A. Test 2. Econ.Anal 13.dec.02. NEPTUN: Name:

1. Find the extremal value (and determine its type) of the function $f(x,y) = 2x^2 - xy + 2y^2 - 2x + 4y + 2x$!
   (a) Compute the partial derivatives of f up to second order!
   (b) Find the location of the extremal value!
   (c) Determine the type of the extremal value!
2. (a) Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!
      i. $\sqrt[3]{4x^8} + \sqrt[4]{(3x)^7} + \frac{8}{x^3}$
      ii. $\frac{3}{1+16x^2}$
      iii. $\cos 3x + \sin(-3x)$
   (b) Compute $\int_0^{\pi} \sin(-x)\,dx$!
3. (a) There are 10 black and 5 white balls in a box. Suppose that we DO put back the balls after the drawings.
      i. What is the chance of drawing firstly 3 white and then 2 black balls?
      ii. What is the chance of drawing 3 white and then 2 black balls if the order is irrelevant?
   (b) Consider a sample space Ω comprising four possible outcomes $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Suppose that to the four possible outcomes the following probabilities are assigned:

   $$P(\omega_1) = 2/10, \quad P(\omega_2) = 3/10, \quad P(\omega_3) = 1/10, \quad P(\omega_4) = 4/10.$$

   Define two events: $E = \{\omega_1, \omega_2\}$, $E = \{\omega_2, \omega_3\}$ Compute $P(E|F)$, the conditional probability of E given F !
4. (a) Compute the following improper integral! $\int_1^{\infty} 1/x^{77}\,dx$
   (b) There are two urns containing colored balls. The first urn contains 100 red balls and 0 blue balls. The second urn contains 10 red balls and 90 blue balls. One of the two urns is randomly chosen (both urns have probability 50% of being chosen) and then a ball is drawn at random from one of the two urns. If a red ball is drawn, what is the probability that it comes from the first urn?

A. Test 2. Econ.Anal 13.dec.02. NEPTUN: Name:

1. Find the extremal value (and determine its type) of the function $f(x,y) = 2x^2 - xy + 2y^2 - 2x + 4y + 2x$!
$= 2x^2 - xy + 2y^2 + 4y$

   (a) Compute the partial derivatives of f up to second order!

   (b) Find the location of the extremal value!

   (c) Determine the type of the extremal value!

ⓐ $f'_x = 4x - y + 0 + 0$

$f'_y = 0 - x + 4y + 4$

$f''_{xx} = 4 \quad f''_{xy} = f''_{yx} = -1 \quad f''_{yy} = 4$

(4points)

ⓑ $\left.\begin{array}{ll} f'_x = 0 & 4x - y = 0 \\ f'_y = 0 & -x + 4y + 4 = 0 \end{array}\right\} \; x = -\frac{4}{15}, \; y = \frac{-16}{15}$ $\quad P\left(-\frac{4}{15}, -\frac{16}{15}\right)$

③

ⓒ $\text{Hessian}(f) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}, \; (\text{Hessian}(f))(P) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

$\begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 4 \cdot 4 - (-1)^2 = 15 > 0$, so the extremal value is either MIN or MAX.

③

$4 > 0 \longrightarrow$ $\boxed{\text{MINIMUM}}$

$\begin{pmatrix} \boxed{4} & -1 \\ -1 & 4 \end{pmatrix}$

2. (a) Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!

   i. $\sqrt[3]{4x^8} + \sqrt[4]{(3x)^7} + \frac{8}{x^8}$

   ii. $\frac{3}{1+16x^2}$

   iii. $\cos 3x + \sin(-3x)$

   (b) Compute $\int_0^\pi \sin(-x)\,dx$!

(3 + 2 + 2) + 3 points

ⓐ i) $\int \sqrt[3]{4x^8} + \sqrt[4]{(3x)^7} + \frac{8}{x^8}\,dx = \int \sqrt[3]{4}\, x^{8/3} + (3x)^{7/4} + 8 \cdot x^{-8}\,dx$

$= \sqrt[3]{4}\,\frac{x^{11/3}}{11/3} + \frac{\frac{(3x)^{11/4}}{11/4}}{3} + 8 \cdot \frac{x^{-7}}{-7} + C$

ii) $\int \frac{3}{1+16x^2}\,dx = 3 \cdot \int \frac{1}{1+(4x)^2}\,dx = 3 \cdot \frac{\operatorname{arctg}(4x)}{4} + C$

iii) $\int \cos(3x) + \sin(-3x)\,dx = \frac{\sin(3x)}{3} + \frac{-\cos(-3x)}{-3} + C$

ⓑ $\int_0^\pi \sin(-x)\,dx = \left[\frac{-\cos(-x)}{-1}\right]_0^\pi = \left[\cos(-x)\right]_0^\pi =$

$= \cos(-\pi) - \cos(-0) = (-1) - 1 = -2$

3. (a) There are 10 black and 5 white balls in a box. Suppose that we DO put back the balls after the drawings.

i. What is the chance of drawing firstly 3 white and then 2 black balls?

ii. What is the chance of drawing 3 white and then 2 black balls if the order is irrelevant?

(b) Consider a sample space Ω comprising four possible outcomes $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Suppose that to the four possible outcomes the following probabilities are assigned:

$$P(\omega_1) = 2/10, \quad P(\omega_2) = 3/10, \quad P(\omega_3) = 1/10, \quad P(\omega_4) = 4/10.$$

Define two events: $E = \{\omega_1, \omega_2\}$, ~~$E$~~ $F = \{\omega_2, \omega_3\}$ Compute $P(E|F)$, the conditional probability of E given F !

a) i) $p(WWWBB) = \frac{5}{15} \cdot \frac{5}{15} \cdot \frac{5}{15} \cdot \frac{10}{15} \cdot \frac{10}{15}$ 3

ii) $p(3\text{ White}, 2\text{ Black}) = p(WWWBB) \cdot \binom{5}{3} = \left(\frac{5}{15}\right)^3 \cdot \left(\frac{10}{15}\right)^2 \cdot \binom{5}{3}$

$\frac{5!}{3!(5-3)!}$

3

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b) $$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(\{\omega_2\})}{P(\{\omega_1, \omega_2\})} =$$

$$= \frac{3/10}{\frac{3}{10} + \frac{1}{10}} = \frac{3}{4} = 75\%$$ 4

$(3+3) + 4$ points

4. (a) Compute the following improper integral! $\int_1^\infty 1/x^{77}\,dx$

(b) There are two urns containing colored balls. The first urn contains 100 red balls and 0 blue balls. The second urn contains 10 red balls and 90 blue balls. One of the two urns is randomly chosen (both urns have probability 50% of being chosen) and then a ball is drawn at random from one of the two urns. If a red ball is drawn, what is the probability that it comes from the first urn?

ⓐ $$\int_1^\infty \frac{1}{x^{77}}\,dx = \lim_{R\to\infty} \int_1^R x^{-77}\,dx = \lim_{R\to\infty}\left[\frac{x^{-76}}{-76}\right]_1^R =$$

$$= \lim_{R\to\infty}\left(-\frac{1}{76}\cdot\underbrace{\frac{1}{R^{76}}}_{\to 0}\right) - \left(-\frac{1}{76}\cdot\frac{1}{1^{76}}\right) = \frac{1}{76}$$

(4)

ⓑ $P(\mathrm{I}\mid R) = ?$ Bayes: $P(\mathrm{I}\mid R) = \dfrac{P(R\mid \mathrm{I})\cdot P(\mathrm{I})}{P(R)}$

1st urn red

$P(R\mid \mathrm{I}) = 100\%$, $P(\mathrm{I}) = 50\%$

$$P(R) = P(R\mid \mathrm{I})\cdot P(\mathrm{I}) + P(R\mid \mathrm{II})\cdot P(\mathrm{II}) =$$
$$= 1\cdot 0.5 + 0.1\cdot 0.5 = 0.55$$

$$P(\mathrm{I}\mid R) = \frac{1\cdot 0.5}{0.55} = \frac{10}{11}$$

(6)

C. Test 2. Econ.Anal 13.dec.09. NEPTUN: Name:

1. (a) Let X be a discrete random variable. Let its support R_X be: $R_X = \{0, 1, 2, 3\}$. Let its probability mass function be:

$$p(x) = \begin{cases} 1/4 & \text{if } x \in R_X \\ 0 & \text{if } x \ni R_X. \end{cases}$$

Compute the mean and the variance of X !

   (b) What is the chance of winning the lottery grand prize if you need to guess 6 numbers correctly out of 60? (Do not compute the numerical answer!)

2. Find the extremal value (and determine its type) of the function $f(x, y) = -x^2 + xy - 2y^2 - 2x + 4y + 2x$!
   (a) Compute the partial derivatives of f up to second order!
   (b) Find the location of the extremal value!
   (c) Determine the type of the extremal value!

3. (a) Compute the $f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy}$ partial derivatives of the following function!

$$f = y^2/x^3. \tag{1}$$

   (b) Suppose that we toss a fair dice two times. The number of heads is counted by the random variable X. Compute the variance of X !

4. (a) Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!
      i. $\sqrt[4]{4x^8} + \sqrt[5]{(3x)^7} + \frac{8}{x^8}$
      ii. $\frac{2}{1+4x^2}$
      iii. $e^{3x} + \sin(-3x)$

   (b) Let X be a discrete random variable. Let its support R_X be: $R_X = \{1, 2, 3\}$. Let its probability mass function be:

$$p(x) = \begin{cases} x/6 & \text{if } x \in R_X \\ 0 & \text{if } x \notin R_X. \end{cases}$$

Compute the mean of X.

C. Test 2. Econ.Anal 13.dec.09. NEPTUN: Name:

1. (a) Let X be a discrete random variable. Let its support R_X be: $R_X = \{0,1,2,3\}$. Let its probability mass function be:

$$p(x) = \begin{cases} 1/4 & \text{if } x \in R_X \\ 0 & \text{if } x \ni R_X. \end{cases}$$

Compute the mean and the variance of X !

(b) What is the chance of winning the lottery grand prize if you need to guess 6 numbers correctly out of 60? (Do not compute the numerical answer!)

3

a mean: $E[X] = \sum_i p_i X_i = \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 3 = 1.5$

Variance: $Var[X] = E[(X - E[X])^2] = \sum_i p_i (X_i - \bar{X})^2 =$

$= \frac{1}{4}(0-1.5)^2 + \frac{1}{4}(1-1.5)^2 + \frac{1}{4}(2-1.5)^2 + \frac{1}{4}(3-1.5)^2$ 3

$= \frac{5}{4}$

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b $N = \binom{60}{6} = \frac{60!}{6!(60-6)!} = \frac{60\cdot 59\cdot 58\cdot 57\cdot 56\cdot 55}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$

4

$P_{jackpot} = \frac{1}{N} = \frac{1}{\binom{60}{6}}$

2. Find the extremal value (and determine its type) of the function $f(x,y) = -x^2 + xy - 2y^2 - 2x + 4y + 2x$!

$$= -x^2 + xy - 2y^2 + 4y$$

   (a) Compute the partial derivatives of f up to second order!

   (b) Find the location of the extremal value!

   (c) Determine the type of the extremal value!

(a) $f'_x = -2x + y - 0 + 0$, $f'_y = 0 + x - 4y + 4$

$f''_{xx} = -2$ $\quad f''_{xy} = f''_{yx} = 1$ $\quad f''_{yy} = -4$ (4)

(b) $f'_x = 0 \quad -2x + y = 0$

$f'_y = 0 \quad x - 4y + 4 = 0$

$\Big\} \to x = \frac{4}{7} \quad y = \frac{8}{7} \quad P\left(\frac{4}{7}, \frac{8}{7}\right)$

(3)

(c) $\text{Hessian}(f) = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}$, $\big(\text{Hessian}(f)\big)(P) = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}$

$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = (-2)\cdot(-4) - 1^2 = 7 > 0$, so there is either a MIN or MAX at P.

$\begin{pmatrix} \boxed{-2} & 1 \\ 1 & -4 \end{pmatrix}$ (2)

$\to -2 < 0 \longrightarrow$ MAXIMUM

3. (a) Compute the $f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy}$ partial derivatives of the following function!

$$f = y^2/x^3. \qquad (1)$$

(b) Suppose that we toss a fair ~~dice~~ coin two times. The number of heads is counted by the random variable X. Compute the variance of X !

a $\left(\frac{y^2}{x^3}\right) = \left(y^2 \cdot x^{-3}\right) = f$

$f'_x = y^2 \cdot (-3) x^{-4}$, $f'_y = 2y \cdot x^{-3}$

$f''_{xx} = y^2 \cdot (-3) \cdot (-4) x^{-5} = 12 y^2 x^{-5}$ 6

$f''_{yy} = (2y \cdot x^{-3})'_y = 2 \cdot x^{-3}$

$f''_{xy} = f''_{yx} = (2y \cdot x^{-3})'_x = 2y \cdot (-3) x^{-4}$

b $\Omega = \{HH, HT, TH, TT\}$ with equal $\frac{1}{4}$ probability.

$X(HH) = 2$, $X(HT) = X(TH) = 1$, $X(TT) = 0$

$\overline{X} = E[X] = \sum_i p_i X_i = \frac{1}{4} \cdot 2 + \left(\frac{1}{4} + \frac{1}{4}\right) \cdot 1 + \frac{1}{4} \cdot 0$ 4

$Var[X] = E\left[(X - \overline{X})^2\right] = \sum_i p_i (X_i - \overline{X})^2 =$

$= \frac{1}{4}(2-1)^2 + \frac{1}{2}(1-1)^2 + \frac{1}{4}(0-1)^2 = \frac{1}{2}$

Or: Compute the variance of the num. of heads for a single toss: $Var[X_{single}] = \frac{1}{2}\left(1 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(0 - \frac{1}{2}\right)^2 = \frac{1}{4}$.

As $X = X_{first} + X_{second}$ and X_{first}, X_{second} are independent,

so $Var[X] = Var[X_{first}] + Var[X_{second}] = 2 \cdot Var[X_{single}]$.

(Here X_{first} counts the num. of heads for the 1^{st} toss, etc.)

4. (a) Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!

   i. $\sqrt[4]{4x^8} + \sqrt[5]{(3x)^7} + \frac{8}{x^6}$
   ii. $\frac{2}{1+4x^2}$
   iii. $e^{3x} + \sin(-3x)$

(b) Let X be a discrete random variable. Let its support R_X be: $R_X = \{1, 2, 3\}$. Let its probability mass function be:

$$p(x) = \begin{cases} x/6 & \text{if } x \in R_X \\ 0 & \text{if } x \notin R_X. \end{cases}$$

Compute the mean of X.

(a) i) $\int \sqrt[4]{4x^8} + \sqrt[5]{(3x)^7} + \frac{8}{x^6}\,dx =$

$= \int \sqrt[4]{4}\; x^{\frac{8}{4}} + (3x)^{7/5} + 8 \cdot x^{-6}\,dx =$

$= \sqrt[4]{4}\; \frac{x^{12/4}}{\underbrace{12/4}_{=3}} + \frac{\frac{(3x)^{12/5}}{12/5}}{3} + 8 \cdot \frac{x^{-5}}{-5} + C$ (3)

ii) $\int \frac{2}{1+4x^2}\,dx = 2 \cdot \int \frac{1}{1+(2x)^2}\,dx = 2 \cdot \frac{\operatorname{arctg}(2x)}{2} + C$ (2)

iii) $\int e^{3x} + \sin(-3x)\,dx = \frac{e^{3x}}{3} + \frac{-\cos(-3x)}{-3} + C$ (2)

(b) $E[X] = \sum_i p_i X_i = \frac{1}{6} \cdot 1 + \frac{2}{6} \cdot 2 + \frac{3}{6} \cdot 3 = \frac{14}{6} = \frac{7}{3}$

(3)