Solution. Quiz 1.

① $\alpha\begin{pmatrix}2\\0\end{pmatrix}+\beta\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}8\\2\end{pmatrix} \iff \begin{matrix}2\alpha+3\beta=8\\0\alpha+1\beta=2\end{matrix}$

$$\Downarrow$$

$\beta=2,\quad 2\alpha+3\cdot 2=8$

$\alpha=1$

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② $\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}=2\cdot 3+1\cdot 4=10$

$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}0\\1\\2\\3\end{pmatrix}=1\cdot 0+2\cdot 1+3\cdot 2+4\cdot 3=20$

$\left\|\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\|=\sqrt{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}}=\sqrt{1^2+2^2+3^2+4^2}=\sqrt{30}$

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③ $\bar{a}=\begin{pmatrix}1\\2\\3\\x\end{pmatrix},\ \bar{b}=\begin{pmatrix}x\\1\\2\\3\end{pmatrix}$. How much is x, if $\bar{a}\perp\bar{b}$?

$0=\bar{a}\bar{b}=1\cdot x+2\cdot 1+3\cdot 2+x\cdot 3=4x+8 \Rightarrow x=2$

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⑤ $\begin{pmatrix}2x+3y\\4x+5y\end{pmatrix}=A\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow A=\begin{pmatrix}2&3\\4&5\end{pmatrix}$

$\begin{pmatrix}y\\x\end{pmatrix}=A\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow A=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ | $\begin{pmatrix}x\\y\end{pmatrix}=E\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$(x+2y)=A\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow A=(1\ 2)$

identity (unit) matrix

$\varphi\left[\begin{pmatrix}x\\y\end{pmatrix}\right]=\begin{pmatrix}x\\2x\\x+y\end{pmatrix}=A\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow A=\begin{pmatrix}1&0\\2&0\\1&1\end{pmatrix}$

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⑥ $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}1\cdot 5+2\cdot 7&1\cdot 6+2\cdot 8\\3\cdot 5+4\cdot 7&3\cdot 6+4\cdot 8\end{pmatrix}$

$(1\ 2)\begin{pmatrix}3\\4\end{pmatrix}=(1\cdot 3+2\cdot 4)=1\cdot 3+2\cdot 4$

$\begin{pmatrix}3\\4\end{pmatrix}(1\ 2)=\begin{pmatrix}3\cdot 1&3\cdot 2\\4\cdot 1&4\cdot 2\end{pmatrix}=\begin{pmatrix}3&6\\4&8\end{pmatrix}$

$\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1\cdot 1+2\cdot 2\\3\cdot 1+4\cdot 2\\5\cdot 1+6\cdot 2\end{pmatrix}$

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⑦ Equation of a plane:

$\bar{r}_0=\begin{pmatrix}1\\2\\3\end{pmatrix};\quad \bar{n}=\begin{pmatrix}5\\6\\7\end{pmatrix}.\quad \bar{n}\cdot(\bar{r}-\bar{r}_0)=0$

$\begin{pmatrix}5\\6\\7\end{pmatrix}\cdot\left[\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}\right]=\begin{pmatrix}5\\6\\7\end{pmatrix}\begin{pmatrix}x-1\\y-2\\z-3\end{pmatrix}$

$=5\cdot(x-1)+6\cdot(x-2)+7\cdot(x-3)=$

$=5x+6y+7z-38=0$

Vectors

n dim. real vector space

prototype: $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} ; x_i \in \mathbb{R} \right\}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

A set V with two operations:

1. addition of vectors
2. multiplication of vectors by real numbers

is an n dim. real vector space, if it is isomorphic to $\mathbb{R}^n$.

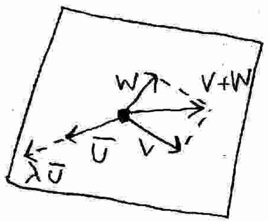
[Isomorphism: a one-to-one mapping φ, which is compatible with the operations:

$\varphi: V \to \mathbb{R}^n$, $\varphi(\bar v + \bar w) = \varphi(\bar v) + \varphi(\bar w)$, $\varphi(\lambda \bar v) = \lambda \varphi(\bar v)$

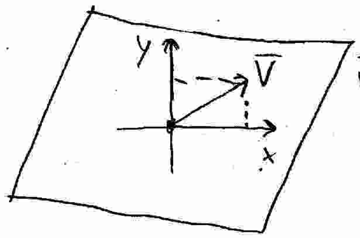
or $\varphi(\alpha \bar v + \beta \bar w) = \alpha \varphi(\bar v) + \beta \varphi(\bar w)$,

i.e. φ is an invertible, linear mapping]

Example: V = plane + origin



A coordinate system on the plane identifies V with $\mathbb{R}^2$:



$\bar V \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$

Econ. example:

Alice buys:	3 apples	4 oranges	2 melons	1 lemons
Bob :	2	7	1	3

$\bar a = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 1 \end{pmatrix}$, $\bar b = \begin{pmatrix} 2 \\ 7 \\ 1 \\ 3 \end{pmatrix}$ Alice + Bob together $\bar a + \bar b = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 3 \\ 4 \end{pmatrix}$

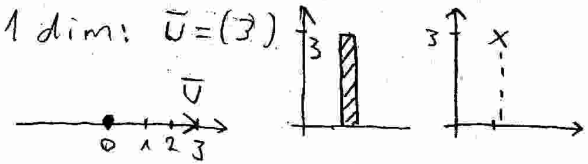
Alic buys the same 4 times in every month:

$$4\bar a = 4 \cdot \begin{pmatrix} 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \cdot 3 \\ 4 \cdot 4 \\ 4 \cdot 2 \\ 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \\ 8 \\ 4 \end{pmatrix}$$

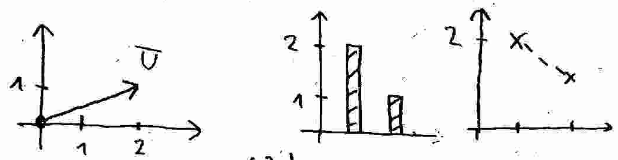
Dim. of the vectorspace can be large.

Pictures:

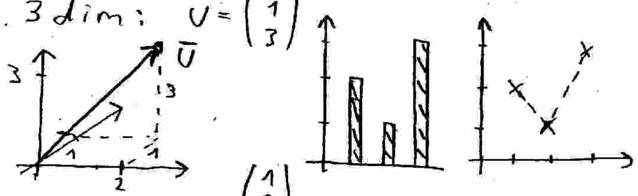
1 dim: $\bar U = (3)$



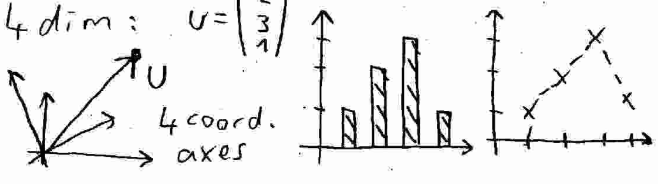
2 dim: $\bar U = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



3 dim: $\bar U = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$



4 dim: $U = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$



Geom. picture is "hard" in dim ≥ 4, but the "chart" or "function" picture works in any dimension.

Scalar product, planes, hyperplanes.

Alice buys 4 apples, one apple costs 3 EUR.

cost = price · quantity = $4\cdot 3 = 12$

Alice buys 4 apples, 2 oranges, price: apple - 5, orange - 1 EUR

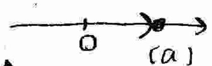
$\text{cost} = \overrightarrow{\text{price}} \cdot \overrightarrow{\text{quantity}} = \begin{pmatrix}5\\1\end{pmatrix}\cdot\begin{pmatrix}4\\2\end{pmatrix} = 5\cdot 4 + 1\cdot 2$

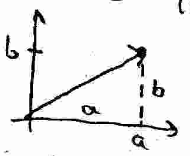
Scalar (inner, dot) product in $\mathbb{R}^n$:

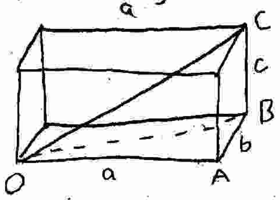
$$\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}\begin{pmatrix}y_1\\ \vdots \\ y_n\end{pmatrix} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

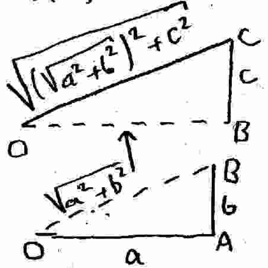
Notations: $\bar{a}\,\bar{b} = \bar{a}\cdot\bar{b} = (a,b)$

Can be used to compute length:

1 dim:  $|(a)|^2 = (a)\cdot(a) = a^2$, $|(a)| = \sqrt{a^2}$

2 dim:  $\left|\begin{pmatrix}a\\b\end{pmatrix}\right| = \sqrt{\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}a\\b\end{pmatrix}} = \sqrt{a^2+b^2}$

3 dim: 



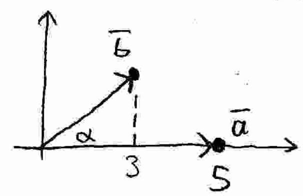
$|\overline{OC}| = \sqrt{a^2+b^2+c^2}$

$$\left|\begin{pmatrix}a\\b\\c\end{pmatrix}\right| = \sqrt{\begin{pmatrix}a\\b\\c\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}} = \sqrt{a^2+b^2+c^2}$$

Higher dim: $|\bar{v}|$ is defined as $\sqrt{\bar{v}\cdot\bar{v}} = \sqrt{\bar{v}^2}$

Vector space + inner product $\longrightarrow$ Euclidean vector space

Geom. def.: $\begin{pmatrix}5\\0\end{pmatrix}\cdot\begin{pmatrix}3\\2\end{pmatrix} = 5\cdot 3 + 0\cdot 2$

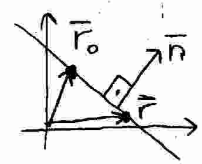


$5 = \left|\begin{pmatrix}5\\0\end{pmatrix}\right|$, $3 = \left|\begin{pmatrix}3\\2\end{pmatrix}\right|\cdot\cos\alpha$

so $\bar{a}\,\bar{b} = |\bar{a}|\cdot|\bar{b}|\cdot\cos\alpha$

$\bar{a}\cdot\bar{b} = 0 \iff \bar{a}\perp\bar{b}$: scalar product defines ortogonality

Line in Plane (linear 1 dim object in 2 dim)



$\bar{r}_0$ a point on the line, $\bar{n}$: normal vector

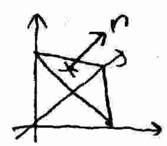
Equation between the coordinates of a generic point $\bar{r} = \begin{pmatrix}x\\y\end{pmatrix}$:

Ex: $\bar{r}_0 = \begin{pmatrix}1\\3\end{pmatrix}$, $\bar{n} = \begin{pmatrix}4\\2\end{pmatrix}$. $\quad\bar{n}\perp(\bar{r}-\bar{r}_0) \iff \bar{n}\cdot(\bar{r}-\bar{r}_0) = 0$

$$\begin{pmatrix}4\\2\end{pmatrix}\cdot\left[\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}1\\3\end{pmatrix}\right] = \begin{pmatrix}4\\2\end{pmatrix}\begin{pmatrix}x-1\\y-3\end{pmatrix} = 0$$

$$4(x-1)+2(y-3) = 4x+2y-10 = 0$$

Plane in 3d space



Ex: $\bar{r}_0 = \begin{pmatrix}1\\3\\2\end{pmatrix}$, $\bar{n} = \begin{pmatrix}4\\2\\5\end{pmatrix} \Rightarrow 4\cdot(x-1)+2(y-3)+5(z-2) = 0$

Plane as a two variables function

$4x+2y+5z-20 = 0 \longrightarrow z(x,y) = \dfrac{20-4x+2y}{5}$

3d hyperplane in 4d space

Ex: $\bar{r}_0 = \begin{pmatrix}1\\3\\2\\4\end{pmatrix}$, $\bar{n} = \begin{pmatrix}4\\2\\5\\6\end{pmatrix} \Rightarrow 4\cdot(x-1)+2(y-3)+5(z-2)+6(v-4) = 0$

Stochastic models (Markovian model of unemployment)

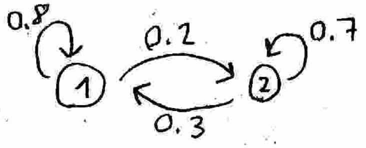
two states: 1. employed
2. unemployed

transition probability: $1 \leftarrow 1: \; W_{1\leftarrow 1} = W_{11} = 0.8$

$1 \leftarrow 2: \; W_{1\leftarrow 2} = W_{12} = 0.3$

$2 \leftarrow 1: \; W_{2\leftarrow 1} = W_{21} = 0.2$

$2 \leftarrow 2: \; W_{2\leftarrow 2} = W_{22} = 0.7$



Evolution of N_1, N_2 (empl., unempl. people)

$$\varphi: \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.8N_1 + 0.3N_2 \\ 0.2N_1 + 0.7N_2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

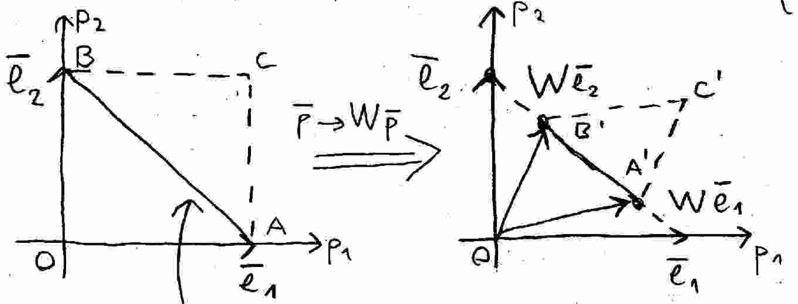
probabilities $p_1 = \dfrac{N_1}{N_1+N_2}, \; p_2 = \dfrac{N_2}{N_1+N_2}$ [Note that $p_i \geq 0$, $p_1+p_2=1$]

$$\varphi: \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

stochastic matrix:
nonnegativ entries
columns sum = 1

Standard base: $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \; W = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$

$W e_1 = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}, \; W e_2 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$ $= \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$



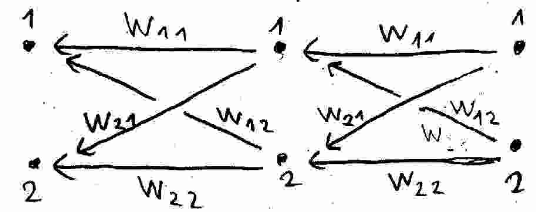
$p_1 + p_2 = 1$ unit square $\longrightarrow$ paralelgramma

Matrix multiplication

Entries of W: trans. prob. in 1 year

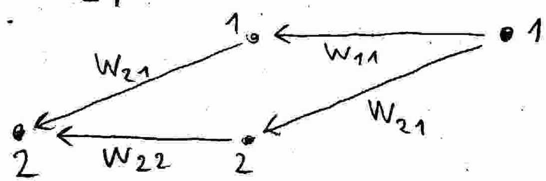
— " — W^2 — " — 2 — " —

$$\begin{tikzcd} W\cdot W\bar{p} & W\bar{p} \arrow[l, "W"', "1\,year"] & \bar{p} \arrow[l, "W"', "1\,year"] \arrow[ll, bend left, "W^2 = W\cdot W", "2\,years"'] \end{tikzcd}$$



transition prob $2 \leftarrow 1$:

$$(W\cdot W)_{21} = W_{21}\cdot W_{11} + W_{22} W_{21}$$



$$(W\cdot W)_{ij} = W_{i1}W_{1j} + W_{i2}W_{2j} = \sum_k W_{ik}W_{kj}$$

So

$$\boxed{(AB)_{ij} = \sum_k A_{ik} B_{kj}}$$

Dynamical system

$\bar{p} \longrightarrow W\bar{p} \longrightarrow W^2\bar{p} \longrightarrow W^3\bar{p} \longrightarrow \ldots$

Fixed point: $\bar{p}_F = W\bar{p}_F$ (steady state)

Theorem: For a stochastic matrix W there exists a nonzero vector $\bar{p}$ such that

$$W\bar{p} = \underbrace{1}_{\text{eigenvalue of } W} \cdot \underbrace{\bar{p}}_{\text{eigenvector of } W}$$

Eigenvectors and eigenvalues of W:

Solutions of: $W\bar{p} = \lambda\bar{p}$, $\bar{p} \neq \bar{0}$

$$\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\begin{matrix} 0.8p_1 + 0.3p_2 = 1p_1 \\ 0.2p_1 + 0.7p_2 = 1 \cdot p_2 \end{matrix} \Rightarrow \begin{matrix} 0.3p_2 = 0.2p_1 \\ p_2 = \frac{2}{3}p_1 \end{matrix} \Rightarrow \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ \frac{2}{3}p_1 \end{pmatrix}$$

Eigenvectors for the $\lambda = 1$ eigenvalue:

$\lambda_1 = 1$, $\bar{v} = \begin{pmatrix} p_1 \\ \frac{2}{3}p_1 \end{pmatrix} = p_1 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$, $\bar{v}_1 = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$

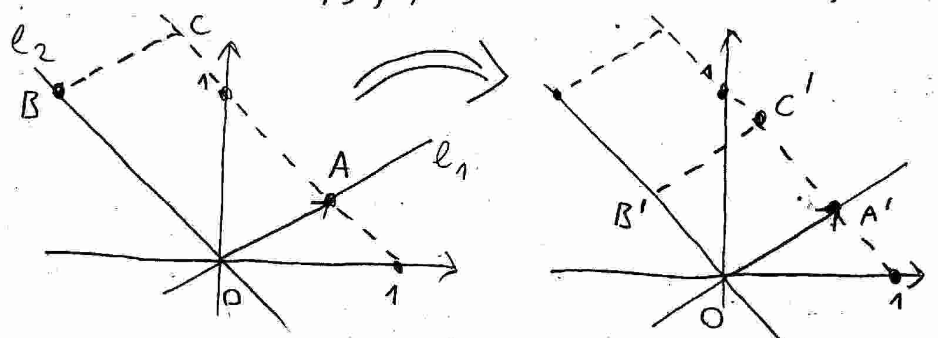
The lines $\left\{ \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} ; N_1 + N_2 = \text{const.} \right\}$ are mapped onto themself: $W \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$0.8 \cdot 1 + 0.3(-1) = \lambda_2 \cdot 1 \Rightarrow \boxed{0.5 = \lambda_2}$$

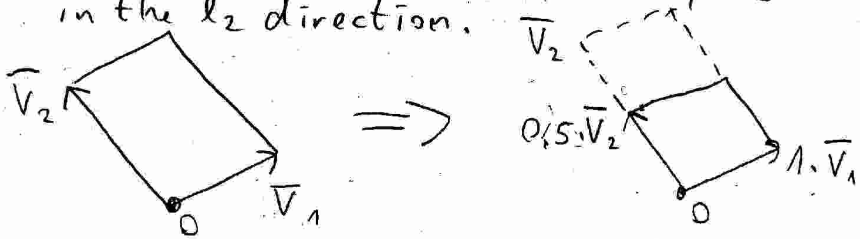
Eigensystem of W:

$\lambda_1 = 1$, $\bar{v}_1 = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$; $\lambda_2 = 0.5$, $\bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

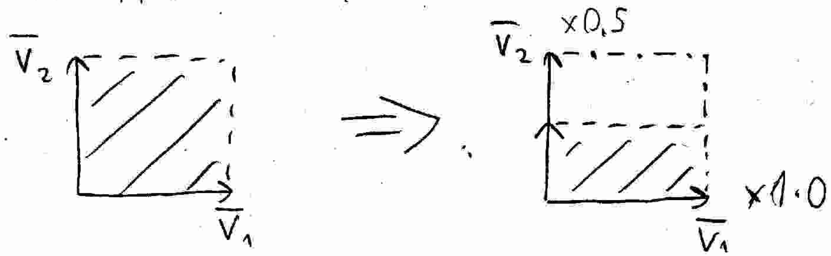


Multiplication by W maps: $OACB \rightarrow OA'C'B'$

Transformation multiplies by $\lambda_1 = 1$ in the l_1 direction, multiplies by $\lambda_2 = 0.5$ in the l_2 direction.



If $\bar{v}_1, \bar{v}_2$ were plotted as the standard basis:



Existence of fixed point

① Detailed balance:

Steady state: (num. of $1 \to 2$) = (num. of $2 \to 1$)

$$W_{2 \leftarrow 1} P_1 = W_{1 \leftarrow 2} P_2$$

$$0.2 \cdot p_1 = 0.3 p_1 \quad \Rightarrow \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$$

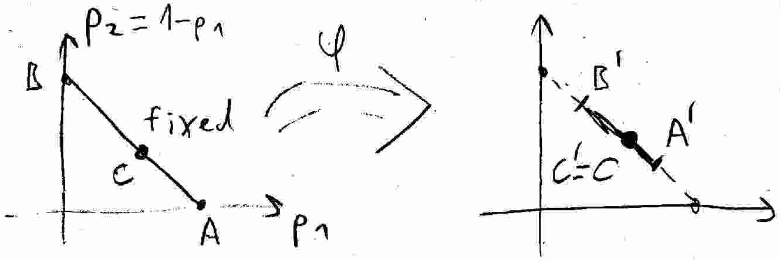
② Contraction:

distance(E,F) < K · distance(E',F'), for some fixed K < 1.

$\Rightarrow$ fixed point exists.

①' It is true because we have only two states.
True for time-reversible dynamics
(uncommon in economics).

③ Brouwer's fixed point theorem:



The Theorem states that any continuous mapping from the simplex:

$$\Delta^{n-1} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} ; \sum x_i = 1, x_i \geq 0 \right\}$$

onto itself has a fixed point.

Proof: (n=2)