1

(4) $f(x,y)=x^2y^5$, $f'_x=2xy^5$, $f'_y=x^2\cdot 5y^4$,

$$f''_{xx}=(2xy^5)'_x=2y^5,\quad f''_{xy}=f''_{yx}=(2xy^5)'_y=2x\cdot 5y^4$$

$$f''_{yy}=(x^2\cdot 5y^4)'_y=x^2\cdot 5\cdot 4y^3$$

(5) $f(x,y)=\sin(x+y^2)$, $f'_x=\cos(x+y^2)\cdot(x+y^2)'_x=\cos(x+y^2)$

$$f'_y=\cos(x+y^2)\cdot(x+y^2)'_y=\cos(x+y^2)\cdot 2y$$

$$f''_{xx}=-\sin(x+y^2)\cdot(x+y^2)'_x=-\sin(x+y^2)$$

$$f''_{xy}=[\cos(x+y^2)]'_y=-\sin(x+y^2)\cdot(x+y^2)'_y=-\sin(x+y^2)\cdot 2y=f''_{yx}$$

$$f''_{yy}=(f'_y)'=[\cos(x+y^2)]'_y\cdot 2y+\cos(x+y^2)\cdot[2y]'=$$

$$=[-\sin(x+y^2)\cdot 2y]\cdot 2y+\cos(x+y^2)\cdot 2$$

(7) $f(x,y)=x/y$, $f'_x=(\frac{1}{y}\cdot x)'_x=\frac{1}{y}$, $f'_y=(x\cdot y^{-1})'_y=x\cdot(-1)\,y^{-2}=-xy^{-2}$

$$f''_{xx}=\left(\frac{1}{y}\right)'_x=0,\quad f''_{xy}=f''_{yx}=\left(\frac{1}{y}\right)'_y=(y^{-1})'_y=-1\cdot y^{-2}$$

$$f''_{yy}=(-xy^{-2})'_y=-x\cdot(-2)\cdot y^{-3}$$

2 (12) $f(x,y)=x^2-xy+4y^2-2x+4y$

$$f'_x=2x-y+0-2+0$$

$$f'_y=0-x+8y-0+4$$

Location of the extremal value:

$$\left.\begin{aligned} f'_x&=0=2x-y-2\\ f'_y&=0=-x+8y+4\end{aligned}\right\}\quad (x,y)=\left(\frac{12}{15},-\frac{6}{15}\right)$$

Type: $f''_{xx}=2$, $f''_{xy}=-1$, $f''_{yy}=8$

$$\text{Hessian}(f)=\begin{pmatrix}2&-1\\-1&8\end{pmatrix},\quad \left(\text{Hessian}(f)\right)\left(\frac{12}{15},-\frac{6}{15}\right)=\begin{pmatrix}2&-1\\-1&8\end{pmatrix}$$

$\det\begin{pmatrix}\boxed{2}&-1\\-1&8\end{pmatrix}=2\cdot 8-(-1)^2=17>0 \Rightarrow$ MINIMUM or MAXIMUM

(if <0 $\rightarrow$ SADDLE POINT)

$2>0 \Rightarrow$ MINIMUM

(if <0 $\rightarrow$ MAXIMUM)

3 (14) $\int 5\,dx = 5x + C$

(16) $\int x^2 - 2\,dx = \frac{x^3}{3} - 2x + C$

(17) $\int x^2 - y\,dx = \frac{x^3}{3} - yx + C$ (here y is not a variable)

(18) $\int \sqrt[3]{2x^7} + \sqrt[3]{(2x)^7} + \frac{7}{x^7}\,dx = \int \sqrt[3]{2}\,x^{7/3} + (2x)^{7/3} + 7 \cdot x^{-7}\,dx$

$$= \sqrt[3]{2}\,\frac{x^{10/3}}{10/3} + \frac{\frac{(2x)^{10/3}}{10/3}}{2} + 7 \cdot \frac{x^{-6}}{-6} + C$$

(19) $\int e^x + \sin x\,dx = e^x - \cos x + C$

(20) $\int e^{3x} + \sin(3x)\,dx = \frac{e^{3x}}{3} - \frac{\cos(3x)}{3} + C$

(21) $\int \frac{3}{1+9x^2}\,dx = \int 3 \cdot \frac{1}{1+(3x)^2}\,dx = 3 \cdot \frac{\operatorname{arctg}(3x)}{3} + C$

---

4 (23) $y' = x+1 \longrightarrow y = \int x+1\,dx = \frac{x^2}{2} + x + C$ (general solution)

$y(2) = 3 \longrightarrow \frac{2^2}{2} + 2 + C = 3 \longrightarrow C = -1$

$y = \frac{x^2}{2} + x + (-1)$ (particular solution)

---

(27) $y' = \frac{1}{x+1} \longrightarrow y = \int \frac{1}{x+1}\,dx = \ln|x+1| + C$

$y(2) = 3 \longrightarrow 3 = \ln|2+1| + C \longrightarrow C = 3 - \ln 3$

$y = \ln|x+1| + (3 - \ln 3)$

(Remark: In this case it is more sensible to say that $y(x)$ is defined only for $x > -1$)

---

5 $y' = y$

(28) $\frac{dy}{dx} = y$

$\frac{dy}{y} = dx$

$\int \frac{dy}{y} = \int dx$

$\ln|y| = x + C$

$e^{\ln|y|} = e^{x+C}$

$|y| = e^{x+C}$

$y = \pm e^{x+C} = \pm e^C \cdot e^x$

or $y = \tilde{C} e^x$ (gen. sol.)

$y(2) = 3$

$3 = \tilde{C} e^2$

$\tilde{C} = \frac{3}{e^2}$

$y(x) = \frac{3}{e^2} \cdot e^x$

(part. sol.)

(5) (33)

$$y' = 2y + 5$$
$$\frac{dy}{dx} = 2y + 5$$
$$\frac{dy}{2y+5} = dx$$
$$\int \frac{dy}{2y+5} = \int dx$$
$$\frac{\ln|2y+5|}{2} = x + C$$

$$\ln|2y+5| = 2(x+C)$$
$$|2y+5| = e^{2(x+C)}$$
$$2y+5 = \pm e^{2(x+C)}$$
$$y = \frac{\pm e^{2(x+C)} - 5}{2}$$

or

$$y = \tilde{C} e^{2x} - \frac{5}{2}$$

$$y(2) = 3$$
$$\tilde{C} e^{2\cdot 2} - \frac{5}{2} = 3$$
$$\tilde{C} = \frac{3 + 5/2}{e^4}$$
$$y = \frac{3 + 5/2}{e^4} \cdot e^{2x} - \frac{5}{2}$$

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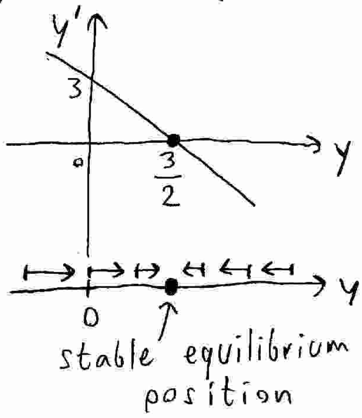
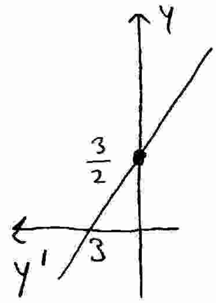
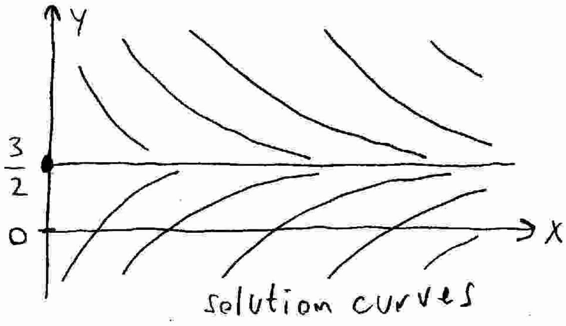
(6) (37) $y' = -2y + 3$ fixpoint: $y' = 0 = -2y + 3 \longrightarrow y_{fix} = \frac{3}{2}$







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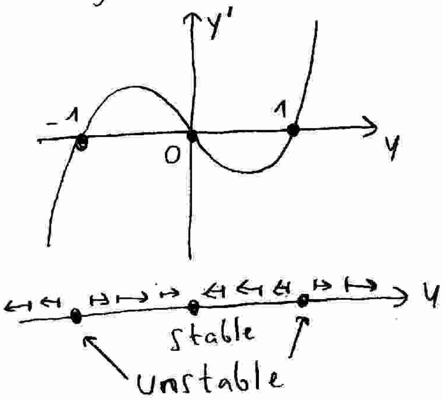
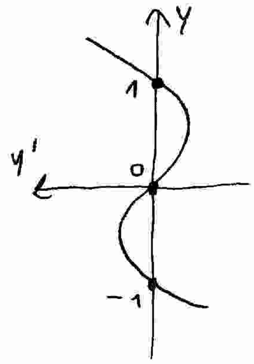
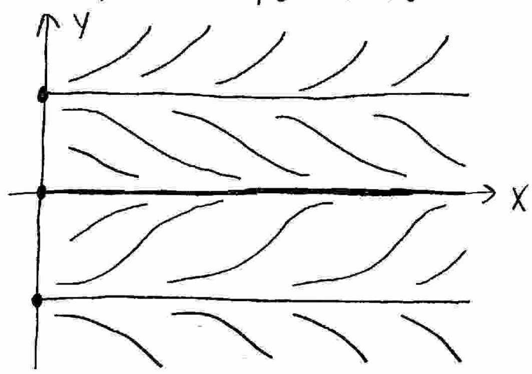
(41) $y' = y^3 - y = (y+1)(y-1)y$ fixpoints: $y_1 = -1$, $y_2 = 0$, $y_3 = 1$







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(5) (33) Second solution: $y' = 2y + 5$

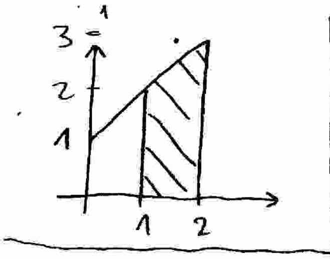
fixed point: $y' = 0 = 2y + 5 \longrightarrow y_{fix} = -\frac{5}{2}$

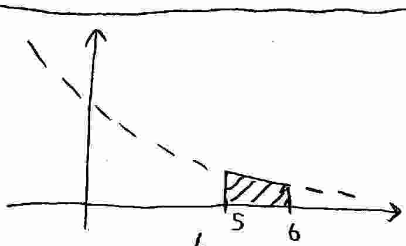
$\Delta y = y - y_{fix} = y - \left(-\frac{5}{2}\right)$. Then $(\Delta y)' = y'$, so $(\Delta y)' = 2\big(\overbrace{\Delta y + \left(-\tfrac{5}{2}\right)}^{y}\big) + 5 = 2\Delta y$.

Consequently $\Delta y = C \cdot e^{2x}$,

i.e. $y = \Delta y + y_{fix} = C \cdot e^{2x} - \frac{5}{2}$

(7) (43) $\int_1^2 x+1\,dx = \left[\frac{x^2}{2}+x\right]_1^2 = \left(\frac{2^2}{2}+2\right)-\left(\frac{1^2}{2}+1\right) = 2\frac{1}{2}$





(45) $\int_5^6 \frac{1}{x+5}\,dx = \left[\ln|x+5|\right]_5^6 = \ln 11 - \ln 10$

(48) $\int_0^{\pi} \cos(2x)\,dx = \left[\frac{\sin(2x)}{2}\right]_0^{\pi} = \frac{\sin(2\cdot\pi)}{2} - \frac{\sin(2\cdot 0)}{2} = 0-0=0$

(8). (50) $\int_1^{\infty} \frac{1}{x+1}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x+1}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_1^R =$

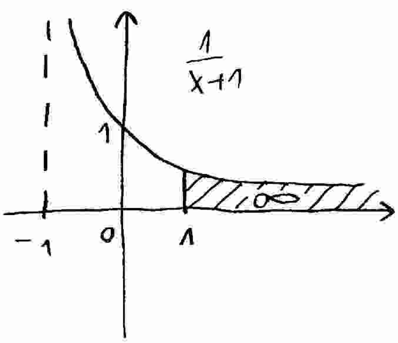
$= \lim_{R\to\infty} \ln R - \ln 1 = \infty - 0 = \infty$

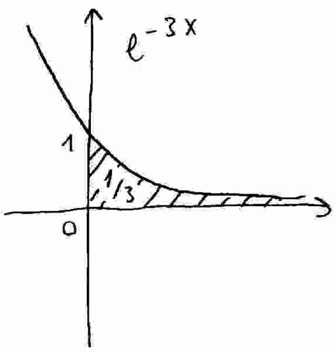
(51) $\int_0^{\infty} e^{-3x}\,dx = \lim_{R\to\infty} \int_0^R e^{-3x}\,dx = \lim_{R\to\infty} \left[\frac{e^{-3x}}{-3}\right]_0^R =$

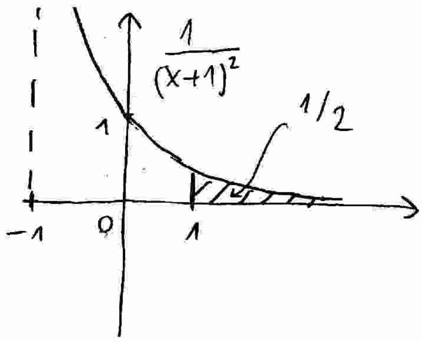
$= \lim_{R\to\infty} \frac{e^{-3R}}{-3} - \frac{e^{-3\cdot 0}}{-3} = \frac{0}{-3} - \frac{1}{-3} = \frac{1}{3}$

(52) $\int_1^{\infty} (x+1)^{-2}\,dx = \lim_{R\to\infty} \int_1^R (x+1)^{-2}\,dx = \lim_{R\to\infty} \left[\frac{(x+1)^{-1}}{-1}\right]_1^R =$

$= \lim_{R\to\infty} \frac{(R+1)^{-1}}{-1} - \frac{(1+1)^{-1}}{-1} = \frac{0}{-1} - \frac{2}{-1} = \frac{1}{2}$







Probability theory.

2. $p=\frac{1}{N}$, where $N=\frac{90\cdot 89\cdot 88\cdot 87\cdot 86}{1\cdot 2\cdot 3\cdot 4\cdot 5}=\frac{90!}{5!(90-5)!}=\binom{90}{5}$

Here 90 = num. of possibilities for the first number
89 = — " — second — " —
⋮

Divide by 5!, as the order of the winning numbers is irrelevant.

---

3. $p(WWWBB)=\frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13}\cdot\frac{10}{12}\cdot\frac{9}{11}$

at the 2^{nd} drawing there are 14 balls in the box, 4 of them are white

at the 4^{th} drawing there are 12 balls in the box, 10 of them are white

If the order does not matter, multiply this by $\binom{5}{3}=\binom{5}{2}$ ← the number of 5 letter words containing 3 W and 2 B.

So $p(\text{3 white and 2 black})=p(WWWBB)\cdot\binom{5}{3}=$

$$=\frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13}\cdot\frac{10}{12}\cdot\frac{9}{11}\cdot\frac{5!}{2!\,3!}=$$

$$=\frac{\binom{10+5}{5}}{\binom{5}{3}\binom{10}{2}}$$

black, white, num. of drawings, black, num. of blacks drawn

---

4. $p(WWWBB)=\frac{5}{15}\cdot\frac{5}{15}\cdot\frac{5}{15}\cdot\frac{10}{15}\cdot\frac{10}{15}\cdot\frac{10}{15}=\left(\frac{5}{15}\right)^3\cdot\left(\frac{10}{15}\right)^2$

$p(\text{3 white and 2 black})=p(WWWBB)\cdot\binom{5}{3}=$

$$=\binom{5}{3}\cdot\left(\frac{5}{15}\right)^3\cdot\left(\frac{10}{15}\right)^2$$

---

8. A and B independent $\Leftrightarrow p(A\cap B)=p(A)p(B)$

$p(\text{girl}\cap\text{brown})=p(\text{girl})\cdot p(\text{brown})=0.6\cdot 0.4=0.24$

Number of brown haired girls $=0.24\cdot 100=24$

11. Possible values of the random variable X : $\{1,2,3,4,5,6\}$

All of them have $\frac{1}{6}$ probability, so

$$E[X] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = 3.5$$

---

12. $Var[X] = E[\cancel{(}(X - E[X])^2] =$

$$= \frac{1}{6}\cdot(1-3.5)^2 + \frac{1}{6}\cdot(2-3.5)^2 + \frac{1}{6}\cdot(3-3.5)^2 + \ldots + \frac{1}{6}\cdot(3-6)^2 = 2.91$$

So $Var[X] \approx 2.91 \approx \sigma^2 \approx (1.70)^2$ [$\sigma \approx 1.70$: standard deviation]

---

15. Sample space $\Omega = \{HH, HT, TH, TT\}$, each prob. is $\frac{1}{4}$.

$p(X_1=1) = p(\{HH, HT\}) = \frac{1}{2}$ | $p(X_2=1) = p(\{HH, TH\}) = \frac{1}{2}$

$p(X_1=0) = p(\{TH, TT\}) = \frac{1}{2}$ | $p(X_2=0) = p(\{HT, TT\}) = \frac{1}{2}$

$$p(X_1+X_2=2) = p(\{HH\}) = \frac{1}{4}$$
$$p(X_1+X_2=1) = p(\{HT, TH\}) = \frac{1}{2}$$
$$p(X_1+X_2=0) = p(\{TT\}) = \frac{1}{4}$$

$E[X_1] = E[X_2] = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{2}$, $E[X_1+X_2] = E[X_1] + E[X_2] = 1$

$$Var[X_1] = Var[X_2] = \frac{1}{2}\left(1-\frac{1}{2}\right)^2 + \frac{1}{2}\left(0-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$Var[X_1+X_2] = \frac{1}{4}(2-1)^2 + \frac{1}{2}(1-1)^2 + \frac{1}{4}(0-1)^2 = \frac{1}{2}$$

So $Var[X_1+X_2] = Var[X_1] + Var[X_2] = \frac{1}{2} = \frac{1}{4} + \frac{1}{4}$, as it should be, since X_1 and X_2 are independent random variables.

16. single voter: $X_H = X_{\text{Huge Party}}$ $X_S = X_{\text{Small Party}}$

$E[X_H] = 0.5 \cdot 1 + 0.5 \cdot 0 = 0.5$

$Var[X_H] = 0.5 \cdot (1-0.5)^2 + 0.5 \cdot (0-0.5)^2 = 0.25$

$E[X_S] = 0.1 \cdot 1 + 0.9 \cdot 0 = 0.1$

$Var[X_S] = 0.1 \cdot (1-0.1)^2 + 0.9 \cdot (0-0.1)^2 = 0.09$

1000 voter: $X_H^{\text{Poll}} = X_H^{(1)} + \ldots + X_H^{(1000)}$,

where $X_H^{(i)}$ is the random variable describing the answer of the i^{th} person of the poll. $X_H^{(i)}$ and X_H have the same expectation value and variance, and $X_H^{(i)}$ and $X_H^{(j)}$ are independent of each other for $i \neq j$.

Consequently $E[X_H^{\text{poll}}] = 1000 \cdot E[X_H] = 500$

$Var[X_H^{\text{poll}}] = 1000 \cdot Var[X_H] = 250 \approx 16^2$

The standard deviation of X_H^{poll} is about 16, so $\frac{16}{1000}$ is the "typical" error of the measured popularity of the Huge Party (compared to the exact 50%).

So the result (in %) is likely to fall in the (50% − 1.6%, 50% + 1.6%) interval.

[With $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx 68\%$ probability.

$\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx 95\%$]

Similary $E[X_S^{\text{poll}}] = 1000 \cdot E[X_S] = 100$

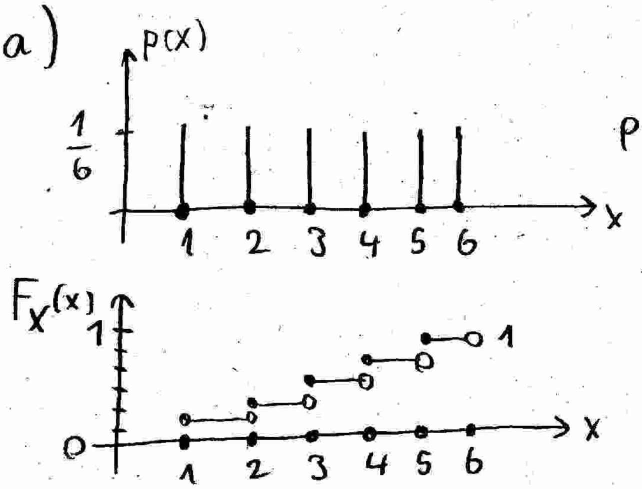
$Var[X_S^{\text{poll}}] = 1000 \cdot Var[X_S] = 90 \approx 9.5^2$

So the measured popularity of the Small Party is likely (with about 68% chance) to fall in the

(10% − 0.95%, 10% + 0.95%) interval.

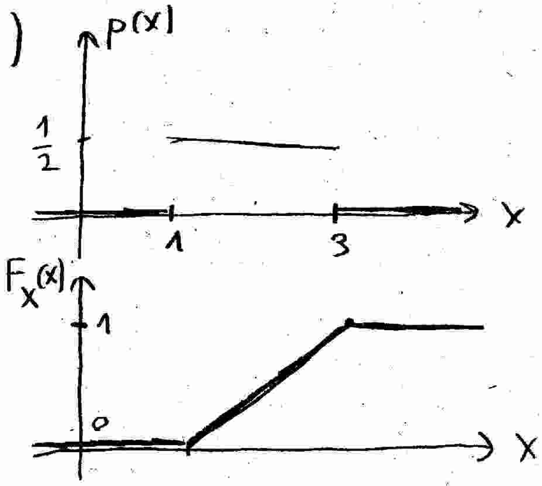
$\uparrow \frac{9.5}{1000}$

17. a)

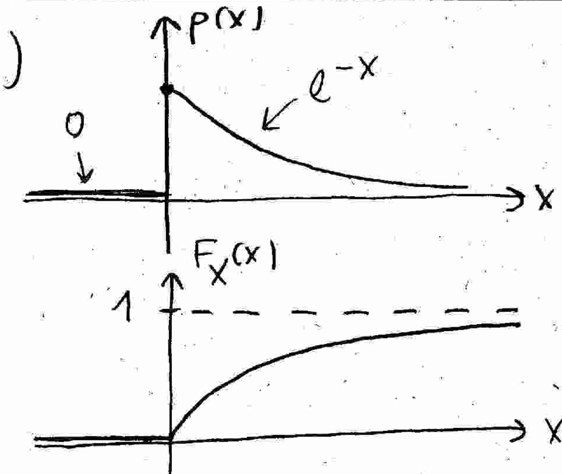


probability mass function

b)



c)



$$F_X(x) = \int_{-\infty}^{x} p(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \int_0^x e^{-z} dz & \text{if } x \geq 0 \end{cases}$$

$$\int_0^x e^{-z} dz = \left[\frac{e^{-z}}{-1}\right]_0^x =$$

$$= \frac{e^{-x}}{-1} - \frac{e^{-0}}{-1} = 1 - e^{-x}$$