A. Compute $f'_x(x,y), f'_y(x,y), f''_{yx}(x,y)$!

- $f(x,y) = x/y$,
- $f(x,y) = e^{xy^2}$.

B. Compute the following indefinite integrals!

- $\int e^{-x} \cdot x \, dx$
- $\int e^{-x^2} \cdot x \, dx$

C. Let

$$f(x) = \begin{cases} 4 - x, & \text{if } x \in [0,2], \\ 2, & \text{otherwise.} \end{cases}$$

Draw a plot of f !
Compute the definite integrals of f on the following intervals:

$$[1,1],\ [0,2],\ [1,3],\ [3,4].$$

(Try to solve this exercise by computing the areas of simple geometric shapes!)

D. Let $f(x,y) = x^2 - xy + 2y^2 - 2x + 3$. Find and classify the critical point of f !

E.

- There are 4 black and 7 white balls in a box. Suppose that we DO put back the balls after the drawings. What is the chance of drawing firstly 6 white and then 2 black balls? What is the chance of drawing 5 white and then 3 black balls if the order is irrelevant?
- Suppose that we roll a fair dice. Six numbers (from 1 to 6) can appear face up with equal chances. So our sample space is: $\Omega = \{1,2,3,4,5,6\}$. Define the events E and F as follows: $E = \{2,4\}$, $F = \{3,4,5,6\}$. Are E and F independent? Prove your answer!
  Now roll the dice twice. What is the chance that both E and F will never happen?
- There were two boxes, one containing 10 silver and 10 gold coins, while the other had 18 gold and only two silver coins. I was allowed to draw a random coin from the box of my choice. I got a silver coin, so I took the other box. What was my chance that I picked the box almost full of gold?

F. Toss a fair coin thrice. What are the chances of getting 0, 1, 2, 3 tails? What is the expected value and the variance of the number of tails?

MAX 39
pass : 19–39
fail : 0–18

Test.2. Econ.Anal. 24.dec.3. NEPTUN: Name:

4 A. Compute $f'_x(x,y), f'_y(x,y), f''_{yx}(x,y)$! $f'_x = \frac{1}{y},\ f'_y = -\frac{x}{y^2},\ f''_{yx} = -\frac{1}{y^2}$

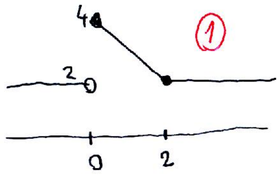
- $f(x,y) = x/y$, (2)
- $f(x,y) = e^{xy^2}$. (2) $f'_x = e^{xy^2}\cdot y^2,\ f'_y = e^{xy^2}\cdot 2yx,\ f'_{yx} = e^{xy^2}\cdot 2y\cdot y^2\cdot x + e^{xy^2}\cdot 2y x$

5 B. Compute the following indefinite integrals!

- (3) $\int e^{-x}\cdot x\,dx = \left| \begin{matrix} f' = e^{-x} & g = x \\ f = -e^{-x} & g' = 1 \end{matrix} \right| = -e^{-x}\cdot x - \int -e^{-x}\,dx = -e^{-x}\cdot x - e^{-x} + C$
- (2) $\int e^{-x^2}\cdot x\,dx = -\frac{1}{2}e^{-x^2} + C$

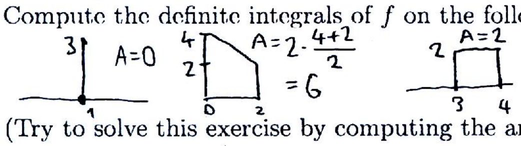
6 C. Let

$$f(x) = \begin{cases} 4 - x, & \text{if } x \in [0,2], \\ 2, & \text{otherwise.} \end{cases}$$



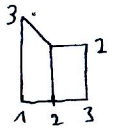
Draw a plot of f !

Compute the definite integrals of f on the following intervals:



$[1,1]$, $[0,2]$, $[1,3]$, $[3,4]$. (1)

0 (1), 6 (1), $1\cdot\frac{3+2}{2} + 1\cdot 2 = 4.5$ (2), 2



(Try to solve this exercise by computing the areas of simple geometric shapes!)

$f'_x = 2x - y - 2,\ f'_y = -x + 4y \quad f'_x = f'_y = 0 \Rightarrow \begin{matrix} 2x - y - 2 = 0 \\ -x + 4y = 0 \end{matrix}$ (2) $\Rightarrow \begin{matrix} x = 8/7 \\ y = 2/7 \end{matrix} \quad P_{crit} = (8/7, 2/7)$ (1)

7 D. Let $f(x,y) = x^2 - xy + 2y^2 - 2x + 3$. Find and classify the critical point of f !

$\mathrm{Hessian}(f)(P_{crit}) = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$ (2) $\quad 2\cdot 4 - (-1)^2 = 7 > 0 \Rightarrow$ MIN or MAX, $2 > 0 \Rightarrow$ MIN (2)

E. $p(WWWWWWBB) = \left(\frac{7}{11}\right)^6\cdot\left(\frac{4}{11}\right)^2$ (2) $\quad p(6W + 2B) = \left(\frac{7}{11}\right)^6\cdot\left(\frac{4}{11}\right)^2\cdot\binom{8}{2}$ (1) or $\binom{8}{6} = \frac{8\cdot 7}{1\cdot 2}$

3
- There are 4 black and 7 white balls in a box. Suppose that we DO put back the balls after the drawings. What is the chance of drawing firstly 6 white and then 2 black balls? What is the chance of drawing 5 white and then 3 black balls if the order is irrelevant?

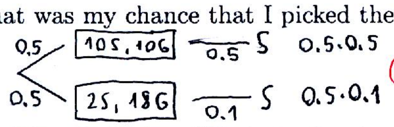
$P(E) = \frac{1}{3},\ p(F) = \frac{2}{3},\ E\cap F = \{4\},\ p(E\cap F) = \frac{1}{6} \neq p(E)p(F) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$

3
- Suppose that we roll a fair dice. Six numbers (from 1 to 6) can appear face up with equal chances. So our sample space is: $\Omega = \{1,2,3,4,5,6\}$. Define the events E and F as follows: $E = \{2,4\}$, $F = \{3,4,5,6\}$. Are E and F independent? Prove your answer! NOT IND.

$\overline{E\cup F} = \{1\} \longrightarrow p(11) = \frac{1}{6}\cdot\frac{1}{6} = 1/36$

Now roll the dice twice. What is the chance that both E and F will never happen?

4
- There were two boxes, one containing 10 silver and 10 gold coins, while the other had 18 gold and only two silver coins. I was allowed to draw a random coin from the box of my choice. I got a silver coin, so I took the other box. What was my chance that I picked the box almost full of gold?



$p = \frac{0.5\cdot 0.5}{0.5\cdot 0.5 + 0.5\cdot 0.1} = \frac{5}{6}$ (2)

7 F. Toss a fair coin thrice. What are the chances of getting 0, 1, 2, 3 tails? What is the expected value and the variance of the number of tails?

0: HHH $\quad p(0\ \text{tails}) = \left(\frac{1}{2}\right)^3 = 1/8$

1: THH, HTH, HHT $\quad p(1\ \text{tails}) = \left(\frac{1}{2}\right)^3\cdot 3 = 3/8$

2: TTH, THT, HTT $\quad p(2\ \text{tails}) = \left(\frac{1}{2}\right)^3\cdot 3 = 3/8$ (3)

3: TTT $\quad p(3\ \text{tails}) = \left(\frac{1}{2}\right)^3 = 1/8$

$E(\text{num.tails}) = 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} = \frac{12}{8} = 1.5$

$Var(\text{num.tails}) = \frac{1}{8}\cdot(0 - 1.5)^2 + \frac{3}{8}\cdot(1 - 1.5)^2 + \frac{3}{8}(2 - 1.5)^2 + \frac{1}{8}(3 - 1.5)^2$

OR: num.tails = (1st toss.num.tails) + (2nd ...) + (3rd ...),

$E(\text{one.toss.num.tails}) = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$

$Var(\ \text{—"—}\ ) = \frac{1}{2}\left(0 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(1 - \frac{1}{2}\right)^2 = \frac{1}{4}$

num.of.tosses

$E(\text{num.tails}) = 3\cdot\frac{1}{2} = 1.5$ (4)

$Var(\text{num.tails}) = 3\cdot\frac{1}{4} = \frac{3}{4} = 0.75$

Econ. Anal. Test 2.

$\boxed{A}$ Compute f'_x, f'_y, f''_{yx}!

① $f(x,y) = x/y$

$f'_x = \frac{1}{y}$, $f'_y = x \cdot (-y^{-2}) = -\frac{x}{y^2}$, $f''_{yx} = (f'_y)'_x = \left(-\frac{x}{y^2}\right)'_x = -\frac{1}{y^2}$

② $f(x,y) = e^{xy^2}$

$f'_x = e^{xy^2} \cdot y^2$, $f'_y = e^{xy^2} \cdot x \cdot 2y$,

$f'_{yx} = (e^{xy^2} \cdot 2xy)'_x = (e^{xy^2} \cdot y^2) \cdot 2xy + e^{xy^2} \cdot 2y = e^{xy^2}(2xy^3 + 2y)$

$\boxed{B}$

① $\int \overset{f' \cdot g}{e^{-x} \cdot x}\, dx = \left| \begin{matrix} f' = e^{-x} & g = x \\ f = -e^{-x} & g' = 1 \end{matrix} \right| = \overset{f \cdot g}{-e^{-x} \cdot x} - \int \overset{f \quad g'}{-e^{-x} \cdot 1}\, dx =$

$= -e^{-x} \cdot x - e^{-x} + C$

② $\int e^{-x^2} \cdot x \, dx = \int e^{-x^2} \cdot \underbrace{(-x^2)' \cdot \left(-\frac{1}{2}\right)}_{-2x \cdot \left(-\frac{1}{2}\right) = x} dx = -\frac{1}{2} e^{-x^2} + C$

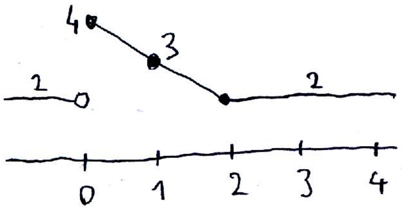
Used formulae: ① $\int f'g = fg - \int fg'$

② $\int f(g(x)) \cdot g'(x)\, dx = F(g(x))$, where $F(x) = \int f(x)\,dx$

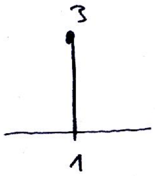
$\boxed{C}$

$f(x) = \begin{cases} 4-x, & \text{if } x \in [0,2] \\ 2, & \text{otherwise} \end{cases}$

Draw a plot of f, and compute a few definite integrals

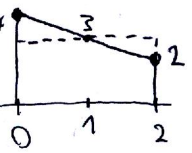


$\int_1^1 f(x)dx$:



int = 0

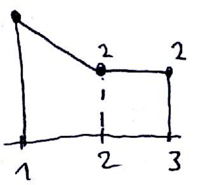
$\int_0^2 f(x)dx$:



int = base · (av. height) =

$(2-0) \cdot \frac{4+2}{2} = 6$

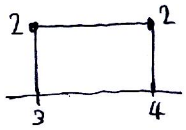
$\int_1^3 f(x)dx$:



int $= (2-1) \cdot \frac{3+2}{2} + (3-2) \cdot 2$

$= 4.5$

$\int_3^4 f(x)dx$:



int $= (4-3) \cdot 2 = 2$

D $f(x,y) = x^2 - xy + 2y^2 - 2x + 3$

Find the critical point of f, and classify it!

Solution:

$f'_x = 2x - y - 2$

$f'_y = -x + 4y$

crit. point: $\left.\begin{array}{l} f'_x = 0 = 2x - y - 2 \\ f'_y = 0 = -x + 4y \rightarrow x = 4y \end{array}\right\} \rightarrow 2\cdot(4y) - y - 2 = 0 \rightarrow y = \frac{2}{7}$

$\rightarrow x = 4 \cdot \frac{2}{7} = \frac{8}{7}$

so the crit. point is at $P_c = (8/7, 2/7)$

Type of crit. point:

$f''_{xx} = 2$

$f''_{yy} = 4$

$f''_{xy} = -1$

$\text{Hessian}(f) = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$, $(\text{Hessian}(f))(P) = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$

① Check the sign of $f''_{xx} f''_{yy} - (f''_{xy})^2$:

$2 \cdot 4 - (-1)^2 = 7 > 0 \longrightarrow P_c$ either MIN, or MAX, not saddle point

② Sign of f''_{xx} discriminate between MIN and MAX:

$f''_{xx} = 2 > 0 \longrightarrow$ critical point is a local minimum

in this case, global

E

① Box

BBBB

WWWWWWW

$p(WWWWWWBB) = (p(W))^6 \cdot (p(B))^2 = \left(\frac{7}{11}\right)^6 \cdot \left(\frac{4}{11}\right)^2$

order is relevant, balls are put back

$p(5W + 3B) = (p(W))^5 \cdot (p(B))^3 \cdot \binom{8}{5} =$

order is irrelevant

$= \left(\frac{7}{11}\right)^5 \cdot \left(\frac{4}{11}\right)^3 \cdot \underbrace{\frac{8!}{5!(8-5)!}}_{\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}}$

E ② $\Omega = \{1, 2, 3, 4, 5, 6\}$, $E = \{2, 4\}$, $F = \{3, 4, 5, 6\}$

fair dice: $p(1) = \ldots = p(6) = \frac{1}{6}$

ⓐ E, F independence:

independent $\Leftrightarrow$ $p(E \cap F) = p(E) \cdot p(F)$

$E \cap F = \{4\}$, so $p(E \cap F) = \frac{1}{6} \neq p(E) \cdot p(F) = \frac{2}{6} \cdot \frac{4}{6} = \frac{2}{9}$

so E and F are not independent

ⓑ Now roll the dice twice. What is the chance that the outcome of the rolls are never in E and F?

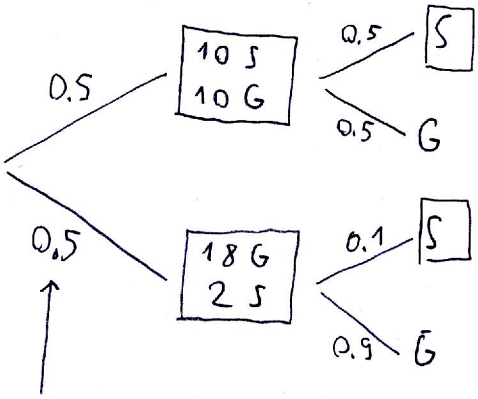
$E \cup F = \{2, 3, 4, 5, 6\}$, $\overline{E \cup F} = \{1\}$

Chance of $p(11) = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

E ③

10 Silver
10 Gold

18 Gold
2 Silver



prior probabilities assigned to the types of the boxes

observed data: the drawn coin is silver

p(silver from 10/10 box) =

= p(10/10 box) · p(silver if box is 10/10)

p(silver from 18/2 box) =

= p(18/2 box) · p(silver if box is 18/2)

$$p(\text{box is } 10/10 \text{ when Silver was drawn}) = \frac{p(\text{silver from } 10/10 \text{ box})}{p(\text{silver from } 10/10 \text{ box}) + p(\text{silver from } 18/2 \text{ box})} =$$

$$= \frac{0.5 \cdot 0.5}{0.5 \cdot 0.5 + 0.5 \cdot 0.1} = \frac{5}{6} =$$

$$= p(\text{"}10/10\text{"} \mid S) = \frac{p(\text{"}10/10\text{"} \cap S)}{p(\text{"}10/10\text{"}) \cdot p(S \mid \text{"}10/10\text{"}) + p(\text{"}18/2\text{"}) \cdot p(S \mid \text{"}18/2\text{"})}$$

Here "10/10" and "18/2" are disjoint and $p(10/10) + p(18/2) = 1$

$\boxed{F}$ $\Omega = \{H, T\}$ $p(H) = \frac{1}{2}$, $p(T) = \frac{1}{2}$

$\Omega^3 = \Omega \times \Omega \times \Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$p(HHH) = p(HHT) = \ldots = p(TTT) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ Fair coin, 3 tosses

Tails: random variable, counting the overall number of tails:

$Tails(HHH) = 0$, $Tails(HHT) = 1$, ... $Tails(TTT) = 3$

Expected value: $E(Tails) = p(Tails=0) \cdot 0 + p(Tails=1) \cdot 1 + p(Tails=2) \cdot 2 + p(Tails=3) \cdot 3 =$

$= \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = 1.5$

Variance: $Var(Tails) = \frac{1}{8} \cdot (0-1.5)^2 + \frac{3}{8} \cdot (1-1.5)^2 + \frac{3}{8}(2-1.5)^2 + \frac{1}{8}(3-1.5)^2 =$

$= 0.75$

The same result with less compudation:

$Tails = tails_1 + tails_2 + tails_3$,

where $tails_i$ is the num. of tails at the i^{th} toss, $i = 1, 2, 3$.

The $tails_i$ random variables are independent of each other, and have the same expeted value and variance as the rand. var "tails" defined on Ω: $tails(H) = 0$, $tails(T) = 1$.

$E(tails) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

$Var(tails) = \frac{1}{2}\left(0 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(1 - \frac{1}{2}\right)^2 = \frac{1}{4}$

Then

$E(Tails) = E(tails_1 + tails_2 + tails_3) = 3 \cdot E(tails) = 3 \cdot \frac{1}{2} = 1.5$

Moreover, since $tails_i$ are independent of each other,

$Var(Tails) = Var(tails_1) + Var(tails_2) + Var(tails_3) = 3 \cdot Var(tails) = 3 \cdot \frac{1}{4}$
$= 0.75$