Mathematics for economic analysis, Homework set 1.

See also the HW1.ods or HW1.xlsx files. Their sheets are: Basic plotting, Newton-Raphson error, Recognizing exponential growth, Euler method.

1 Plotting of sequences

1.1 Sample exercise of plotting a sequence

Plot the sequence a_n! Is it seems to be true/false for a_n: a) monoton increasing/decreasing, b) bounded (maybe only from above/below), c) convergent/divergent, d) if is divergent, does it converge to $\pm\infty$?

$$a_n = (-1)^n \frac{n+3}{2n+5}, \quad n = 0 \ldots \infty, \qquad \text{plot for } n = 0 \ldots 20.$$

Solution:

LibreOffice:

```
A2 : enter 0,
A3 : enter  =A2+1,
select A3:A22,
Sheet -> Fill Cells -> Fill Down
B2 : enter = (-1)^A2 * (A2+3)/(2*A2+5),
select B2:B22,
Sheet -> Fill Cells -> Fill Down,
select A2:B22,
Insert -> Chart ... Choose Chart Type: XY (Scatter) .. Lines only ... Finish
```

Octave (Matlab):

```
for i = 1:21
 n(i) = i;
 an(i) = (-1)^(i-1)*((i-1)+3)/(2*(i-1)+5);
endfor;
plot (n,an);
```

Mathematica (Mathics):

```
an = Table[ (-1)^(i-1)*((i-1)+3)/(2*(i-1)+5), {i,1,21} ]
ListPlot[ an ]
DiscreteLimit[ (-1)^(i-1)*((i-1)+3)/(2*(i-1)+5), i->Infinity ]
```

SAGE:

```
point( [ (n, (-1)^n*(n+3)/(2*n+5)) for n in [0..20] ] )
var( 'n', domain=ZZ )
limit( (-1)^n*(n+3)/(2*n+5), n = +Infinity )
```

WolframAlpha:

```
listplot  table (-1)^n*(n+3)/(2*n+5)  for  n from 0 to 20
limit (-1)^n*(n+3)/(2*n+5)   n goes to infinity
```

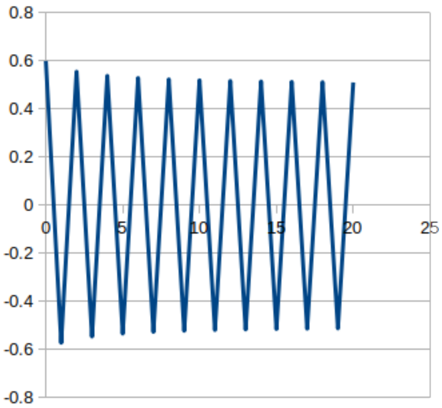


Figure 1: Plot of the sequence $a_n = (-1)^n \frac{n+3}{2n+5}$.

Result: The sequence seems to be a) bounded, since its values are in the range $[-0.6, 0.6]$, b) not monoton, since the curve of the plot goes both up and down, c) not convergent, since it takes infinitely many values around both of ± 0.5.

Remarks:

- Links to online software: Octave, SAGE, WolframAlpha. Mathics installer: Mathics.
- Another (relatively small) symbolic algebra package: wxMaxima
- Octave can do symbolic calculations, that requires to load an extra package "symbolic".
- In the online SAGE calculator run separately the first (point...) line and the next two (var.., limit..) ones.
- The limit of a sequence (which can be regarded as a function defined on integers) might be different from the limit of the "same" function defined on the reals. In Mathematica this is handled by using "DiscreteLimit", as opposed to "Limit". (Mathics has only "Limit".)

  In SAGE try

  ```
  var( 'n', domain=ZZ )
  sin(n*pi)
  ```

  as opposed to

  ```
  var( 'n', domain=RR )
  sin(n*pi)
  ```

  Here ZZ means the integers, while RR denotes the real numbers. The first result is zero, since $\sin(n\pi) = 0$ for all $n \in \mathbb{Z}$, while the second one is "ind" (indefinite), as $\sin(\pi x)$ oscillates between ± 1 for $x \in \mathbb{R}$.
- It is not always easy to find the correct form of the query to WolframAlpha. Try

  ```
  listplot of the table (-1)^n*(n+3)/(2*n+5)  for  n from 0 to 20
  ```

  with or without "of the"!

  Try also queries like "Hungarian GDP last ten years".

1.2 Exercise (plotting)

Repeat at least the spreadsheet part of the previous sample exercise for a few of the following a_n sequences:

$$\frac{n+3}{2n+5}, \quad \frac{3}{2n+5}, \quad (-1)^n \frac{3}{2n+5}, \quad n^2+n, \quad \sqrt[2]{n^2+7}-n, \quad \sqrt[2]{n^2+7n}-n,$$
$$\frac{2^{2n}+3^n}{5\cdot 4^n+7^n}, \quad \frac{5\cdot 4^n+7^n}{2^{2n}+3^n}, \quad \left(1+\frac{1}{n}\right)^n, \quad \left(2+\frac{1}{n}\right)^n, \quad \left(0.5+\frac{1}{n}\right)^n, \quad \left(1+\frac{1}{n}\right)^{n^2}, \quad \left(1+\frac{1}{n}\right)^{\sqrt{n}},$$
$$\sin(n), \quad cos\left(n^2\right), \quad cos\left(2^n\right).$$

Remark: You should be able to solve this exercise even using only paper and pencil for the following cases: 1-4, 7-12. The Hungarian course covers 5-6, too. The last three are not at all trivial, the last one is in fact way beyond my background in mathematics.

2 Recursive sequences, discrete dynamical systems

Let us suppose that given the current state s_n of a system at time n, we can compute what will be the s_{n+1} state of the system a unit time later:

$$s_{n+1} = f(s_n), \tag{1}$$

where f is the function generating the time evolution of the system. Then, starting with an initial s_0 state, we can generate the sequence s_n:

$$s_0,\ s_1 = f(s_0),\ s_2 = f(s_1) = f(f(s_0)) = f^2(s_0),\ s_3 = f(s_2) = f(f(f(s_0))) = f^3(s_0),\ \ldots$$
$$s_0 \to s_1 \to s_2 \to s_3 \to \ldots$$

Here the state s_n might consist of several numbers.

s_{n+1} might depend on several previous states, however these more complicated cases can be reduced to (1). For example

$$s_{n+1} = f(s_{n-1}, s_n) \qquad \Longleftrightarrow \qquad \tilde{s}_{n+1} = \begin{pmatrix} s_{n+1} \\ s_n \end{pmatrix} = \tilde{f}(\tilde{s}_n) = \begin{pmatrix} f(s_{n-1}, s_n) \\ s_n \end{pmatrix}.$$

2.1 Recognizing exponential (geometric) behaviour

The behaviour of a dynamical system around a fixed point is often described (at least approximately as long as the system is close to the fixed point) by geometric sequences:

$$x_{n+1} = f(x_n), \quad x_{fix} = f(x_{fix}), \quad \Delta x_n = x_n - x_{fix},$$
$$\Delta x_{n+1} \approx \lambda \Delta x_n \quad \Longleftrightarrow \quad \Delta x_n \approx \lambda^n \Delta x_0, \quad \text{where} \quad \lambda = f'(x_{fix}).$$

The multidimensional variant is

$$\overrightarrow{\Delta x_n} \approx \sum_i c_i \lambda_i^n \overrightarrow{v_n},$$

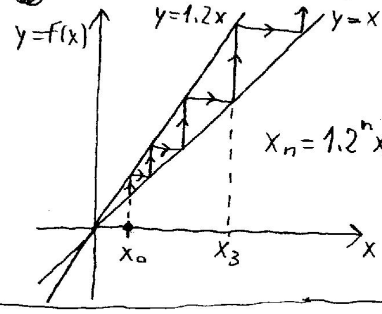
Here the summation index i runs from one to the dimension of the system. The computations of $c_i, \lambda_i, \overrightarrow{v_n}$ are standard exercises in linear algebra. Let us remark that this formalism can describe oscillating behaviour if one is willing to use complex numbers like $\sqrt{-1}$.

How can we recognize that the sequence $a_n = cq^n$ is geometric from its plot?

Solution: plot n versus $\lg(a_n) = \lg(cq^n) = \lg(c) + \lg(q) \cdot n$! Then the graph will be a straight line with slope $\lg(q)$!

Discrete dynamical system, graphical iteration of functions.

1 Geometric sequences: $x_{n+1} = q x_n = f(x_n) \implies x_n = f^n(x_0) = q^n x_0$



2 $x_{n+1} = f(x_n) = 1.2x_n - 20$

Find x_n!

Solution:

(A) Find the fixed point of the dynamics:

$$x_{fix} = f(x_{fix})$$

$$x_{fix} = 1.2x_{fix} - 20 \implies x_{fix} = 100$$

(B) Introduce the $\Delta x \leftrightarrow \Delta y$ coord. system:

$$\Delta x = x - x_{fix} = x - 100, \quad \Delta x_0 = x_0 - 100$$

(C) In the new coord. sys. the dynamics is geometric:

$$\Delta x_{n+1} = 1.2\,\Delta x_n, \quad \Delta x_n = 1.2^n \Delta x_0$$

Indeed

$$\underbrace{\underbrace{(1.2x_n - 20)}_{x_{n+1}} - 100}_{\Delta x_{n+1}} = 1.2\underbrace{(x_n - 100)}_{\Delta x_n}$$

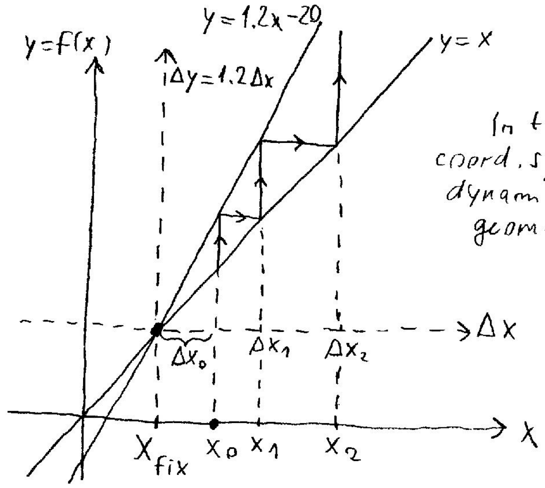
(D) Go back to the $x \leftrightarrow y$ coord. sys.

$$x_n = \Delta x_n + x_{fix} = \Delta x_n + 100$$

So

$$x_n = \underbrace{1.2^n \underbrace{(x_0 - 100)}_{\Delta x_0}}_{\Delta x_n} + 100$$

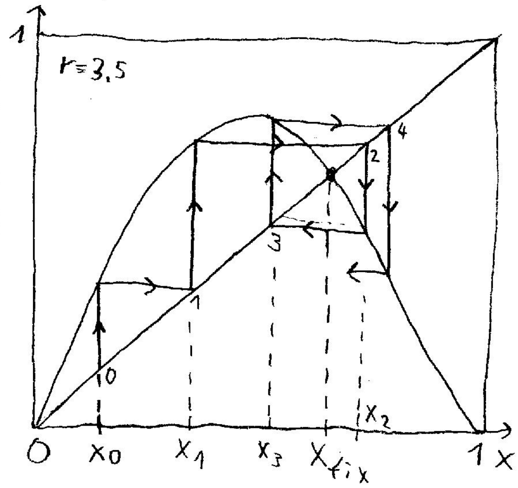
x_{fix} is unstable, dynamics goes away from it, since $|1.2| > 1$



In the $\Delta x \leftrightarrow \Delta y$ coord. sys. the dynamics is geometric.

3 Graphical iteration for nonlinear systems: logistic map

$$f(x) = r(1-x) \cdot x, \quad r \in [0,4]$$



2.1.1 Sample exercise

A bank account pays 20% interest and charges 20HUF yearly fee. What will be the balance after n years if the initial deposit is 222HUF ?

The mathematical formulation of the problem: Let

$$x_{n+1} = 1.2x_n - 20, \;\; x_0 = 222 \quad \Longleftrightarrow \quad x_n = 1.2^n(222-100)+100, \quad \text{where } \; x_{fix} = 100.$$

Plot $n \leftrightarrow (x_n - 100)$ and also $n \leftrightarrow (\lg(x_n - 100))$!
Now suppose that the $x_{fix} = 100$ is unknown to us, so we plot $n \leftrightarrow x_n$ and $n \leftrightarrow (\lg(x_n))$. How can we use this last plot to recover at least approximately the numbers 1.2 and 122=222-100 ?

Solution:

LibreOffice:

```
A2 : enter 0,
A3 : enter  =A2+1,
select A3:A22,
Sheet -> Fill Cells -> Fill Down
B2 : enter  222,
B3 : enter  1.2*B2-20,
select B3:B22,
Sheet -> Fill Cells -> Fill Down,
C2 : enter =log10(B2),
select C2:C22,
Sheet -> Fill Cells -> Fill Down,
... repeat these for columns D and E .....

select A2:B22,
Insert -> Chart ... Choose Chart Type: XY (Scatter), Lines only ... Finish
... Insert a similar chart for the data of columns A and C

G1 : enter =linest(c2:c22;a2:a22;true;true) with "Return"
 this will give the 0.067733 slope of the best fitting line to the n<->lg(xn) graph

G3 : enter =linest(c2:c22;a2:a22;true;true) with "Ctrl+Shift+Return"
 now we get a more detailed data set:
0.0677338911947     2.28370712755946
0.001017424410523 0.011894151178379
0.995731374569587 0.02823238162055
4432.08157408962 19
3.53266761260906 0.015144280067399
```

Most of these numbers are coming from statistical analysis, not all of them are relevant for us. The best fitting line for the $(n, \lg(x_n))$ data points is

$$\lg(x_n) \approx 2.28370712755946 + 0.0677338911947 \cdot n,$$

so the best approximation of the x_n sequence with a geometric one is

$$x_n \approx 10^{2.283707127755946} \cdot \left(10^{0.0677338911947}\right)^n = 192.18 \cdot 1.16878^n.$$

This is a poor approximation of $122 \cdot 1.2^n$, not too surprising since $x_n = 122 \cdot 1.2^n + 100$. For smaller values of n the extra 100 does matters, however when n is larger, for example when $n > 13$, we get a more accurate picture of the asymptotic geometric behaviour of x_n. Executing "linest(c16:c22;a16:a22;true;true)" will give

$$x_n \approx 142.154 \cdot 1.19196^n.$$

All these illustrates that even if the asymptotic behaviour is described by a geometric sequence, there might be a transitional interval where the geometric sequence approximation does not work very well.

The results (and much more irrelevant staff) can be obtained by using *Data* → *Statistics* → *Regression*. By editing a chart, one can insert a "Trend line" for a set of data points.

We must address the question that what the meaning of $x_n \approx geom.seq(n) = 122 \cdot 1.2^n$ might be. Two reasonable choices are:

- $x_n - geom.seq(n) \to 0$, which is false in our case,
- $x_n / geom.seq(n) \to 1$, that is true.

Other commonly used notations expressing that x_n behaves similarly to a geometric sequence with quotient 1.2 are:

- $x_n = O(1.2^n)$: there exists $c, n_0 > 0$ constants such that $n > n_0$ implies $|x_n| < c \cdot 1.2^n$,
- $x_n = \Theta(1.2^n)$: there exists $c_1, c_2, n_0 > 0$ constants such that $n > n_0$ implies $c_1 \cdot 1.2^n < |x_n| < c_2 \cdot 1.2^n$

2.1.2 Exercise

Repeat the previous exercise for the sequence

$$x_{n+1} = 0.9x_n + 10, \quad x_0 = 222.$$

- Plot $n \leftrightarrow x_n$.
- Should we plot $n \leftrightarrow \lg(x_n)$?
- Find the x_{fix} fixed point of this recursive law! Can you do it by using "*Tools* $\rightarrow$ *Solver*" ?
- Plot $n \leftrightarrow (x_n - x_{fix})$ and $n \leftrightarrow \lg(x_n - x_{fix})$!
- Find the best linear fit to the second graph! Figure out what would be the best geometric approximation

  $$(x_n - x_{fix}) \approx geom.seq(n) = ab^n.$$

  Actually we have equality here.

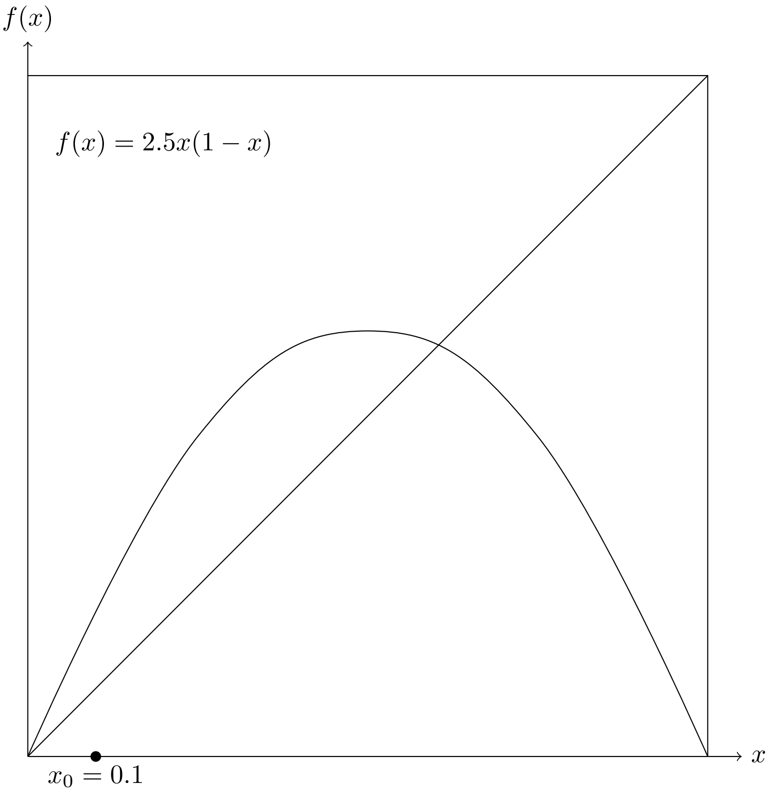
2.2 Nonlinear systems: logistic map

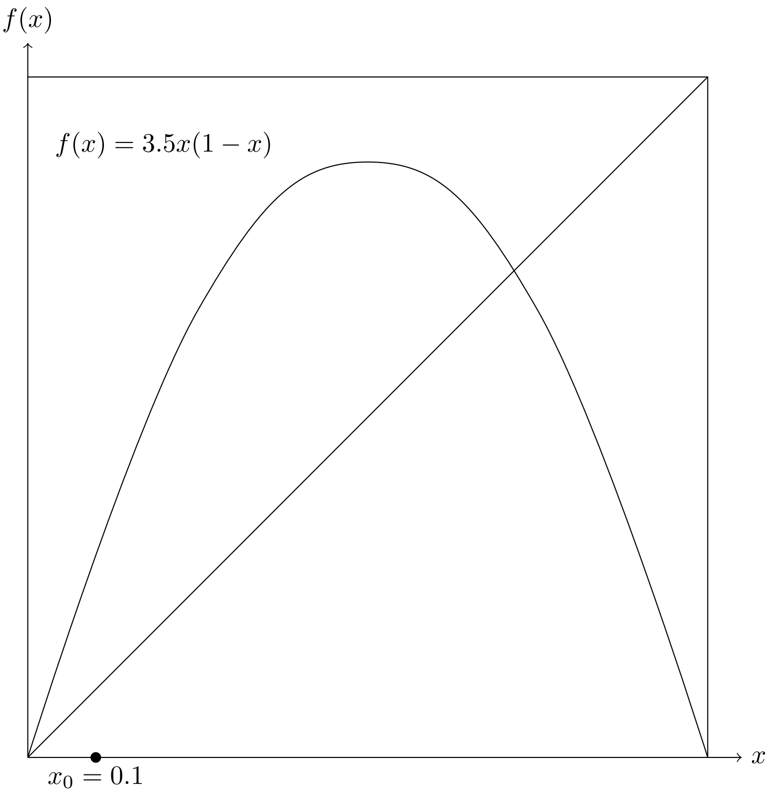
Problem: Study the behaviour of the logistic dynamical system.

$$x_{n+1} = f(x) = rx_n(1 - x_n), \quad x_0 \in [0, 1], \;\; r \in [0, 4].$$

2.2.1 Exercise

Perform graphical iteration on the following figures and compare the result with the plots of x_n. (Generate the plots either by a spreadsheet program or by a computer algebra system.)





2.3 Fibonacci sequence

The Fibonacci sequence is a linear recursive sequence

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 1,\ F_1 = 1.$$

Linear (even multicomponent) dynamical systems can be solved explicitly using linear algebra tools. That should be compared to the complicated behaviour of the logistic map, which is a very simple nonlinear system.

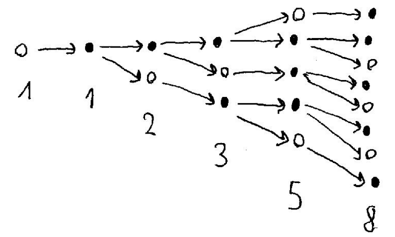
Linear dynamical systems

Fibonacci sequence: $F_0 = 1$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$

1 1 2 3 5 8 13 21 34 55 ...

• old rabbit
o young rabbit

rule: $\bullet \to \bullet, \circ$; $\circ \to \bullet$



1 year: young → old
old → old, young offspring

F_{n+1} depends on F_n and F_{n-1}, that is the previous two years, F_n scalar (a single number)

Describe state as $\vec{F}_n = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} \text{all rabbits} \\ \text{old rabbits} \end{bmatrix}$

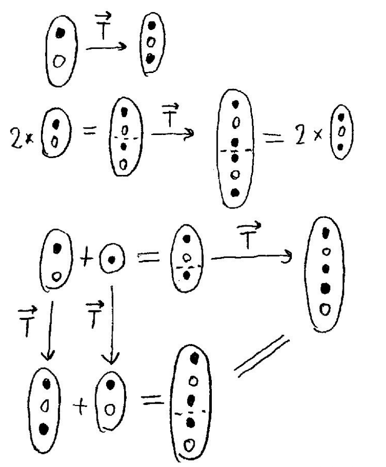
then $\vec{F}_{n+1}$ can be computed from $\vec{F}_n$:

$$\vec{F}_{n+1} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \vec{T}\Bigg(\underbrace{\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}}_{\vec{F}_n}\Bigg),$$

where $\vec{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \end{bmatrix} = \begin{bmatrix} 1\cdot x + 1\cdot y \\ 1\cdot x + 0\cdot y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\vec{T}$ is linear:

for example



$\vec{T}(2\vec{v}) = 2\vec{T}(\vec{v})$

$\vec{T}(\vec{u}+\vec{v}) = \vec{T}(\vec{u}) + \vec{T}(\vec{v})$

Iteration of linear mappings like $\vec{T}$ has a well developed theory called linear algebra: vectors, matrices

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Fibonacci's method for $G_0 = 1$, $G_1 = 1$, $G_{n+1} = G_n + 2G_{n-1}$ $\quad \bullet \to \bullet, \circ, \circ \quad \circ \to \bullet$

① Rule $G_{n+1} = G_n + 2G_{n-1}$ solved by $G_n = q^n$, if $q = 1 + 2q^{-1} \to q_1 = 2,\ q_2 = -1$

② Rule — " — also solved by $G_n = \alpha_1 q_1^n + \alpha_2 q_2^n$, so pick $\alpha_{1,2}$ such that $G_0 = G_1 = 1$

$n=0$: $1 = \alpha_1 \cdot q_1^0 + \alpha_2 q_2^0 = \alpha_1 + \alpha_2$
$n=1$: $1 = \alpha_1 \cdot q_1^1 + \alpha_2 q_2^1 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$ $\Rightarrow \alpha_1 = \frac{2}{3},\ \alpha_2 = \frac{1}{3}$

③ So the solution of the rule and the initial conditions: $G_n = \frac{2}{3} \cdot 2^n + \frac{1}{3}(-1)^n$

2.3.1 Exercise

Generate the first few members of the Fibonacci sequence two ways.

- Regard it as a scalar one component system, where the next value depends on the previous two states.
- Regard it as a two component system, where the next value depends only on the previous state (which consists of two numbers).

What would be the best fitting geometric sequence approximation of the Fibonacci sequence?

2.3.2 Exercise

Repeat the previous exercise for the sequences where the recursive law of the Fibonacci sequence is modified as

- $F_{n+1} = F_n + 2F_{n-1}$,
- $F_{n+1} = F_n + 6F_{n-1}$.

Find an explicit expression for F_n!

Remark: These kind of recurrence relations are explicitly solvable.

- SAGE

```
from sympy import Function,rsolve
from sympy.abc import n
u = Function('u')
f = u(n-1)+2*u(n-2)-u(n)
rsolve(f, u(n), {u(0):1,u(1):1})
     ...  RETURNS  ....
(-1)**n/3 + 2*2**n/3
```

  Unfortunately it is very hard to figure this out of the documentation of the program package, I just copied this from the internet.

- In Mathematica

```
RSolve[{f[n + 1] == f[n] + 2 f[n - 1], f[0] == 1, f[1] == 1}, f[n], n]
```

  provides the same result

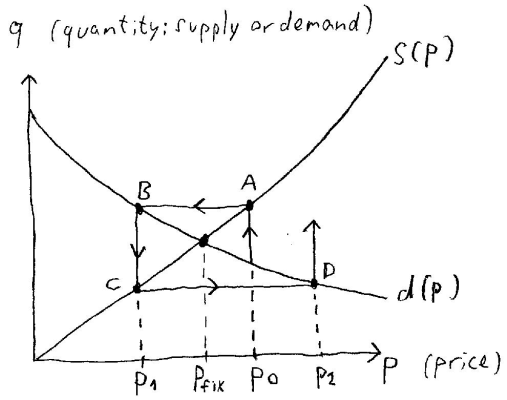
$$\frac{1}{3}\left((-1)^n + 2^{n+1}\right).$$

- In WolframAlpha this works:

```
f[n+1]=f[n]+2f[n-1], f[0]=1,f[1]=1
```

2.4 Cobweb model of price fluctuation

See Cobweb model and Mike Rosser: Basic Mathematics For Economists, Chapter 13.



p: price

$s(p)$: supply (price)

$d(p)$: demand (price)

p_0: price last year

A: supply this year, $s(p_0)$

B: $s(p_0)$ supply meets demand at price p_1:
$s(p_0) = d(p_1) \Longrightarrow p_1 = d^{-1}(s(p_0))$

C: supply next year

D: next year supply will be sold at price p_2

Example:

$$s(p) = p, \quad d(p) = 3p^{-2} = \frac{3}{p^2}$$

$d^{-1}(q)$ Inverse of $d(p)$: $\quad q = d(p) = \frac{3}{p^2}$

$$p = (3/q)^{1/2} \quad \text{so } d^{-1}(q) = (3/q)^{1/2} = 3^{1/2} q^{-1/2}$$

Dynamics of the price is generated by

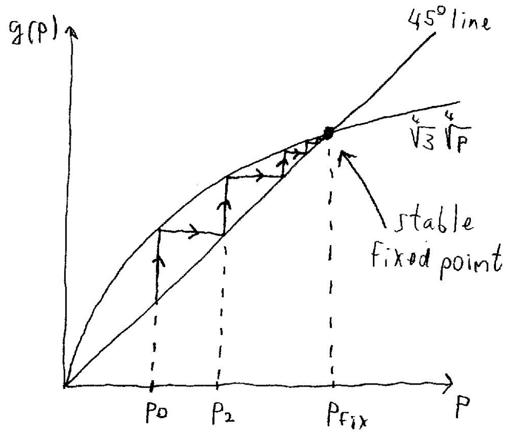
$$p_{n+1} = d^{-1}(s(p_n)) = f(p_n) = 3^{1/2}(\underbrace{s(p_n)}_{= p_n})^{-1/2} = 3^{1/2} p_n^{-1/2}$$

so $f(p) = 3^{1/2} p^{-1/2} = \sqrt{\frac{3}{p}}$

Price oscillates around p_{fix} (equilibrium price, fixed point of the dyn. sys.), it is easier to understand what happens in every two years:

$$p_{n+2} = f(f(p_n)) = g(p_n) = 3^{1/2}\left(3^{1/2} p_n^{-1/2}\right)^{-1/2} = 3^{1/2 - 1/4} p_n^{1/4} = \sqrt[4]{3}\sqrt[4]{p_n}$$

so $g(p) = \sqrt[4]{3}\sqrt[4]{p}$



$\boxed{\text{Cobweb model of price fluctuation}}$

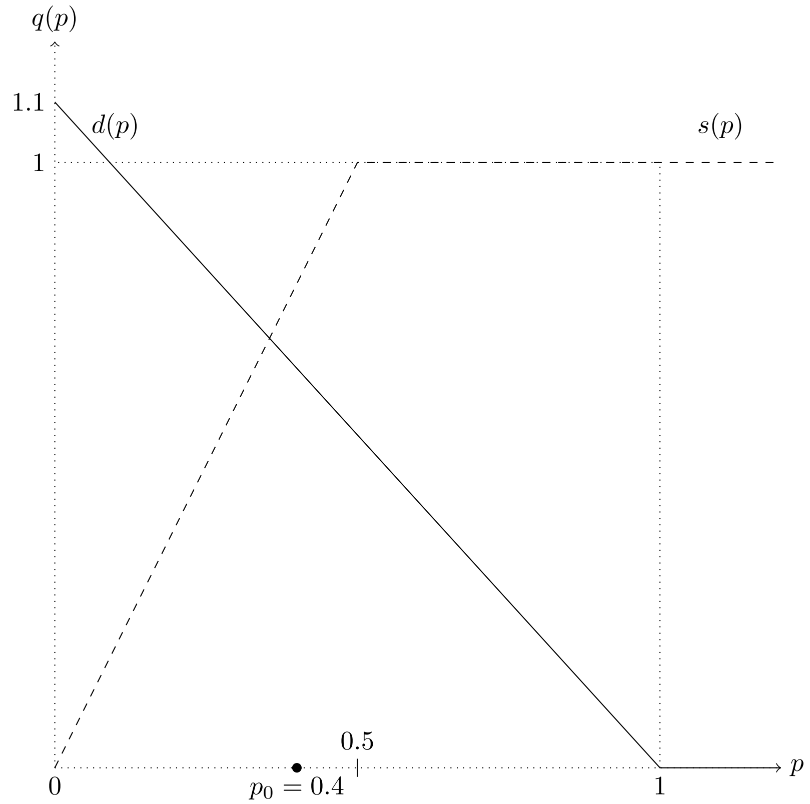
2.4.1 Exercise

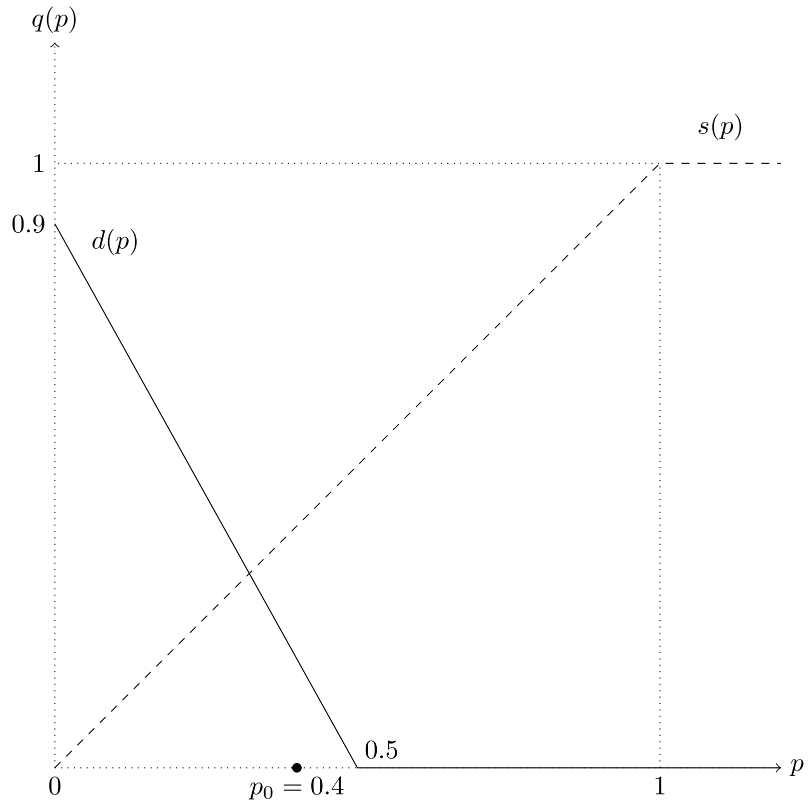
Repeat the analysis of the previous page for $s(p) = p$, $d(p) = 2/\sqrt{p}$. We need inverse functions and lines for that, see the next three pages.

2.4.2 Exercise

Perform graphical iteration for the following supply and demand functions starting at $p_0 = 0.4$. Compare your result with computer generated sequences! How should you handle the following problems: a) $d^{-1}(q)$ does not exist, b) the value of $s(p)$ might be outside of the range $R_d = D_{d^{-1}}$.

Introduce a new coordinate system $\Delta p \leftrightarrow \Delta q$ whose origin is at the intersection of the demand and supply functions, i.e. find the fixed point (equilibrium price) of the system. What will be then $\Delta d(\Delta p)$ and $\Delta s(\Delta p)$ when we are close to the fixed point? What will be $\Delta d^{-1}(\Delta s(\Delta p))$? Can you determine the stability of the fixed point?



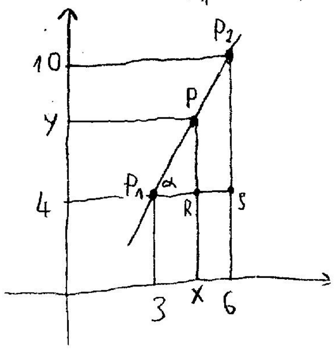


Lines

① Find the equation of the straigh line through the points

$$P_1 = (3,4), \quad P_2 = (6,10)$$

Ⓐ



$$tg\,\alpha = \frac{|\overline{P_2 S}|}{|\overline{P_1 S}|} = \frac{|\overline{PR}|}{|\overline{P_1 R}|} = \frac{10-4}{6-3} = \frac{y-4}{x-3} = \frac{6}{3} = 2$$

$$\frac{y-4}{x-3} = 2 \rightarrow y-4 = 2(x-3) \rightarrow \boxed{y = 2x-2}$$

So $y(x) = 2x - 2$

Ⓑ Alternatively:

The equation of a (non vertical) line: $y(x) = ax + b$

$(3,4) = P_1$ on the line: $4 = a \cdot 3 + b$ I.

$(6,10) = P_2$ on the line: $10 = a \cdot 6 + b$ II.

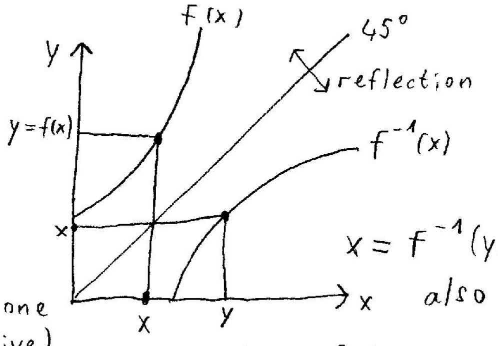
$$\xrightarrow{II.-I.} (10-4) = a \cdot (6-3) + 0 \longrightarrow a = \frac{6}{3} = 2 \xrightarrow{I.} 4 = 2 \cdot 3 + b \longrightarrow b = -2$$

So $y(x) = 2x - 2$

Inverse functions

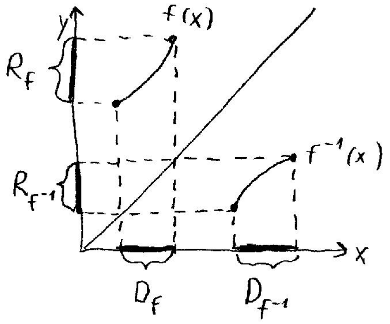
$f: x \to y = f(x)$

$f^{-1}: y \to x = f^{-1}(y)$



$x = f^{-1}(y) = f^{-1}(f(x))$ also $y = f(f^{-1}(y))$

f^{-1} exists if f is one-to-one (bijective)



f has domain D_f, range R_f

f^{-1} has domain $D_{f^{-1}} = R_f$, range $R_{f^{-1}} = D_f$

Example: $f(x) = 3x+2$. Find f^{-1}!

$y = 3x+2$

solve for x: $x = \frac{y-2}{3} = \frac{1}{3}y - \frac{2}{3}$

so $f^{-1}(y) = \frac{1}{3}y - \frac{2}{3}$

rename y: $f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$

Example: $f(x) = \sqrt{x-3} + 2$, $D_f = [3, \infty)$

Then $R_f = [2, \infty) = D_{f^{-1}}$

$y = \sqrt{x-3} + 2$

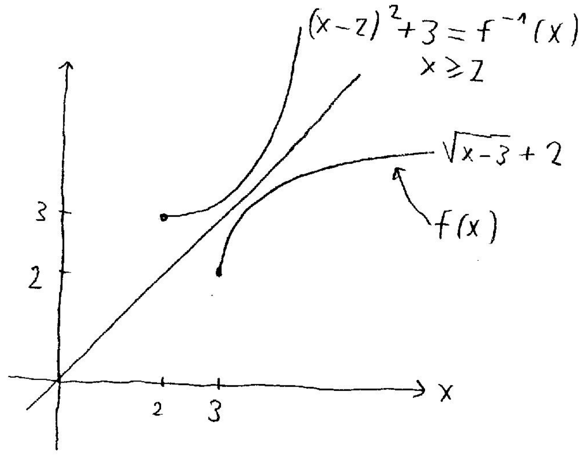
solve for x: $y - 2 = \sqrt{x-3}$

$(y-2)^2 = x - 3$ where $y \in [2, \infty)$

$x = (y-2)^2 + 3$ —"—

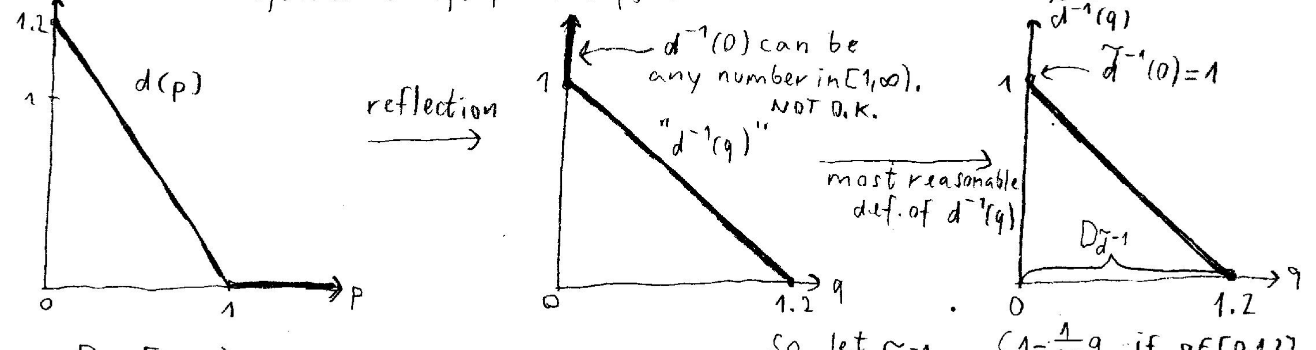
so $f^{-1}(y) = (y-2)^2 + 3$ —"—

rename: $f^{-1}(x) = (x-2)^2 + 3$, $D_{f^{-1}} = [2, \infty)$



Example: demand(price) function $d(p) = \begin{cases} 1.2(1-x) & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

cobweb model requires $d^{-1}(q)$, but $d(p)$ is not one-to-one (not bijective)



$D_d = [0, \infty)$

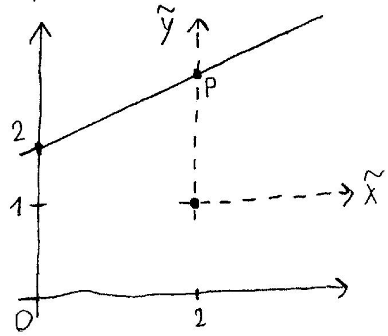
$R_d = [0, 1.2]$

So let $\tilde{d}^{-1}(q) = \begin{cases} 1 - \frac{1}{1.2}q & \text{if } q \in [0, 1.2] \\ \text{otherwise not defined} \end{cases}$

Note that $d(\tilde{d}^{-1}(q)) = q$, but $\tilde{d}^{-1}(d(p)) = p$ can be false

Coordinate transformation

Problem: Let $f(x) = \frac{1}{3}x + 2$. Instead of the original $x \leftrightarrow y$ coord. sys., introduce a new $\tilde{x} \leftrightarrow \tilde{y}$ system, where $\tilde{x} = x - 2$, $\tilde{y} = y - 1$. What function describes the graph of $f(x)$ in the new system?



Coordinates of P: $x \leftrightarrow y$ $\quad (2, 2\frac{2}{3})$ $\quad \frac{1}{3} \cdot 2 + 2$

$\tilde{x} \leftrightarrow \tilde{y}$ $\quad (0, 1\frac{2}{3})$

$\tilde{x} = x - 2 = 0$

$\tilde{y} = y - 1 = 1\frac{2}{3}$

Slopes are the same in both systems, so $\tilde{y}(\tilde{x}) = \tilde{f}(\tilde{x}) = \frac{1}{3}\tilde{x} + 1\frac{1}{3}$

Alternatively: $\begin{aligned} x &= \tilde{x} + 2 \\ y &= \tilde{y} + 1 \end{aligned} \longrightarrow$

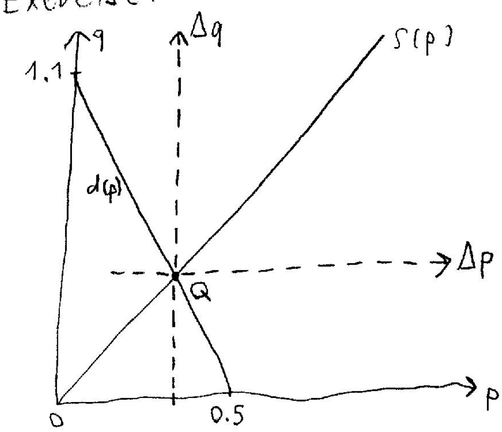
$$f(x) = y = \frac{1}{3}x + 2$$

$$f(\tilde{x}+2) = \tilde{y} + 1 = \frac{1}{3}(\tilde{x}+2) + 2$$

$$\tilde{y} = \frac{1}{3}(\tilde{x}+2) + 2 - 1$$

$$= \frac{1}{3}\tilde{x} + 1\frac{1}{3} = \tilde{f}(\tilde{x})$$

Exercise: cobweb model of price fluctuation



$s(p) = p$, $d(p) = -\frac{1.1}{0.5}p + 1.1$

$\text{price.next.year}(\text{price.this.year}) = f(p) = d^{-1}(s(p))$

Compute $d^{-1}(q)$

$$d(p) = q = -\frac{1.1}{0.5}p + 1.1$$

$$p = \frac{q - 1.1}{-\frac{1.1}{0.5}} = -\frac{0.5}{1.1}q + 0.5$$

$$d^{-1}(q) = -\frac{0.5}{1.1}q + 0.5$$

Compute $f(p)$

$$f(p) = d^{-1}(s(p)) = -\frac{0.5}{1.1}\underbrace{p}_{s(p)=p} + 0.5$$

Dynamics is simpler in the dashed system:

1. Find the Q Fixedpoint: $s(p) = d(p) \longrightarrow p = -\frac{1.1}{0.5}p + 1.1 \longrightarrow p_{fix} = \frac{1.1}{3.2}$
2. The graphs of $s(p)$ and $d(p)$ are described by $\Delta s(\Delta p)$, $\Delta d(\Delta p)$ in the dashed system.
   $$\Delta s(\Delta p) = \Delta p, \quad \Delta d(\Delta p) = -\frac{1.1}{0.5}\Delta p$$
3. $\Delta f(\Delta p) = \Delta d^{-1}(\Delta s(\Delta p)) = \left(-\frac{1.1}{0.5}\right)^{-1} \cdot 1 \cdot \Delta p = -\frac{0.5}{1.1}\Delta p$
4. So the Δp_n seq. is geometric, with limit $= 0$, stable fixedpoint

2.4.3 Exercise

One of unrealistic features of the cobweb model is that the supply next year depends only on a single number, the price this year. What would happen if the s_{n+1} supply next year was $s(p_{average}) = s((p_n + p_{n-1})/2)$ instead of just $s(p_n)$? Can this change the stability of the equilibrium price?

Study this question for the previous exercise! What would happen if you were using a weighted average like $p_{w.av} = 0.8p_n + 0.2p_{n-1}$ instead of $(p_n + p_{n-1})/2$? What would happen if you were using the average price for the last three years?

2.4.4 Exercise

Our cobweb model seems to be a fairly well known toy model in mathematical economy. Try to find a few scholarly papers discussing the questions of the previous exercise! (I have not tried this.)

2.4.5 Exercise

Perform graphical iteration on the following figures! The were generated using the following demand and supply functions:

$$d(p) = \begin{cases} \frac{1}{p+0.5} - 0.5 & \text{if } p \in [0, 1.5], \\ 0 & \text{otherwise,} \end{cases} \qquad s(p) = c\tanh(3p), \text{ where } c \text{ is either } 0.7 \text{ (green) or } 1.4 \text{ (red).}$$

Explain the meaning of the marked intersection points with regard to price dynamics! What can be said about their satbility?

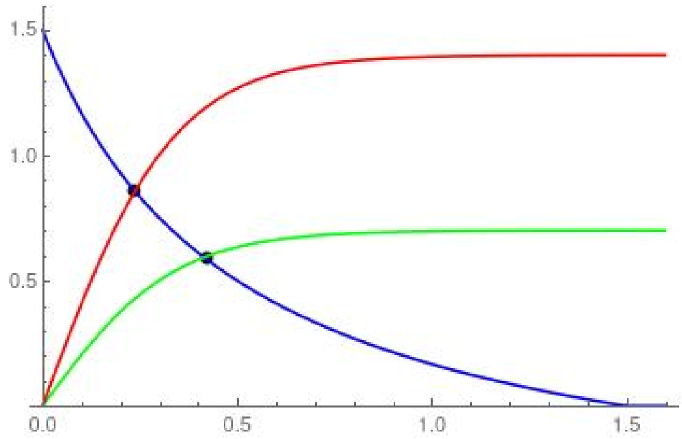


Figure 2: Plots of the $q(p)$ quantity(price) functions, where $q(p)$ is either the demand $d(p)$ or the two supply $s(p)$ functions.

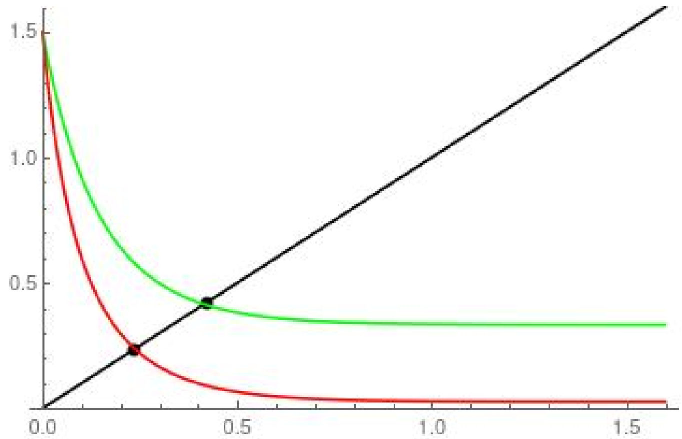


Figure 3: Plots of the price-next-year(price-this-year)) $f(p) = d^{-1}(s(p))$ functions for the two different supply functions. The figure also contains a black auxiliary 45° line.

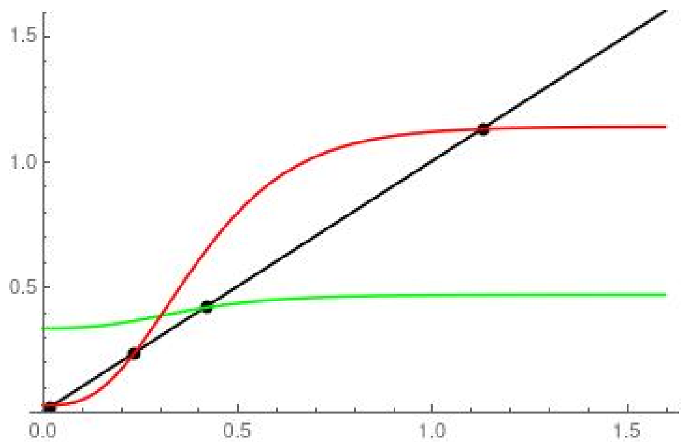


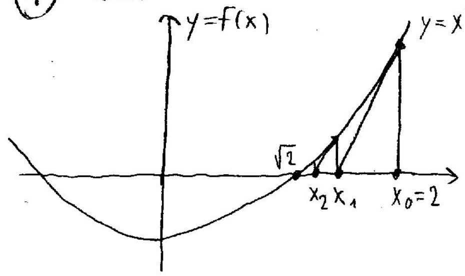
Figure 4: Plots of the price-two-years-later(price-this-year)) $f^2(p) = f(f(p))$ functions for the two different supply functions. The figure also contains a black auxiliary 45° line.

2.5 Solving equations. Newton-Raphson method

See Newton-Raphson.

Newton-Raphson method of solving equations

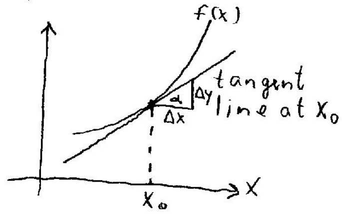
(1) Solve $x^2-2=0$, i.e. compute $\sqrt{2}$!



(A) Guess a value for x_0, hopefully close to $\sqrt{2}$.

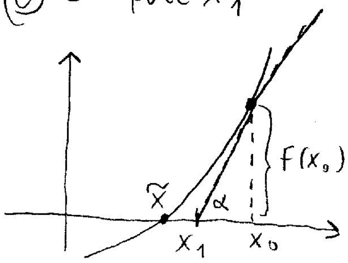
$x_0 = 2$

(B) Compute $f(x_0)$, and the slope of the graph of f at x_0, i.e. compute $f'(x_0)$



$$f'(x_0) = \frac{\Delta y}{\Delta x} = \tan\alpha$$

(C) Compute x_1



$$\underbrace{x_0 - x_1} = \frac{f(x_0)}{\tan\alpha} = \frac{f(x_0)}{f'(x_0)}$$

So $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

In our case: $x_1 = x_0 - \dfrac{x_0^2-2}{2x_0}$ (since $(x^2)' = 2x$) derivation

(D) Iterate this process

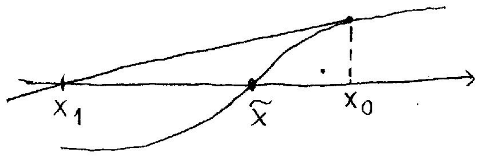
$$x_0 = 2, \quad x_{n+1} = x_n - \frac{x_n^2-2}{2x_n}$$

Until the required accuracy is achieved.

Remarks:

(A) Method finds only the root close to x_0.

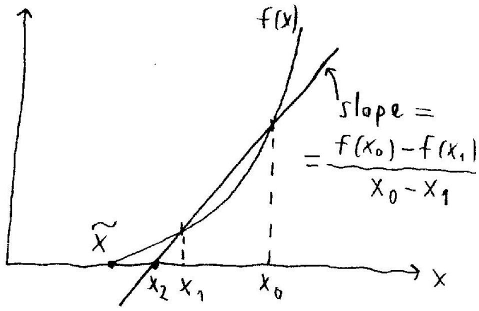
(B) Process might not converge to the root



x_1 is worse approximation than x_0

(2) Two point variant

What to do, when we do not know $f'(x)$.



$$x_2 = x_0 - \frac{f(x_0)}{\dfrac{f(x_0)-f(x_1)}{x_0-x_1}} =$$

$$= x_0 - \frac{f(x_0)(x_0-x_1)}{f(x_0)-f(x_1)}$$

So start with

$x_0 = 2, \ x_1 = 1$

and use the recursive law

$$x_{n+1} = x_{n-1} - \frac{f(x_{n-1})(x_{n-1}-x_n)}{f(x_{n-1})-f(x_n)}$$

2.5.1 Sample exercise

Compute $\sqrt[2]{2}$ by applying the method of Newton and Raphson!

- Approximate the function $f(x) = x^2 - 2$ by its tangent line at the initial guess $x_0 = 2$ for the root of the equation $x^2 - 2 = 0$. Then find the intersection of the tangent line with the x axis, and use it as your next x_1 guess for the root. Iterate this process until the error of the difference between the two sides of the equation $x^2 - 2 = 0$ is sufficiently small.

- The Newton-Raphson method requires the knowledge of the slope of $f(x)$. Let us suppose that we are not smart enough to get that right, so we erroneously believe that the slope of $x^2 - 2$ is $f'(x) = 2x \neq 3x$. The incorrect guess $f'(x) = 3x$ is still not completely unreasonable, it has a correct sign and is off the mark only by 50%. Use this not quite correct slope function, then study the error of the method. Is it seems to be true, that the error decays like a convergent (to zero) geometric sequence?

2.5.2 Exercise

- Repeat the previous exercise for $\sqrt[3]{2}$! The slope function (i.e. the f' derivative of f) of $f(x) = x^3 - 2$ is $f'(x) = 3x^2$.

- Use the two point Newton-Raphson method for the computation of $\sqrt[3]{2}$ with initial guesses $x_0 = 3$, $x_1 = 2$.

2.6 Continuous versus discrete dynamical systems. Differential equations

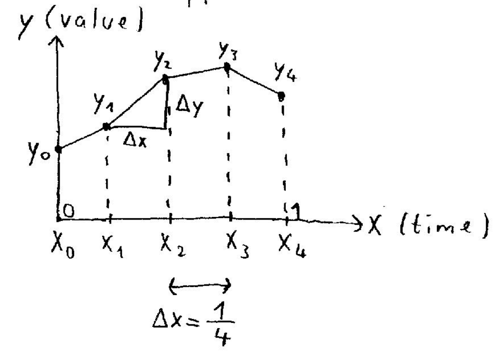
Continuous versus discrete dynamical systems

discrete: recursive sequences   cont.: differential equations

Diff. eq: $\dfrac{dy(x)}{dx} = f(x, y(x))$ Initial condition: $y(0) = y_0$

Problem: Find $y(1)$, or more generally, $y(x)$.

Numerical approximation (Euler method):



time	value	avarage velocity between x_n and x_{n+1}
$x_0 = 0 \cdot \Delta x = 0$	y_0	$(y_1 - y_0) \cdot \frac{1}{\Delta x}$
$x_1 = 1 \cdot \Delta x = \frac{1}{4}$	y_1	$(y_2 - y_1) \cdot \frac{1}{\Delta x}$
$x_2 = 2 \cdot \Delta x = \frac{2}{4}$	y_2	$(y_3 - y_2) \cdot \frac{1}{\Delta x}$
$x_3 = 3 \cdot \Delta x = \frac{3}{4}$	y_3	$(y_4 - y_3) \cdot \frac{1}{\Delta x}$
$x_4 = 4 \cdot \Delta x = 1$	y_4	

The velocity function $f(x, y(x))$ can be calculated if we know x and y, so use (for example) $f(x_1, y_1)$ for the average velocity on $[x_1; x_2]$:

$$\frac{y_2 - y_1}{\Delta x} = f(x_1, y_1) \Longrightarrow y_2 = y_1 + f(x_1, y_1) \cdot \Delta x,$$

and do the same for the other subintervals.

The recursive sequence approximating the solution of the diff. eq.:

x_0, y_0 are given, $x_{n+1} = x_n + \Delta x$, $\underbrace{y_{n+1}}_{\text{next value}} = \underbrace{y_n}_{\text{original value}} + \underbrace{f(x_n, y_n)}_{\text{velocity}} \cdot \underbrace{\Delta x}_{\text{elapsed time between } x_n \text{ and } x_{n+1}}$

given initial condition

time	value
$\boxed{x_0 \quad y_0}$	
$x_1 = x_0 + \Delta x$	$y_1 = y_0 + \boxed{f}(x_0, y_0) \cdot \Delta x$
$x_2 = x_1 + \Delta x$	$y_2 = y_1 + f(x_1, y_1) \cdot \Delta x$
$x_3 = x_2 + \Delta x$	$y_3 = y_2 + f(x_2, y_2) \cdot \Delta x$
$x_4 = x_3 + \Delta x$	$y_4 = y_3 + f(x_3, y_3) \cdot \Delta x$

given function

This method is reasonable if the velociti function does not change by too much in the subintervals, so for example $f(x_1, y_1) \approx f(x_2, y_2)$.

2.6.1 Sample exercise

Compute approximately $y(1)$ with Euler's method for the following:

$$\text{Diff.eq.: } y'(x) = y(x),\ y(0) = 1,$$
$$\text{Euler method: } N = 20,\ \Delta x = (1-0)/20 = 0.05,\ \ x_0 = 0,\ x_{n+1} = x_n + \Delta x,\ y_{n+1} = y_n + y_n \cdot \Delta x.$$

Then $y(1) = y(20 \cdot \Delta x) \approx y_{20}$. What is the error

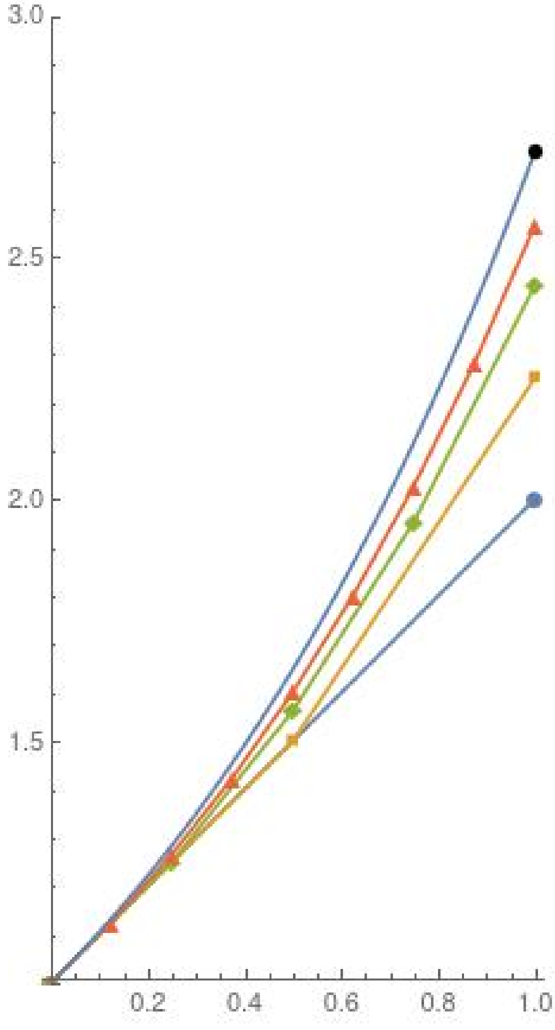
$$error(\Delta x) = |y_{20} - \exp(1)|$$

The exact solution of the DE is $y(x) = \exp(x) = e^x$.

2.6.2 Exercise

Repeat the previous exercise for $N = 1, 2, 4, 8$ and plot the obtained y_n sequences together with the function $\exp(x)$.

Remark: Your solution should look like this:



The black point at the end of the exact solution is at $(1, e) = (1,\ 2.71828182845904523 54\ldots)$.

2.6.3 Exercise (recognizing power like behaviour, error of the Euler method)

Make a table of the values

$$\left(\ \ n,\ k = 2^n,\ \ 1/k,\ \ (1+1/k)^k,\ \ |(1+1/k)^k - e|,\ \ \lg(1/k),\ \ lg\,|(1+1/k)^k - e|\ \ \right)$$

for $n = 0 \ldots 20$.

Plot the points ($\lg(1/k)$, $lg\,|(1+1/k)^k - e|$) ! Find the slope of the best fitting straight line (maybe only for the last few of the points) ! How does the error of Euler's method behaves in this case as a function of $\Delta x = 1/k$?

Remark: Assume that the error

$$error(\Delta x) = |(1+\Delta x)^{1/\Delta x} - e| = |(1+1/k)^k - e|$$

can be well approximated by $error(\Delta x) \approx c\Delta x^\alpha$, at least when Δx is small enough. How can we read off α from our data set?

If $err(\Delta x) = c\Delta x^\alpha$, then $\lg(err(\Delta x)) = \lg(c\Delta x^\alpha) = \lg(c) + \alpha \lg(\Delta x)$, so the plot of $\lg(\Delta x) \leftrightarrow \lg(err(\Delta x))$ is a straight line. So from the last two columns of the table, plot the last few rows (in the first few rows $\Delta x = 1/k$ is not very small), fit a line on these points, then the slope of this trend line will approximate α. (The exact value is 1 in for $\Delta x \to 0$.)

2.6.4 Sample exercise (computer algebra systems for diff.eqs.)

Solve for $y(1)$ and also plot $y(x)$ for a few DE like

- $y'(x) = 2 - 2y(x),\ y(0) = 2,$
- $y'(x) = \sin(y^3(x)) + x,\ y(0) = 0.5.$

Remarks:

- Explicit solution formulae are rarely available, however linear equations like the first one can be solved explicitly even for multicomponent systems.
- There are highly efficient numerical methods, they can treat nonlinear equations like the second example.

Solution:

WolframAlpha:

```
y'(x)=2-2y(x), y(0)=2
```

returns and the plot of $y(x) = e^{-2x} + 1$. I have not managed to force WA to get $y(1)$ in one step.

Unfortunately it is not completely trivial to use CAS systems to solve DE, I think there is not much point to get into this subject with no knowledge of programming these systems.