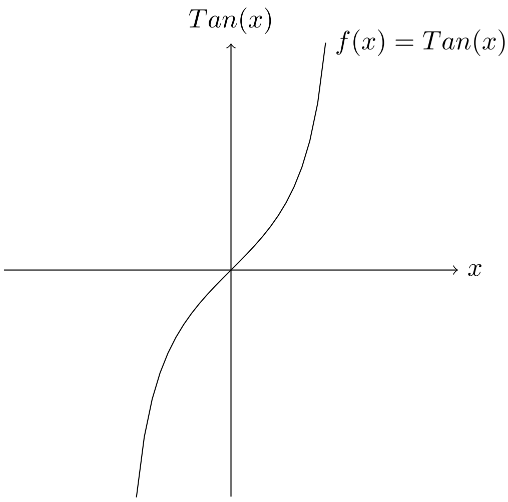
1 Inverse functions

1.0.1 Exercise: Inverse trigonometric functions

Plot the inverse of the sin, cos, tan functions! You need to restrict their domains so they become bijective (one-to-one) functions. However we still want the same ranges as that of the unrestricted trigonometric functions have. We call these restricted functions as Sin, Cos, Tan. What should be the coordinates of the endpoints of the graphs of Sin and Cos? What is the locations of the vertical asymptotes of Tan?







The inverses are called $arcsin(x)$, $arccos(x)$, $arctan(x)$. In this course you do not need to know them, regard this as a sort of drawing exercise.

1.0.2 Sample exercise: Inverse of a multicomponent multivariable function

Let

$$\vec{f}(x_1, x_2) = (x_2, x_1 + x_2).$$

(This rule generates the Fibonacci sequence if $F_0 = 1,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$.)
What is $\vec{f}^{-1}(x_1, x_2)$?
Solution:

$$\begin{aligned} y_1 = x_2, \qquad & y_2 = x_1 + x_2, \\ \text{solve for } x_{1,2}: \qquad x_2 = y_1, \qquad & x_1 = y_2 - x_2 = y_2 - y_1, \\ \text{so}: \qquad & \vec{f}^{-1}(y_1, y_2) = (y_2 - y_1, y_1), \\ \text{after renaming the arguments}: \qquad & \vec{f}^{-1}(x_1, x_2) = (x_2 - x_1, x_1). \end{aligned}$$

1.0.3 Exercise

Let

$$\vec{f}(x_1, x_2) = (x_1 + x_2, 2x_1 + 4x_2).$$

What is $\vec{f}^{-1}(x_1, x_2)$?

Remark: You will learn next semester in Linear Algebra that this exercise is the same as computing the inverse matrix

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}^{-1}.$$

1.0.4 Exercise

Let

$$\vec{f}(x_1, x_2) = (x_1 + 2x_2, 2x_1 + 4x_2).$$

Can you compute $\vec{f}^{-1}(x_1, x_2)$?

Remark: You will learn next semester in Linear Algebra that this exercise is the same as attempting to compute the inverse matrix

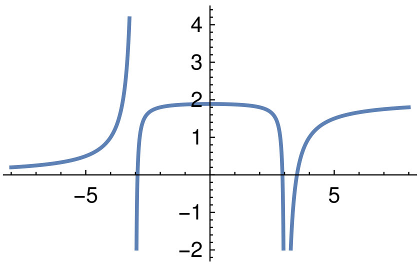
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^{-1}.$$

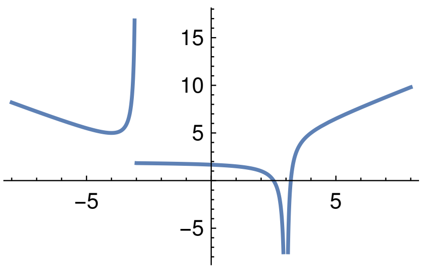
2 Continuous functions

2.1 Limits

2.1.1 Exercise

For the following two $f(x)$ functions





read off from the plots the following limits:

$$\lim_{x\to\infty} f(x), \qquad \lim_{x\to-\infty} f(x), \qquad \lim_{x\to-3} f(x), \qquad \lim_{x\to3} f(x),$$

If you know about left and right limits, solve the exercise for $x \to \pm 3 \pm 0$, too!

2.2 Fixed points

2.2.1 Sample exercise: Fibonacci sequence, stabilization of the ratio of the old rabbits in the population

Check at least the first of the next three pages.

Fixed points

(1) Fibonacci sequence: 1 1 2 3 5 8 13 21 ... $F_{n+1} = F_n + F_{n-1}$

two dimensional description: $\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}2\\1\end{bmatrix}$ $\begin{bmatrix}3\\2\end{bmatrix}$ $\begin{bmatrix}5\\3\end{bmatrix}$ $\begin{bmatrix}8\\5\end{bmatrix}$ $\begin{bmatrix}13\\8\end{bmatrix}$ $\begin{bmatrix}21\\13\end{bmatrix}$ ...

Exercise: Prove that $\lim_{n\to\infty} \frac{F_n}{F_{n+1}}$ exists!

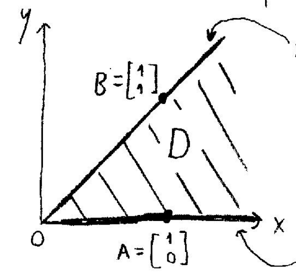
Solution (draft):

time evolution $T: \begin{bmatrix}x\\y\end{bmatrix} \longrightarrow \begin{bmatrix}x+y\\x\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix} \sim \begin{bmatrix}\text{number of all rabbits}\\ \text{number of old rabbits}\end{bmatrix}$, all ≥ 0, old ≥ 0, all $\geq$ old, $x, y \geq 0$, $x \geq y$

T continuous and maps a line through the origo to another such one.

T maps the $\left\{\begin{bmatrix}x\\y\end{bmatrix} \mid x, y \geq 0, x \geq y\right\} = D$ region into D

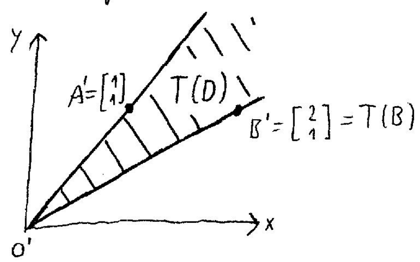


x=y: all rabbits are old

$T(A) = A'$ $\quad T(D) \subset D$

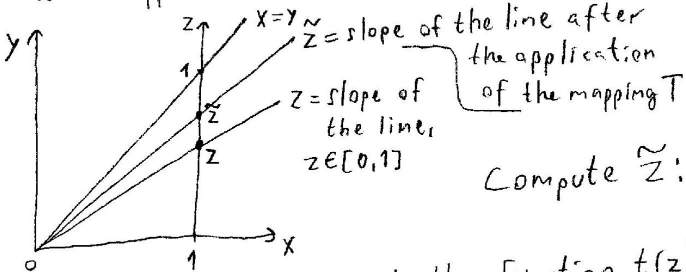
$T(B) = B'$

y=0: all rabbits are young



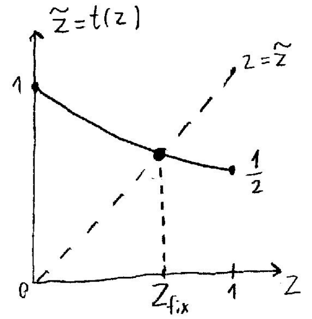
The $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear map has a single, trivial fixed point: $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$

What happens to the directions of the lines through the origo in D?



Compute $\tilde{z}$: $T: \begin{bmatrix}1\\z\end{bmatrix} \to \begin{bmatrix}1+z\\1\end{bmatrix}$, so $\tilde{z} = \frac{1}{1+z}$.

Now let us prove that the function $t(z) = \tilde{z} = \frac{1}{1+z}$, $z \in [0,1]$ has a fixed point:



(1) $t(z)$ continuous on $[0,1]$

(2) $t(0) = 1$, so at $z=0$ the plot of $t(z)$ is above the $z=\tilde{z}$ line

(3) $t(1) = \frac{1}{2}$, so at $z=1$ the plot of $t(z)$ is below the $z=\tilde{z}$ line

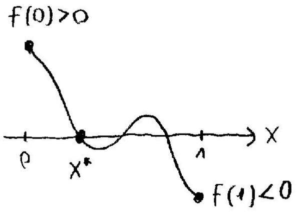
Consequently* there must be a value z_{fix} such that at that point the plot of $t(z)$ and the $z=\tilde{z}$ line intersect: $t(z_{fix}) = z_{fix}$.

Now it is quite reasonable to believe that $\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = z_{fix}$

*see next page!

What theorems ensure the existence of the fixed point z_{fix} ?

(A) Bolzano theorem: Let $f(x)$ be continous on $[0,1]$ and assume that $f(0)>0$ and $f(1)<0$. Then there exists x^* such that $f(x^*)=0$.



Remarks: It is possible that several x^* exist.
Same holds for $f(0)<0$, $f(1)>0$.
For $t(x)$, take $f(x)=t(x)-x$

(B) Contraction mapping: Assume the following:

① $f(x)$ continuous on $[0,1]$
② $f(x)\in[0,1]$ for all $x\in[0,1]$
③ there exist q such that:
$0\le q<1$, $|f(x)-f(y)|<q|x-y|$ for all $x,y\in[0,1]$

Then there exists a unique fixed point x^* such that:
$f(x^*)=x^*$ and $\lim_{n\to\infty} f^n(x_0)=x^*$ for all $x_0\in[0,1]$

Remarks: Can we apply this theorem for $t(x)$?

① $t(x)=\frac{1}{1+x}$ continuous on $[0,1]$ o.k.
② t maps $[0,1]$ to $[\frac{1}{2},1]$ which is included in $[0,1]$ o.k.
③ $t'(x)=-\frac{1}{(1+x)^2}$, so the maximal absolute value of the slope is at $x=0$ is $|t'(0)|=|-1|=1$, this unfortunately violates the third assumption on the existence of $0\le q<1$.
However this problem can be avoided by applying the theorem for $t(x)$ only on (let us say) $[0.1, 1]$. Such application would prove the existence of the limit $\lim_{n\to\infty}\frac{F_n}{F_{n+1}}$.

Let us finally remark, that the theorem has a more general form, it can be formulated for any so called "complete metric space".

The proof of the Contraction Mapping Principle (B) is not very hard:

(We assume that $q = \frac{1}{2}$ for the sake of simplicity)

x_0 $\quad f(x_0)=x_1 \quad f(x_1)=x_2 \quad x_3 \quad x_4 \; \dots$

Apply $|f(x)-f(y)| < \frac{1}{2}|x-y|$ firstly for $\begin{matrix} x=x_0 \\ y=x_1 \end{matrix}$, secondly for $\begin{matrix} x=x_1 \\ y=x_2 \end{matrix}$, etc.:

$$|x_1-x_2| = |f(x_0)-f(x_1)| < \frac{1}{2}|x_0-x_1|$$

$$|x_2-x_3| = |f(x_1)-f(x_2)| < \frac{1}{2}|x_1-x_2| < \frac{1}{2^2}|x_0-x_1|$$

$$\vdots$$

$$|x_n-x_{n+1}| < \frac{1}{2^n}|x_0-x_1|$$

How much can be $|x_n-x_{n+k}|$?

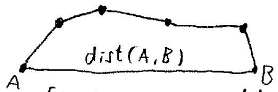
$$|x_n-x_{n+k}| \leq |x_n-x_{n+1}|+|x_{n+1}-x_{n+2}|+\dots+|x_{n+k-1}-x_{n+k}|$$

$$< \left(\frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{n+k-1}}\right)\cdot|x_0-x_1|$$

$$= \frac{1}{2^n}\left(1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{k-1}}\right)|x_0-x_1|$$

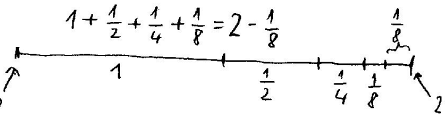
$$< \frac{1}{2^n}\cdot 2\cdot(x_0-x_1).$$

Triangle inequality:



for any reasonable distance function dist(A,B) is no more then the sum of the distances in the upper path

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=2-\frac{1}{8}$



So if $n,m\to\infty$, then $|x_n-x_m|\to 0$, consequently $\lim_{n\to\infty} x_n = x^*$ exists.

(And also unique: for assume $f(x^*)=x^*$, $f(x^{**})=x^{**}$, $x^*\neq x^{**}$, but that cannot be:

$$|f(x^*)-f(x^{**})| = |x^*-x^{**}| < \frac{1}{2}|x^*-x^{**}| \leftarrow \text{contradiction}$$)

The theorem holds for <u>complete</u> metric spaces, like $\mathbb{R}^2$, or D^2 where D^2 is the closed unit ball: $D^2=\{(x,y) \mid \sqrt{x^2+y^2}\leq 1\}$.

It is false for incomplete metric spaces like $(0,1)$:

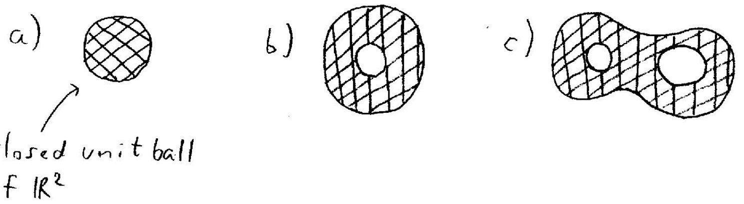
$f(x)=\frac{1}{3}x$ satisfies $|f(x)-f(y)|<\frac{1}{2}|x-y|$, but $x=\frac{1}{3}x \to x=0$, however $0\notin(0,1)$.

(C) Brouwer fixed point theorem: Any continuous mapping of $[0,1]$ into itself has a fixed point. (This is a special case relevant for us)

Generally: Any cont. mapping of the unit ball D^n into itself has a fixed point.

Here $D^n=\{(x_1,\dots,x_n) \mid \sqrt{x_1^2+\dots+x_n^2}\leq 1\}$

Exercise: Guess if it is true that any continuous map of X into itself <u>must</u> have a fixed point, where the closed region X is:



Remark: For a) the fixed point must exist by Brouwer's theorem.
b) is not very hard, you should be able to guess the correct answer
For c) even guessing is nontrivial

2.2.2 Exercise

Repeat the previous sample exercise for the Fibonacci like sequences:

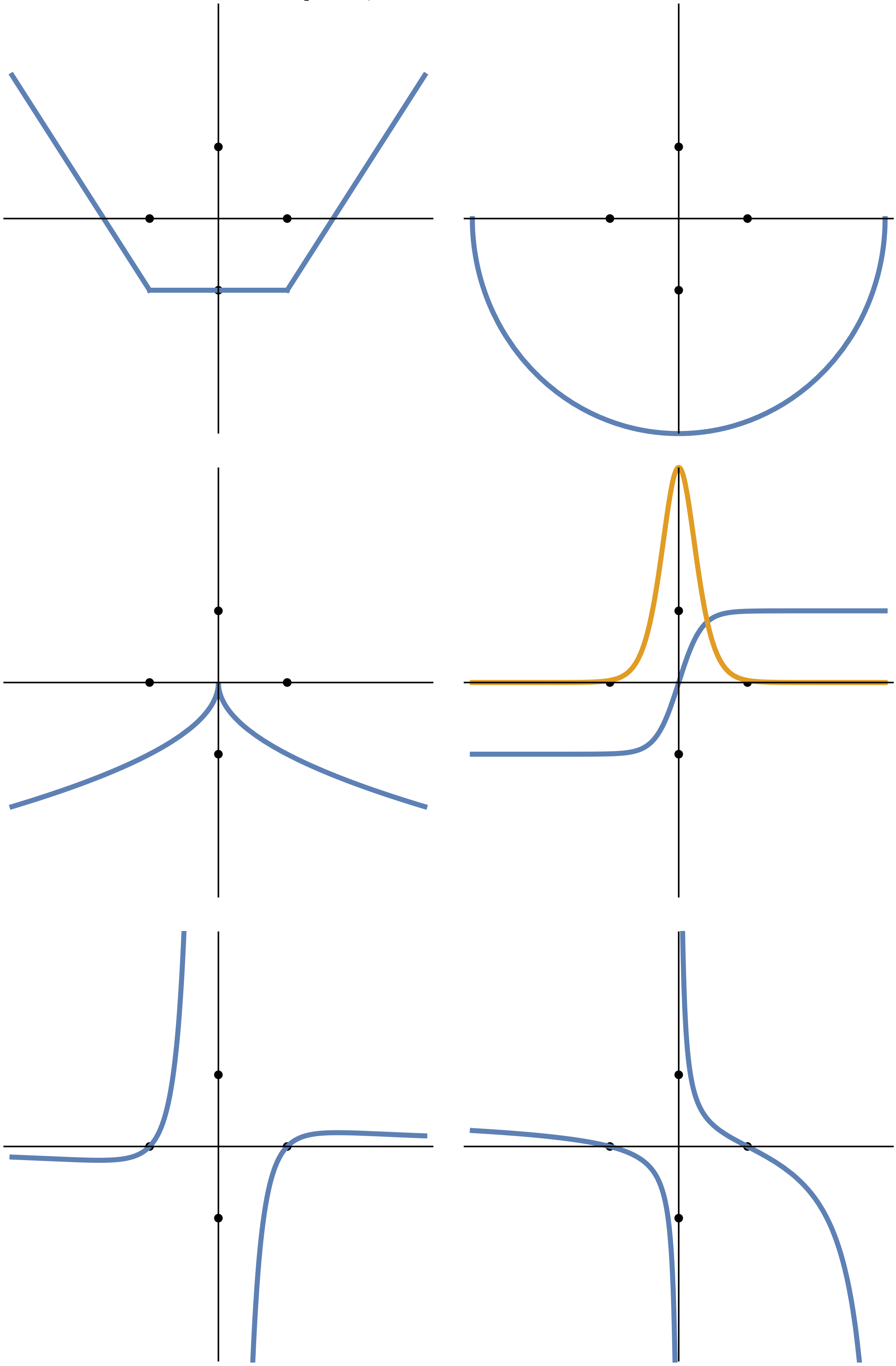
$$F_0 = 1, \; F_0 = 1,$$

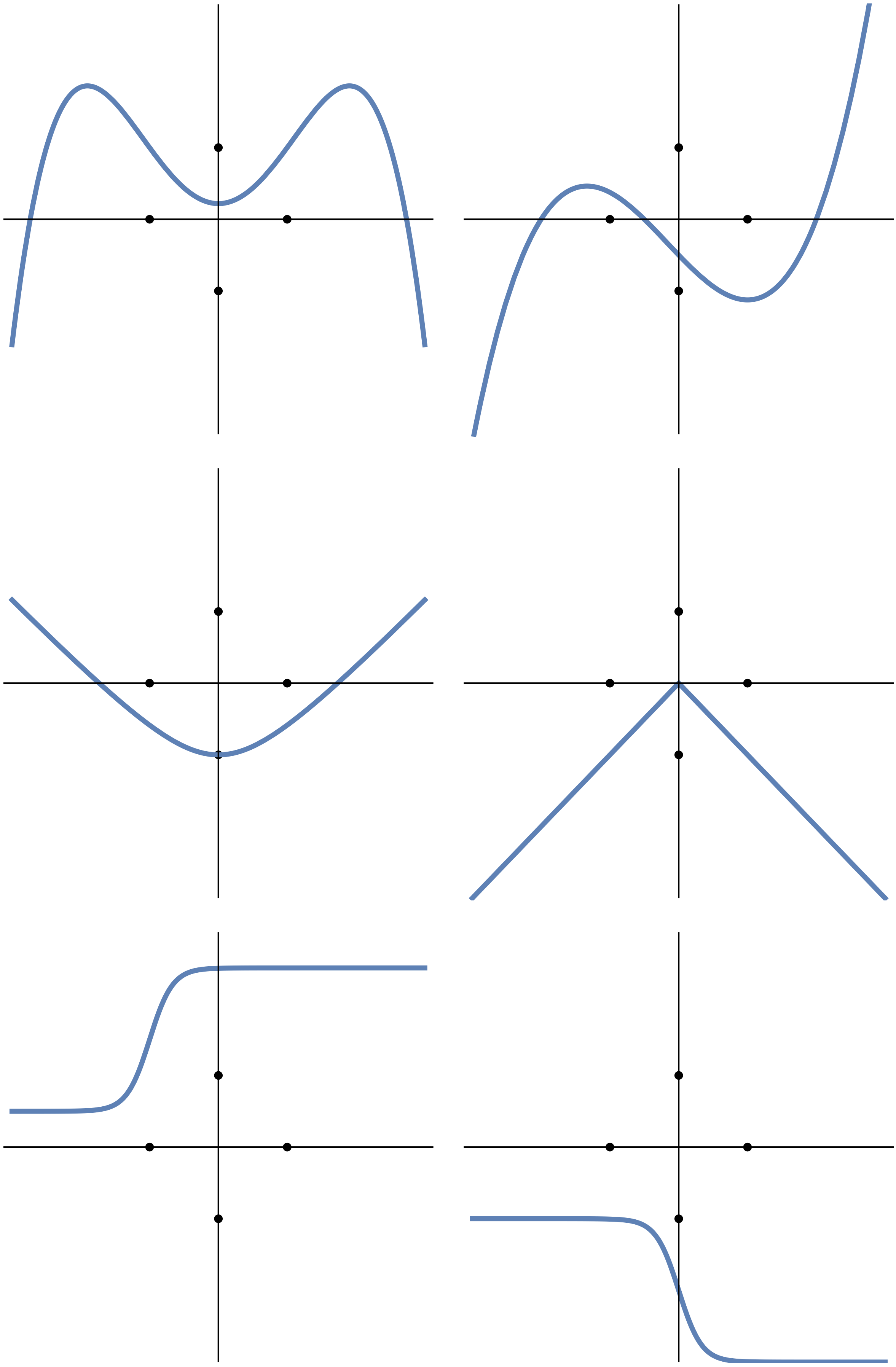
a) $\quad F_{n+1} = F_n + 2F_{n-1}, \qquad$ b) $\quad F_{n+1} = F_n + 6F_{n-1}, \qquad$ b) $\quad F_{n+1} = 2F_n + F_{n-1}.$

3 Derivation

3.0.1 Exercise

Draw (plot) the $f'(x)$ derivative function for $f(x)$ on the same plot! (The black dots on the axes are at where x or y is ± 1.)
For the 4^{th} function the derivative is plotted, too.





Remarks:

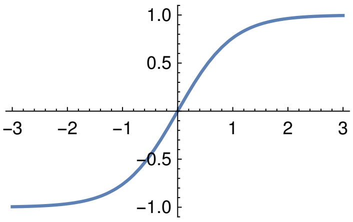
- The first of these functions is

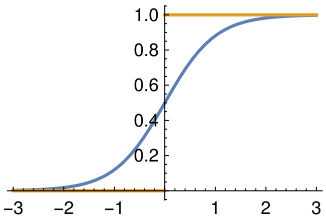
$$f(x) = \begin{cases} -\frac{3x}{2} - 2.5 & x < -1 \\ -1 & -1 < x < 1 \\ \frac{3x}{2} - 2.5 & x > 1 \end{cases}$$

  while the rest are

$$\frac{0.5(x-1)(x+1)}{x^3}, \qquad \sqrt{x^2+1} - 2,$$
$$-\sqrt{|x|}, \qquad -\sqrt{9-x^2},$$
$$\tanh(3(x+1)) + 1.5, \qquad \tanh(3x),$$
$$-\tanh(3x) - 2, \qquad 1 - 0.125(x-2.5)(x-1)(x+1)(x+2.5),$$
$$\frac{0.5(x-1)(x+1)}{(x-3)x}, \qquad -|x|, \qquad 0.25(x-2)(x+0.5)(x+2).$$

  One can has a quite happy life without knowing what is the *tanh* (tangens hyperbolicus) function, it is defined as $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$. Apart from things like hyperbolic geometry, it can be used to get a smooth approximation of the unit step (aka Heaviside theta) function, which is 0 for negative numbers, while 1 for positive ones. The plots of $tanh(x)$ and "$unit.step(x) = \theta(x),\ 0.5(\tanh(x)+1)$" are





  I deliberately scrambled the order of the functions, can you match the functions with their graphs?

- To plot a function in WolframAlpha, try for example

```
plot -sqrt(-x^2+9) from -3 to 3
```

  The derivative is plotted as

```
plot  (-sqrt(-x^2+9))' from -3 to 3
```

- The same in SAGE is

```
var('x');
plot(-sqrt(-x^2+9),x,(-3,3))
```

  and

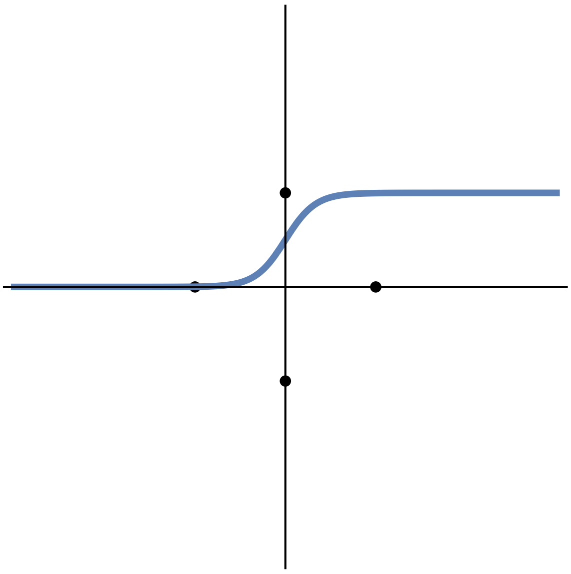
```
var('x');
plot( derivative(-sqrt(-x^2+9),x), x,(-3,3))
```

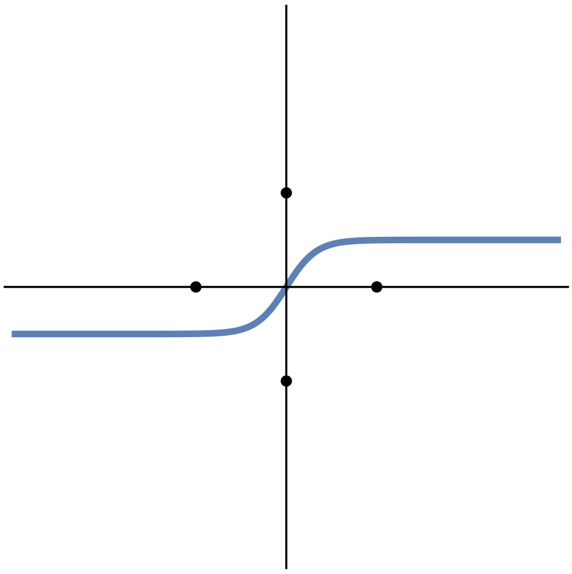
3.0.2 Exercise: Plotting $F(x) = \int f(x)\,dx$.

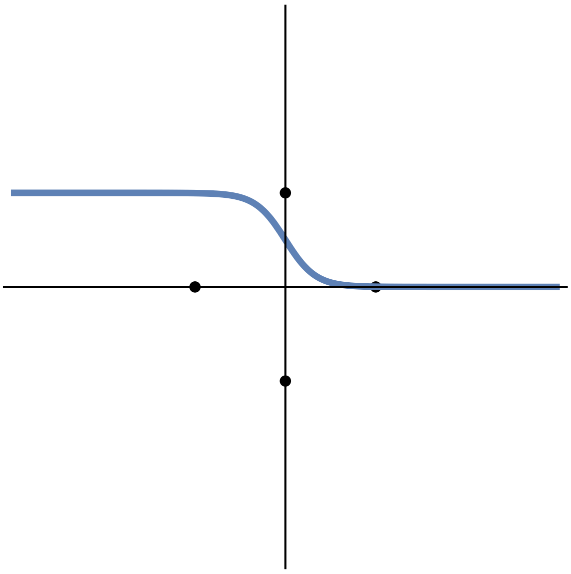
Given $f(x)$, find a function $F(x)$ whose derivative is $f(x)$. F is determined only up to a constant:

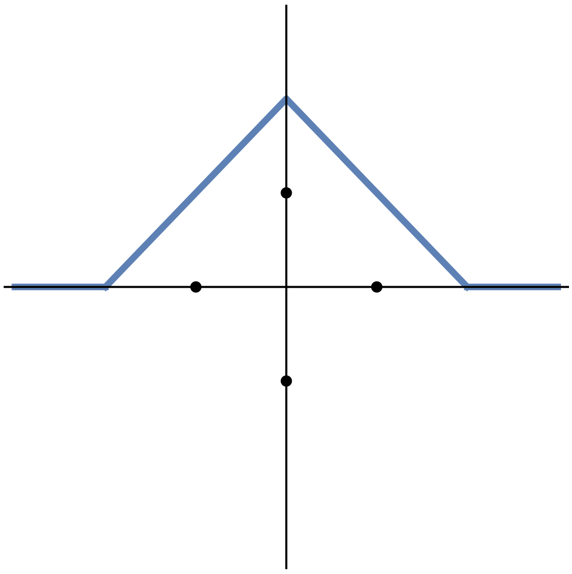
$$F'(x) = f(x) \qquad \Longrightarrow \qquad (F(x) + C)' = f(x).$$

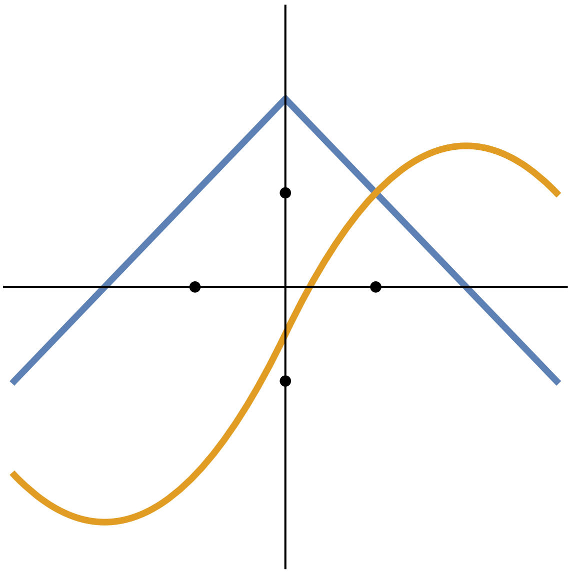
Draw (plot) one of these function for each $f(x)$! For the 5^{th} function a primitive function $F(x)$ is plotted, too.

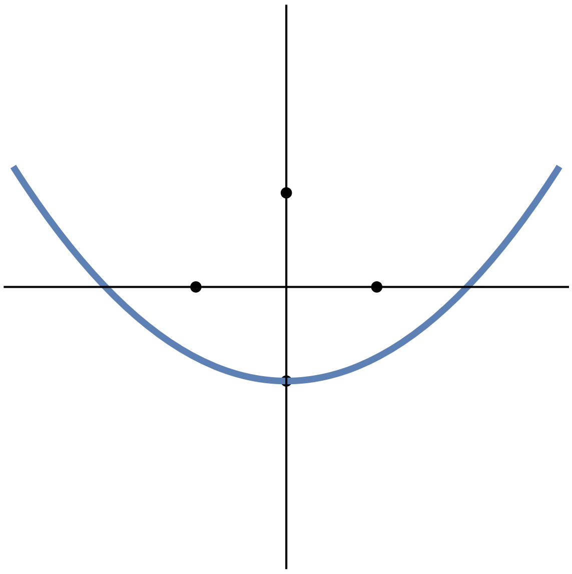


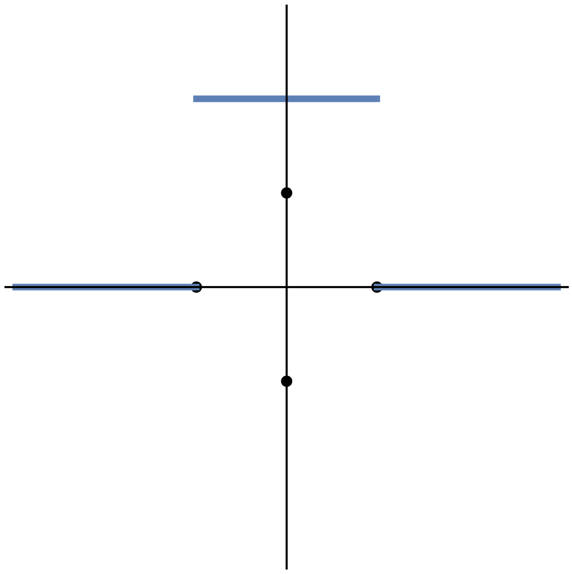
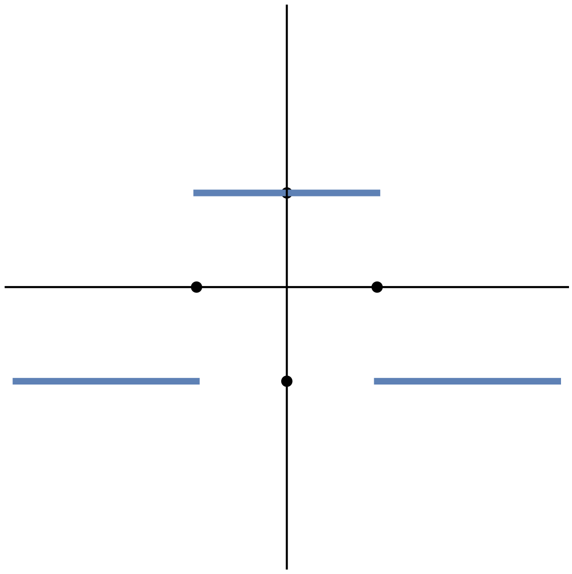
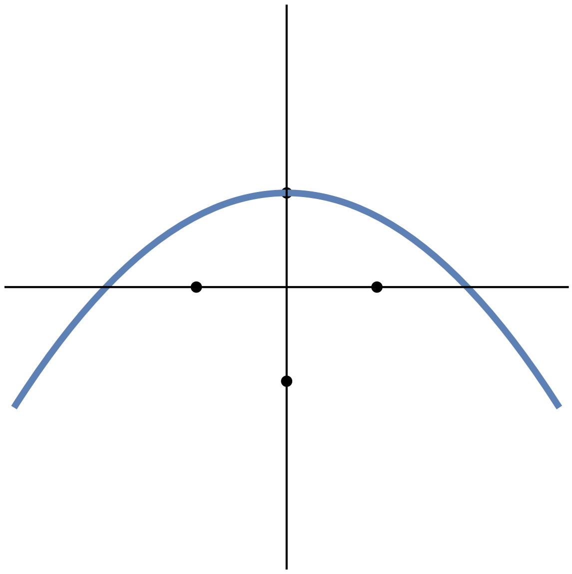
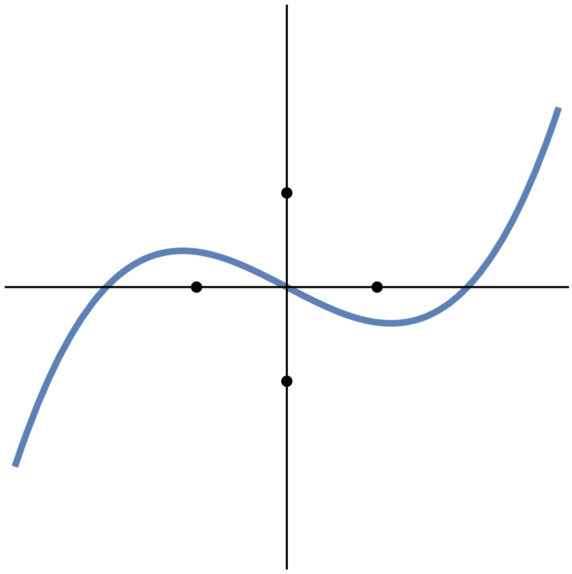
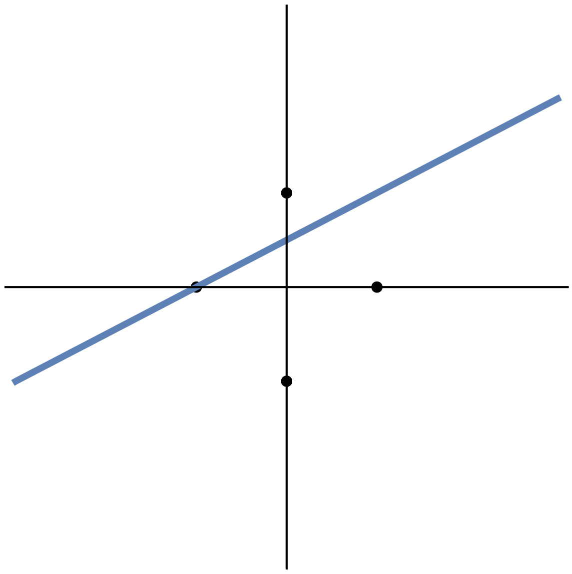
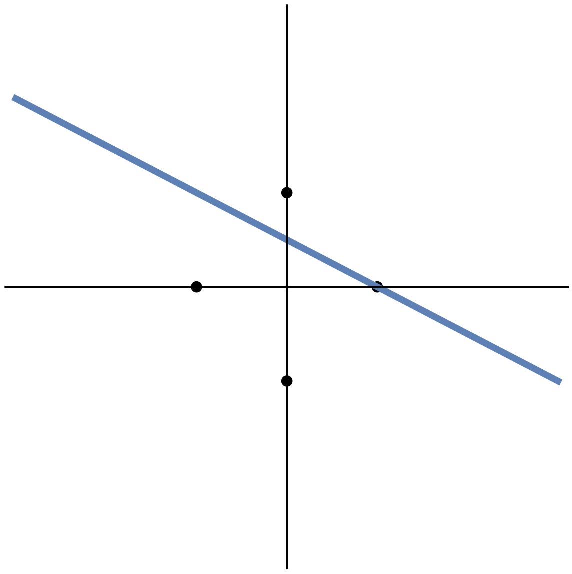










Remarks:

- The list of these functions is

$$
\begin{gathered}
-|x|, \quad 0.25(2-x)(x+2), \quad \frac{1}{2}(1-\tanh(3x)), \\
\frac{1}{2}\tanh(3x), \quad 0.5(x+1), \quad (2-|x|)\theta(2-x)\theta(x+2), \\
0.5(1-x), \quad \frac{1}{2}(\tanh(3x)+1), \quad 2\theta(1-x)\theta(x+1), \\
0.25(x-2)(x+2), \quad 0.125(x-2)x(x+2), \quad 2\theta(1-x)\theta(x+1)-1.
\end{gathered}
$$

  I deliberately scrambled the order of the functions, can you match the functions with their graphs?

- To plot a function in WolframAlpha, try for example

```
plot -sqrt(-x^2+9) from -3 to 3
```

  The integral is obtained by

```
integrate -sqrt(-x^2+9)
```

- The same in SAGE is

```
var('x');
integrate(-sqrt(-x^2+9),x)
```

3.1 Properties of $f(x)$

Searching for the roots, critical and inflection points of $f(x)$ requires the solution of equations like

$$
\begin{aligned}
\text{roots:} &\quad f(x)=0, \\
\text{critical points:} &\quad f'(x)=0, \\
\text{inflection points:} &\quad f''(x)=0.
\end{aligned}
$$

- In WolframAlpha, try for example

```
solve x^4-x^2 = 0 for x
solve (x^4-x^2)' = 0 for x
solve (x^4-x^2)'' = 0 for x
```

- The same in SAGE is

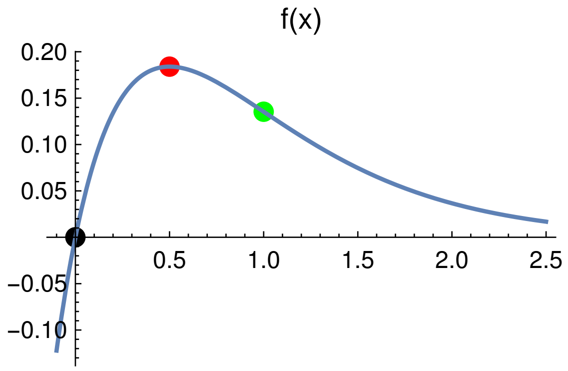
```
var('x');
solve(  x^4-x^2 == 0, x)
solve( derivative(x^4-x^2, x) == 0, x)
solve( derivative(x^4-x^2, x, 2) == 0, x)
```

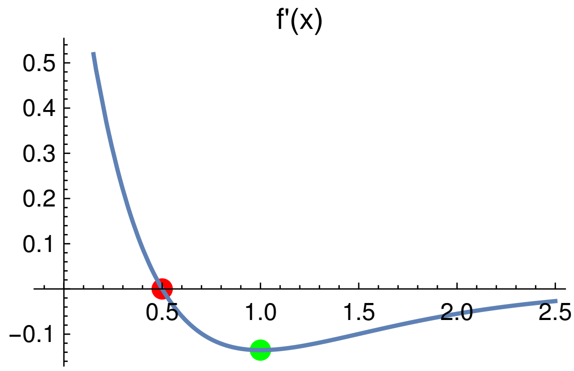
3.1.1 Sample exercise

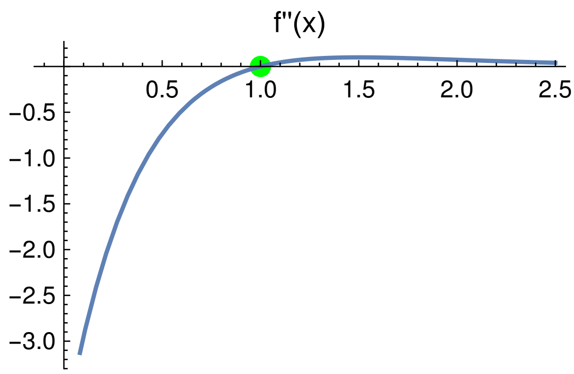
Tell all that you know about the function $f(x) = e^{-2x}x$.

Solution:

$$f(x) = e^{-2x}x$$
$$f'(x) = e^{-2x}(1-2x)$$
$$f''(x) = 4e^{-2x}(x-1)$$
$$roots : (0,0)$$
$$critical\ points : \left(\frac{1}{2}, \frac{1}{2e}\right)$$
$$inflection\ points : \left(1, \frac{1}{e^2}\right)$$







Black dots are the roots of $f(x)$, i.e. solutions of $f(x) = 0$, red ones are the possible critical points (local min. or max.) of $f(x)$, i.e solutions of $f'(x) = 0$, while the green ones denote the inflection points, i.e. solutions of $f''(x) = 0$.

Your solution should also include the following items, too:
D_f, R_f, intervals where f is increasing/decreasing and convex/concave, limits at $\pm\infty$, parity, periodicity.

These should be obvious if you take a look at the plot of $f(x)$.

3.1.2 Exercise

Repeat the previous sample exercise for $f(x) = \frac{x}{e^{2x}}$!
(First compare this function to the sample exercise's one, then decide if you really need to do some complicated analysis.)

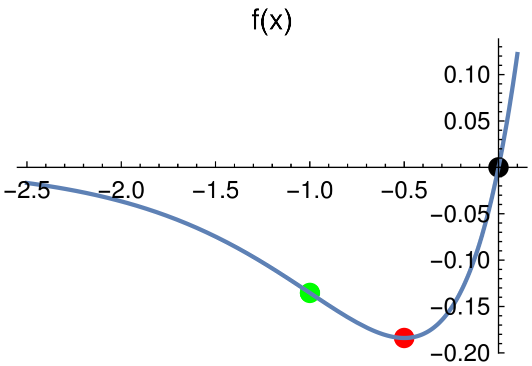
3.1.3 Exercise

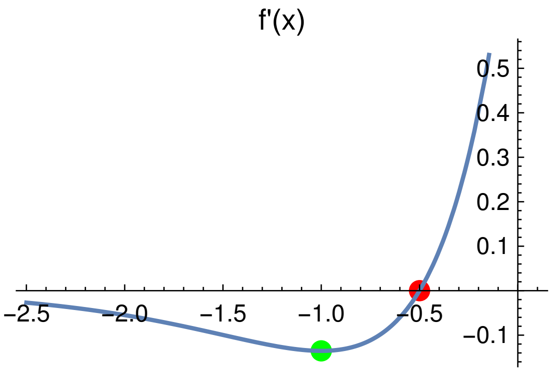
Repeat the previous exercise for $f(x) = xe^{-x}$!

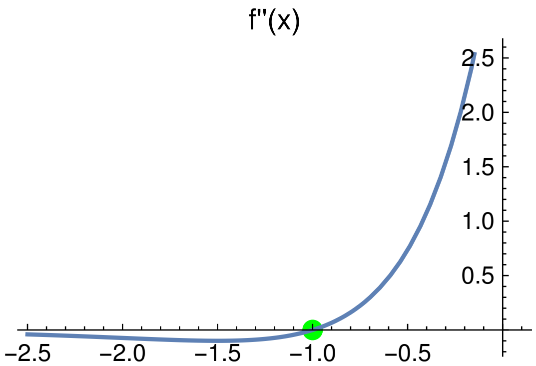
3.1.4 Sample exercise

Tell all that you know about the function $f(x) = e^{2x}x$. Solution:

$$f(x) = e^{2x}x$$
$$f'(x) = e^{2x}(2x+1)$$
$$f''(x) = 4e^{2x}(x+1)$$
$$roots : (0,0)$$
$$critical\ points : \left(-\frac{1}{2}, -\frac{1}{2e}\right)$$
$$inflection\ points : \left(-1, -\frac{1}{e^2}\right)$$







3.1.5 Exercise

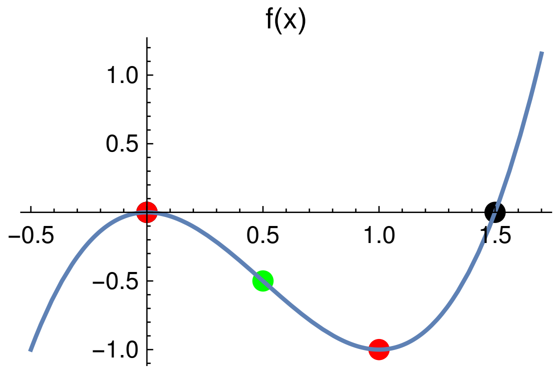
Repeat the previous sample exercise for $f(x) = -xe^{3x}$!

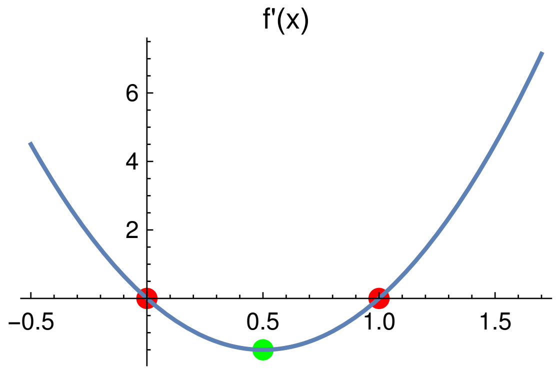
3.1.6 Sample exercise

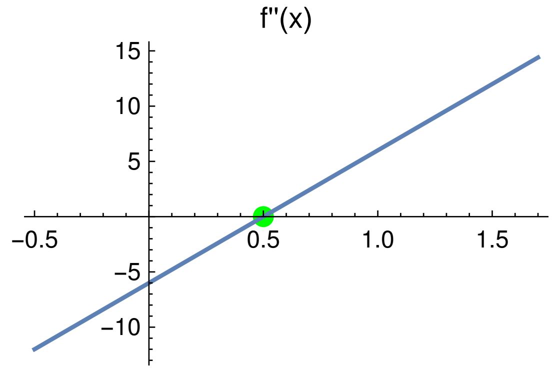
Tell all that you know about the function $f(x) = 2x^3 - 3x^2$.

Solution:

$$f(x) = 2x^3 - 3x^2$$
$$f'(x) = 6(x-1)x$$
$$f''(x) = 12x - 6$$
$$roots : (0,0)(0,0)\left(\frac{3}{2},0\right)$$
$$critical\ points : (0,0)(1,-1)$$
$$inflection\ points : \left(\frac{1}{2},-\frac{1}{2}\right)$$







3.1.7 Exercise

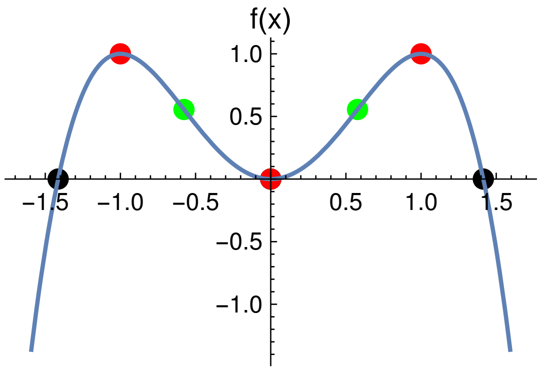
Repeat the previous sample exercise for $f(x) = -x^3 + x$!

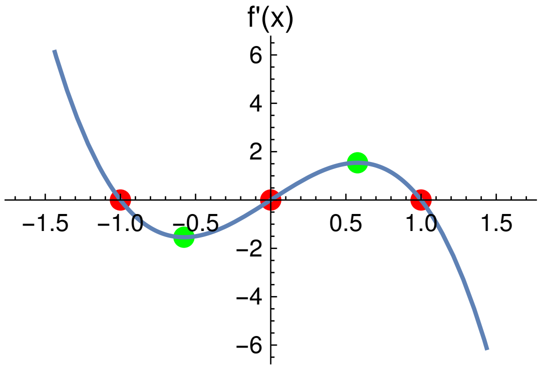
3.1.8 Sample exercise

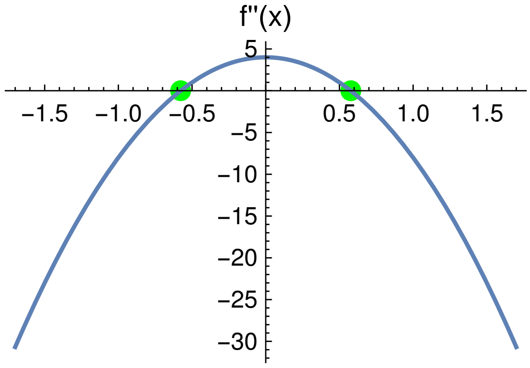
Tell all that you know about the function $f(x) = 2x^2 - x^4$.

Solution:

$$f(x) = 2x^2 - x^4$$
$$f'(x) = -4x\left(x^2 - 1\right)$$
$$f''(x) = 4 - 12x^2$$
$$roots: \ (0,0), \ (0,0), \ \left(-\sqrt{2}, 0\right), \ \left(\sqrt{2}, 0\right),$$
$$critical\ points: (-1,1), \ (0,0), \ (1,1),$$
$$inflection \ \ points: \left(-\frac{1}{\sqrt{3}}, \frac{5}{9}\right), \ \left(\frac{1}{\sqrt{3}}, \frac{5}{9}\right),$$







3.1.9 Exercise

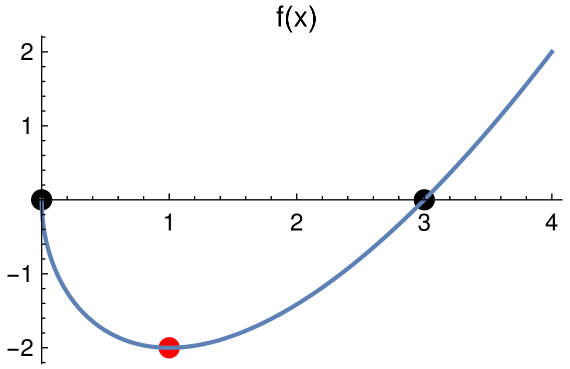
Repeat the previous sample exercise for $f(x) = x^4 - x!$

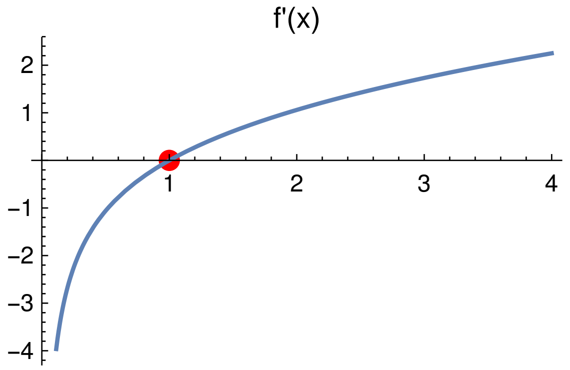
3.1.10 Sample exercise

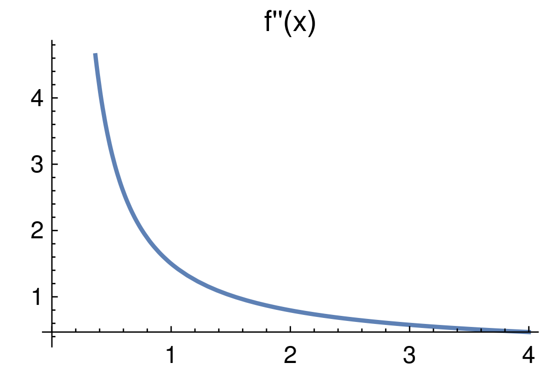
Tell all that you know about the function $f(x) = (x-3)\sqrt{x}$.

Solution:

$$f(x) = (x-3)\sqrt{x}$$
$$f'(x) = \frac{3(x-1)}{2\sqrt{x}}$$
$$f''(x) = \frac{3\ (x+1)}{4x^{3/2}}$$
$$roots:(0,0),\ (3,0),$$
$$critical\ points:\ (1,-2),$$
$$inflection\ points:$$







3.1.11 Exercise

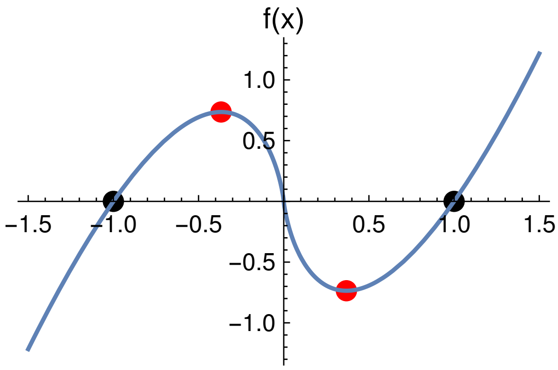
Repeat the previous sample exercise for $f(x) = -x\sqrt{3-x}$!

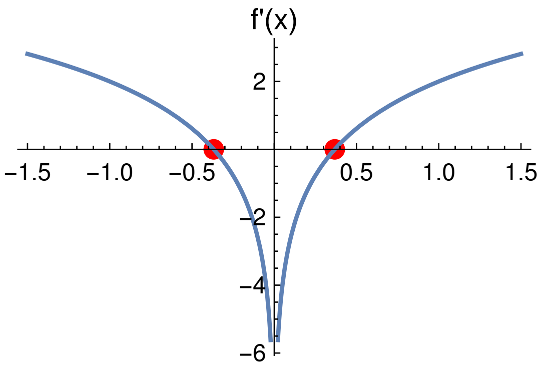
3.1.12 Sample exercise

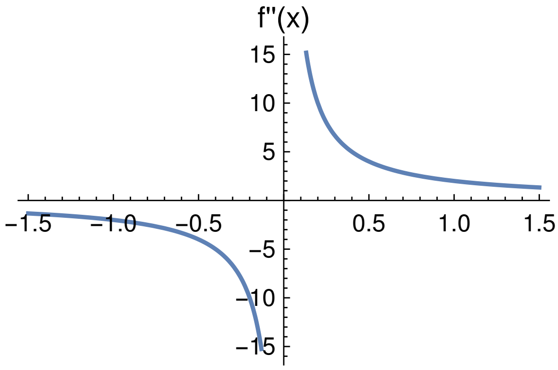
Tell all that you know about the function $f(x) = x \log\left(x^2\right)$.

Solution:

$$f(x) = x \log\left(x^2\right)$$
$$f'(x) = \log\left(x^2\right) + 2$$
$$f''(x) = \frac{2}{x}$$
$$roots : (-1, 0),\ (1, 0),$$
$$critical\ \ points : \left(-\frac{1}{e}, \frac{2}{e}\right),\ \left(\frac{1}{e}, -\frac{2}{e}\right),$$
$$inflection\ \ points :$$







Remark: $x = 0$ is not in the domain $D_f = (-\infty, 0) \cup (0.\infty)$, so at $(0, 0)$ there should be a hole on the plot of f. Indeed $(0, 0)$ is not listed among the roots.

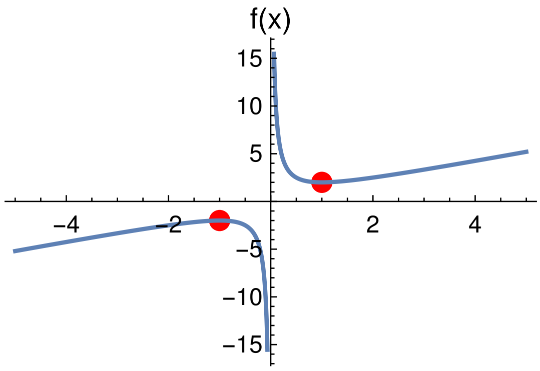
3.1.13 Exercise

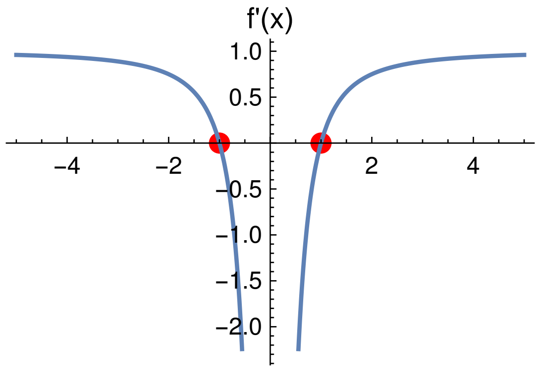
Repeat the previous sample exercise for $f(x) = -x \log(x^4)$!

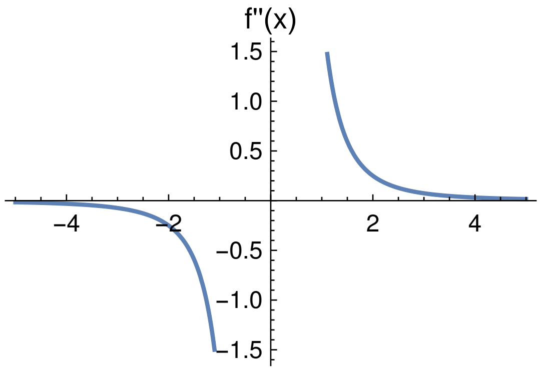
3.1.14 Sample exercise

Tell all that you know about the function $f(x) = x + \frac{1}{x}$.
Solution:

$$f(x) = x + \frac{1}{x}$$
$$f'(x) = 1 - \frac{1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$
$$roots:$$
$$critical\ points: (-1, -2),\ (1, 2),$$
$$inflection\ points:$$







3.1.15 Exercise

Repeat the previous sample exercise for $f(x) = -\frac{2}{x+3} - 5x$!

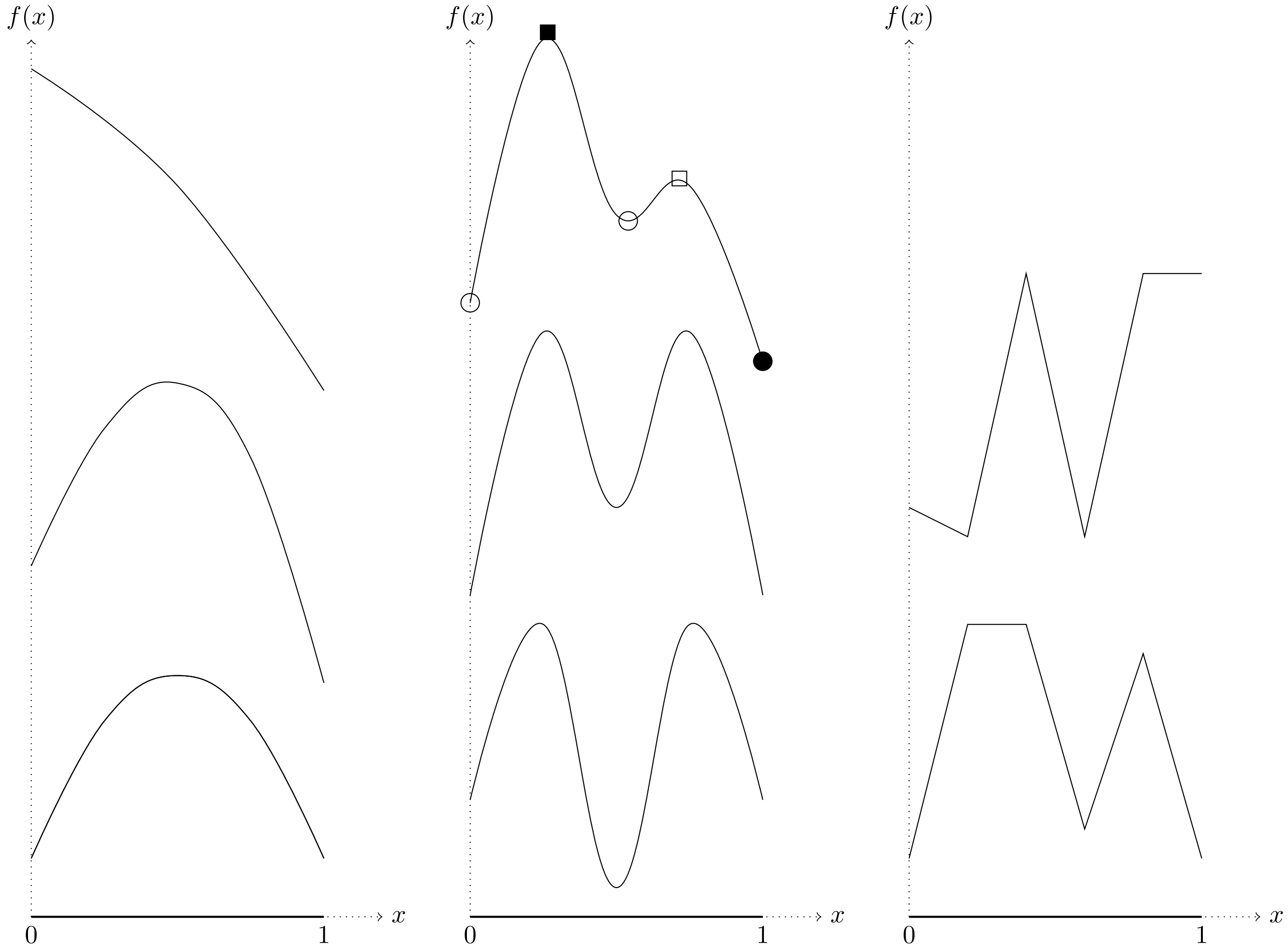
3.1.16 Remarks

We searched for critical an inflection points by solving $f'(x) = 0$ and $f''(x) = 0$. Note however that a solution might not corresponds to a local minimum/maximum or to an inflection point.

- $f(x) = x^3$: Here $f'(0) = 0$, nevertheless $(0, 0)$ neither a local minimum nor a maximum.
- $f(x) = x^4$: Here $f''(0) = 0$, nevertheless $(0, 0)$ is not an inflection point.

3.1.17 Exercise: Local versus global minimum/maximum

Mark the local and global minimums and maximums of the following functions having the same domain $D_f = [0, 1]$! Use the markers of the solved case of the middle top graph!



3.1.18 Exercise

Solve the previous exercise when $D_f = (0, 1)$ only!

Remark:

For the solution we need to know how to treat the missing endpoints of $[0, 1]$. What should be the minimum of $f(x) = x$ on $D_f = (0, 1)$? It can not be 0, as that value would be attained at $x = 0$, which is not inside of D_f. Nevertheless 0 is the largest possible number such that $f(x) \geq 0$, so we say that

$$\inf_{x \in (0,1)} f(x) = 0,$$

instead of the not quite correct

$$\min_{x \in (0,1)} f(x) = 0$$

statement. Here "inf" stands for "infimum".

Similary

$$\sup_{x \in (0,1)} f(x) = 1,$$

where "sup" means "supremum". Usually his distinction hardly matters.

3.1.19 Exercise

Which of the following problems have a unique, definitive and reasonable solution? In case of doubts, defend your opinion!

- Find the rectangle having the largest area, if its circumference is 4.
- Find the rectangle having the smallest area, if its circumference is 4.

3.1.20 Exercise

Let $f(x)$ be a nonnegative, continuous function defined on $[0,1]$ and assume that $f(0) = 0,\ f(1) = 1$.

Which of the following problems have a unique, definitive and reasonable solution? In case of doubts, defend your opinion!

- Find that f for which the area between the x-axis and $f(x)$ is the smallest possible.
- Find that f for which the area between the x-axis and $f(x)$ is the largest possible.
- Find that f for which the arc length of the plot of f is the smallest possible.
- Find that f for which the arc length of the plot of f is the largest possible.

3.2 Numerical approximations

3.2.1 Sample exercise: Plotting

Use a spreadsheet program to plot together $f(x) = 2x^2 - x^4$ and its derivative $f'(x)$ on $[-1,1]$! Pretend that you do not know how to compute the derivative, use instead the numerical approximation

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

3.2.2 Exercise

Repeat the previous sample exercise for $f(x) = xe^{-x}$ on $[0,3]$!

3.2.3 Sample exercise: Error of numerical derivation

Let

$$f(x) = e^x, \quad x_0 = 2, \quad \Delta x = \frac{1}{2^n}, \quad n = 2 \ldots 20.$$
$$f'(x_0) \approx \frac{\Delta f}{\Delta x} = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x},$$
$$error(\Delta x) = \left| \frac{\Delta f}{\Delta x} - f'(x_0) \right|.$$

Assume that $error(\Delta x) \approx c \cdot \Delta x^\alpha$ for some c, α constants when $\Delta x \to 0$. To extract α, plot $\lg(\Delta x) \leftrightarrow \lg(error(\Delta x))$, since

$$\lg(error(\Delta x) \approx \lg(c \cdot \Delta x^\alpha) = \lg(c) + \alpha \lg(\Delta x),$$

so α would be just the slope of the *loglog* plot. Since the power like behaviour of the error is accurate only when Δx is small, use only the points corresponding to (let us say) $n = 13 \ldots 20$.

- How much are 2^{10} and 2^{20}? You should know the first one exactly, while the second at least approximately. (How many bytes is a kilobyte?)
- How much are 2^{-10} and 2^{-20} approximately?

Use the *linest* function to fit a line to the data points! So what is your estimate of α? Your result should be close to an integer, what would that be?

Solution: Make a spreadsheet table with columns

$$n = 2 \ldots 20, \ \ \Delta x = 1/2^n, \ \ \Delta f/\Delta x, \ \ error(\Delta x), \ \ lg(\Delta x), \ \ \lg(error(\Delta x)).$$

Then applying *linest* to the last two columns for $n = 13 \ldots 20$ (on my spreadsheet it is =LINEST(F13:F20,E13:E20)) will produce two numbers: 1.000025708 and 0.567681845. So according to our measurements

$$error(\Delta x) \approx 10^{0.567681845} \cdot \Delta x^{1.000025708} \approx 3.69557 \cdot \Delta x.$$

In the $\Delta x \to 0$ limit the exponent of Δx would be just 1.

3.2.4 Exercise

Repeat the previous exercise for

$$\sin x, \quad x = 0 \ \text{ or } \ x = \pi/4.$$

You should get different result at the two x_0 values!

3.2.5 Exercise

Repeat the previous exercise for

$$f(x) = \exp(x), \quad x = 2,$$

but now use the following approximation of the derivative f':

$$f'(x_0) \approx \frac{\Delta f}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}$$

3.2.6 Exercise

Try to explain why the numerical approximation of f' is better in the last exercise compared to the sample exercise! Draw a pictures of f and on that figure draw straight lines with slopes corresponding to the two approximation schemes! Which one seems to be a better approximation?