Vectors

n dim. real vector space

prototype: $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} ; x_i \in \mathbb{R} \right\}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} , \quad \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

A set V with two operations:

1. addition of vectors
2. multiplication of vectors by real numbers

is an n dim. real vector space, if it is isomorphic to $\mathbb{R}^n$.

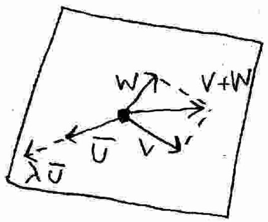
[Isomorphism: a one-to-one mapping φ, which is compatible with the operations:

$\varphi: V \to \mathbb{R}^n$, $\varphi(\bar v + \bar w) = \varphi(\bar v) + \varphi(\bar w)$, $\varphi(\lambda \bar v) = \lambda \varphi(\bar v)$

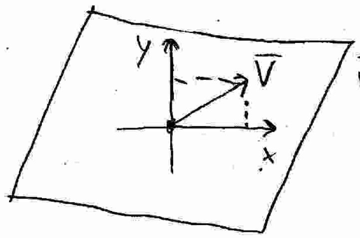
or $\varphi(\alpha \bar v + \beta \bar w) = \alpha \varphi(\bar v) + \beta \varphi(\bar w)$,

i.e. φ is an invertible, linear mapping]

Example: V = plane + origin



A coordinate system on the plane identifies V with $\mathbb{R}^2$:



$\bar V \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$

Econ. example:

Alice buys:	3 apples	4 oranges	2 melons	1 lemons
Bob:	2	7	1	3

$\bar a = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 1 \end{pmatrix}$, $\bar b = \begin{pmatrix} 2 \\ 7 \\ 1 \\ 3 \end{pmatrix}$ Alice + Bob together $\bar a + \bar b = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 3 \\ 4 \end{pmatrix}$

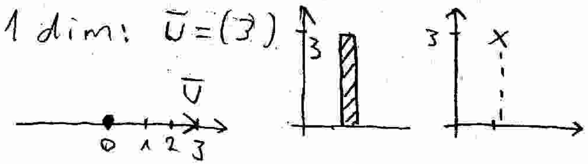
Alic buys the same 4 times in every month:

$$4\bar a = 4 \cdot \begin{pmatrix} 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \cdot 3 \\ 4 \cdot 4 \\ 4 \cdot 2 \\ 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \\ 8 \\ 4 \end{pmatrix}$$

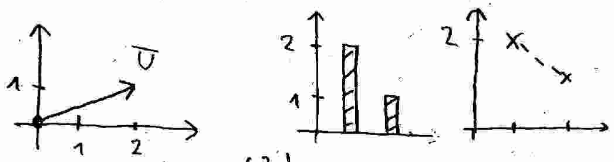
Dim. of the vectorspace can be large.

Pictures:

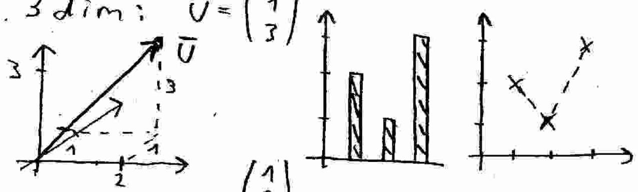
1 dim: $\bar u = (3)$



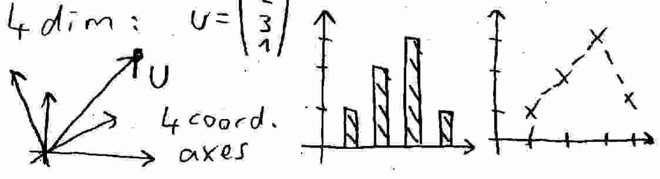
2 dim: $\bar u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



3 dim: $\bar u = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$



4 dim: $u = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$



Geom. picture is "hard" in dim ≥ 4, but the "chart" or "function" picture works in any dimension.

Scalar product, planes, hyperplanes.

Alice buys 4 apples, one apple costs 3 EUR.

cost = price · quantity = 4 · 3 = 12

Alice buys 4 apples, 2 oranges, price: apple - 5, orange - 1 EUR

$\text{cost} = \overrightarrow{\text{price}} \cdot \overrightarrow{\text{quantity}} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 5 \cdot 4 + 1 \cdot 2$

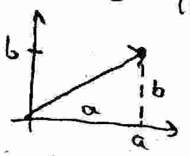
Scalar (inner, dot) product in $\mathbb{R}^n$:

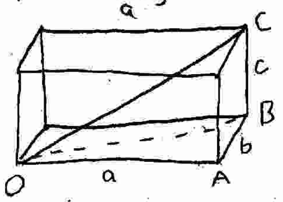
$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$

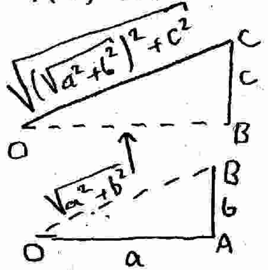
Notations: $\bar{a}\,\bar{b} = \bar{a} \cdot \bar{b} = (a, b)$

Can be used to compute length:

1 dim:  $|(a)|^2 = (a) \cdot (a) = a^2, \; |(a)| = \sqrt{a^2}$

2 dim:  $\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = \sqrt{\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}} = \sqrt{a^2 + b^2}$

3 dim: 



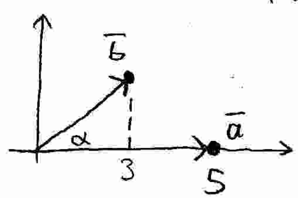
$|\overline{OC}| = \sqrt{a^2 + b^2 + c^2}$

$\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \sqrt{a^2 + b^2 + c^2}$

Higher dim: $|\bar{v}|$ is defined as $\sqrt{\bar{v} \cdot \bar{v}} = \sqrt{\bar{v}^2}$

Vector space + inner product $\longrightarrow$ Euclidean Vector Space

Geom. def.: $\begin{pmatrix} 5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 5 \cdot 3 + \cancel{0 \cdot 2}$

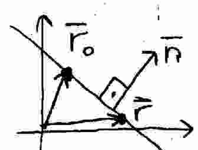


$5 = \left|\begin{pmatrix} 5 \\ 0 \end{pmatrix}\right|, \quad 3 = \left|\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right| \cdot \cos\alpha$

so $\bar{a}\,\bar{b} = |\bar{a}| \cdot |\bar{b}| \cdot \cos\alpha$

$\bar{a} \cdot \bar{b} = 0 \iff \bar{a} \perp \bar{b}$: scalar product defines ortogonality

Line in plane (linear 1 dim object in 2 dim)



$\bar{r}_0$ a point on the line, $\bar{n}$: normal vector

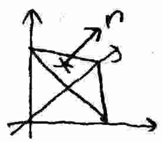
Equation between the coordinates of a generic point $\bar{r} = \begin{pmatrix} x \\ y \end{pmatrix}$:

Ex: $\bar{r}_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \bar{n} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. $\quad \bar{n} \perp (\bar{r} - \bar{r}_0) \iff \bar{n} \cdot (\bar{r} - \bar{r}_0) = 0$

$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} x-1 \\ y-3 \end{pmatrix} = 0$

$4(x-1) + 2(y-3) = 4x + 2y - 10 = 0$

Plane in 3d space



Ex: $\bar{r}_0 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \bar{n} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} \Rightarrow 4 \cdot (x-1) + 2(y-3) + 5(z-2) = 0$

Plane as a two variables function

$4x + 2y + 5z - 20 = 0 \longrightarrow z(x,y) = \dfrac{20 - 4x + 2y}{5}$

3d hyperplane in 4d space

Ex: $\bar{r}_0 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}, \bar{n} = \begin{pmatrix} 4 \\ 2 \\ 5 \\ 6 \end{pmatrix} \Rightarrow 4 \cdot (x-1) + 2(y-3) + 5(z-2) + 6(v-4) = 0$

A. Test 1. Econ.Anal 13.Oct.21. NEPTUN: Name: A

1. A. Compute the derivatives of the following functions!

1. $e^x \cos(2x-1)$
2. $e^7 \ln(2x-1)$
3. $\frac{\ln(2x)}{\ln(x)}$

B. What is the prediction of the linear approximation of the function $f(x)$ at $x = x_0$ for the value of $f(x_0 + \Delta x)$?
$f(x) = \ln x$, $x_0 = e$, $\Delta x = 0.1$.

A. 1. $\left[e^x\cos(2x-1)\right]' = \left[e^x\right]'\cos(2x-1) + e^x\left[\cos(2x-1)\right]'$

2 $= e^x \cos(2x-1) + e^x(-\sin(2x-1)\cdot 2)$

2 2. $\left[e^7 \ln(2x-1)\right]' = e^7 \frac{1}{2x-1}\cdot 2$ (as e^7 is just a constant)

3 3. $\left[\frac{\ln(2x)}{\ln x}\right]' = \frac{[\ln(2x)]'\cdot \ln x - \ln(2x)\cdot[\ln x]'}{(\ln x)^2} = \frac{\left[\frac{1}{2x}\cdot 2\right]\ln x - \ln(2x)\cdot\frac{1}{x}}{(\ln x)^2}$

B. $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\,\Delta x$

$f(x) = \ln x$, $f'(x) = \frac{1}{x}$

3 $f(e) = \ln e = 1$, $f'(e) = \frac{1}{e}$

$f(e+0.1) \approx 1 + \frac{1}{e}\cdot 0.1$

A

2. A. Study the monotonicity, convexity and local extremal values of the following function!
$f(x) = 2x^3 - 3x^2$.
Draw its graph!

B. Study the monotonicity of the following sequence!
$\frac{3n+4}{5n+6}$.

A. $f = 2x^3 - 3x^2 = x^2(2x-3)$

2

$f' = 6x^2 - 6x = x(6x-6)$

$f'' = 12x - 6$

MIN/MAX:

$f' = 0 = x(6x-6)$

2

$x_1 = 0$ $\quad$ $x_2 = 1$

$f''(0) = 12\cdot 0 - 6 = -6$ $\quad$ $f''(1) = 12\cdot 1 - 6 = 6$

$-6 < 0$ $\quad$ $6 > 0$

MAX $\quad$ MIN

$x<0$	$x=0$	$0<x<1$	$x=1$	$x>1$
↗	MAX	↘	MIN	↗

2

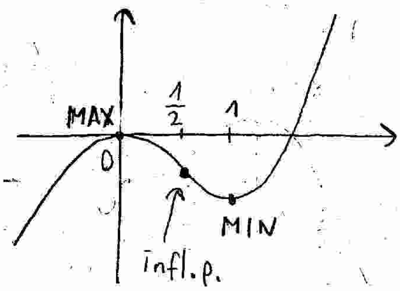
convexity

inflection point: $f'' = 0$

$12x - 6 = 0$ $\quad$ $x_{inf} = \frac{1}{2}$

$x<\frac{1}{2}$	$x=\frac{1}{2}$	$x>\frac{1}{2}$
$\frown$	INFL.	$\smile$
concave		convex

1



B. $a_{n+1} - a_n = \frac{3(n+1)+4}{5(n+1)+6} - \frac{3n+4}{5n+6} = \frac{(3n+7)(5n+6) - (3n+4)(5n+11)}{(5n+11)(5n+6)}$

3

$= \frac{-2}{(5n+11)(5n+6)} < 0 \Rightarrow$ decreasing sequence

2

B. Test 1. Econ.Anal 13 Oct.21. NEPTUN: Name: B

1. A. Compute the derivatives of the following functions!

   1. $\sqrt[3]{\sin(3x)}$
   2. $\sqrt[3]{x}\, tg(2x-1)$
   3. $\frac{x^7}{\sin(3x)}$

B. Let $f(x) = -x^2 - 2x$. Compute $\frac{f(5+\Delta x)-f(5)}{\Delta x}$! What is the limit of this fraction as $\Delta x \to 0$? What is $f'(5)$?

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2 A. 1 $\left[\sqrt[3]{\sin(3x)}\right]' = \left[(\sin(3x))^{1/3}\right]' =$

$= \frac{1}{3}(\sin(3x))^{-3/3} \cdot (\sin(3x))' = \frac{1}{3}(\sin(3x))^{-2/3} \cdot \cos(3x) \cdot 3$

2 2 $\left[x^{1/3} \cdot tg(2x-1)\right]' = (x^{1/3})' \cdot tg(2x-1) + x^{1/3} \cdot [tg(2x-1)]' =$

$= \frac{1}{3} x^{-2/3} \cdot tg(2x-1) + x^{1/3} \cdot \frac{1}{\cos^2(2x-1)} \cdot 2$

2 3 $\left[\frac{x^7}{\sin(3x)}\right]' = \frac{(x^7)' \sin(3x) - x^7 \cdot (\sin(3x))'}{(\sin(3x))^2}$

$= \frac{7x^6 \sin(3x) - x^7 \cos(3x) \cdot 3}{(\sin(3x))^2}$

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4 B. $\frac{\Delta f}{\Delta x} = \frac{[-(5+\Delta x)^2 - 2\cdot(5+\Delta x)] - [-5^2 - 2\cdot 5]}{\Delta x} = \frac{(-5\cdot 2 - 2)\Delta x - \Delta x^2}{\Delta x}$

$= -5\cdot 2 - 2 + \Delta x$

$\lim\limits_{\Delta x \to 0} -5\cdot 2 - 2 + \Delta x = -5\cdot 2 - 2 = -12$, so $f'(5) = -12$

B

3.A. Compute the limit of the following sequence! $a_n = \frac{2^{2n-88}}{3^{n+77}5^n}$.

B. Let $\phi(x) = 4x + 16$, $x_0 = 13$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\cancel{\phi^n(1) = x_n}$? $\varphi^n(13) = x_n$?

1. Find the fixed point x_f of ϕ !
2. Introduce $\Delta x = x - x_f$ and $\tilde{\phi}(\Delta x) = \phi(x_f + \Delta x) - x_f$. Calculate $\tilde{\phi}$ and $\tilde{\phi}^n$!
3. Compute x_n !

A. $$\lim_{n\to\infty} \frac{2^{2n-88}}{3^{n+77}\cdot 5^n} = \lim_{n\to\infty} \left(\frac{2^2}{3\cdot 5}\right)^n \cdot \frac{2^{-88}}{3^{77}} =$$

$$= \lim_{n\to\infty} \left(\frac{4}{15}\right)^n \cdot \left(\frac{2^{-88}}{3^{77}}\right) = 0$$ 3

B. 1 fixpoint: $\varphi(x_f) = x_f$, $4x_f + 16 = x_f \longrightarrow x_f = -\frac{16}{3}$ 2

2 $\Delta x = x - \left(-\frac{16}{3}\right)$ $\quad \tilde{\varphi}(\Delta x) = 4\cdot\Delta x$

(or $\tilde{\varphi}(\Delta x) = \left[4\cdot\left(-\frac{16}{3} + \Delta x\right) + 16\right] - \left(-\frac{16}{3}\right) = 4\Delta x$) 1

$\tilde{\varphi}^n(\Delta x) = 4^n \Delta x$

3 $x_0 = 13 \longrightarrow \Delta x_0 = 13 - \left(-\frac{16}{3}\right)$

$\Delta x_n = 4^n\left(13 - \left(-\frac{16}{3}\right)\right)$

$x_n = 4^n\left(13 - \left(-\frac{16}{3}\right)\right) + \left(-\frac{16}{3}\right)$

[Remark: if $x_n = \varphi^n(1)$,
then $x_n = 4^n\left(1 - \left(-\frac{16}{3}\right)\right) + \left(-\frac{16}{3}\right)$] 3

φ^{-1}: $\varphi(x) = 4x + 16$ 1

$y = 4x + 16$

$x = \frac{y-16}{4} = \frac{1}{4}y - 4$

$\varphi^{-1}(y) = \frac{1}{4}y - 4 \qquad \varphi^{-1}(x) = \frac{1}{4}x - 4$

4. (a) Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!

   i. $\sqrt[4]{4x^8} + \sqrt[5]{(3x)^7} + \frac{8}{x^6}$

   ii. $\frac{2}{1+4x^2}$

   iii. $e^{3x} + \sin(-3x)$

   (b) Let X be a discrete random variable. Let its support R_X be: $R_X = \{1,2,3\}$. Let its probability mass function be:

$$p(x) = \begin{cases} x/6 & \text{if } x \in R_X \\ 0 & \text{if } x \notin R_X. \end{cases}$$

Compute the mean of X.

a i) $\int \sqrt[4]{4x^8} + \sqrt[5]{(3x)^7} + \frac{8}{x^6}\,dx =$

$= \int \sqrt[4]{4}\, x^{\frac{8}{4}} + (3x)^{7/5} + 8\cdot x^{-6}\,dx =$

$= \sqrt[4]{4}\,\frac{x^{12/4}}{\underbrace{12/4}_{=3}} + \frac{\frac{(3x)^{12/5}}{12/5}}{3} + 8\cdot\frac{x^{-5}}{-5} + C$ 3

ii) $\int \frac{2}{1+4x^2}\,dx = 2\cdot\int\frac{1}{1+(2x)^2}\,dx = 2\cdot\frac{\operatorname{arctg}(2x)}{2} + C$ 2

iii) $\int e^{3x} + \sin(-3x)\,dx = \frac{e^{3x}}{3} + \frac{-\cos(-3x)}{-3} + C$ 2

b $E[X] = \sum_i p_i X_i = \frac{1}{6}\cdot 1 + \frac{2}{6}\cdot 2 + \frac{3}{6}\cdot 3 = \frac{14}{6} = \frac{7}{3}$

3

only a!

2. A. Let $f(x) = 3x^2 + 1$. Compute $\frac{f(5+\Delta x_n) - f(5)}{\Delta x_n}$! What is the limit of this fraction if $\Delta x_n = 1/n$? What is $f'(5)$?

B. Study the monotonicity and the limit of the following sequences!

a) $\frac{4-1}{5n+6}$, b) $\frac{3n}{5}(-1)^n$.

A $\frac{\Delta f_n}{\Delta x_n} = \frac{[3\cdot(5+\Delta x_n)^2+1]-[3\cdot 5^2+1]}{\Delta x_n} = 3\cdot 2\cdot 5 + 3\Delta x_n$

$\lim_{n\to\infty} 3\cdot 2\cdot 5 + 3\cdot\frac{1}{n} = 3\cdot 2\cdot 5 + 3\cdot 0 = 30$

$f'(5) = 30$ 4

$4-1=3$

B a) $a_{n+1} - a_n = \frac{3}{5(n+1)+6} - \frac{3}{5n+6} = \frac{3\cdot(5n+6) - 3\cdot(5n+11)}{(5n+11)\cdot(5n+6)}$

$= \frac{-15}{(5n+11)(5n+6)} < 0$ if $n = 0, 1, 2 \ldots$ decreasing 2

$\lim_{n\to\infty} \frac{3}{5n+6} = \lim_{n\to\infty} \frac{3/n}{5+6/n} = \frac{0}{5+0} = 0$ 1

b) $\frac{3n}{5}(-1)^n$ not monotone, as its sign is alternating. 1

divergent sequence, as $\frac{3n}{5} \to \infty$, but the $(-1)^n$ factor turns it to an alternating sequence.

2

3.A. Compute the limit of the following sequence! $a_n = \left(1 - \frac{1}{3n}\right)^{-3n+7}\left(\frac{2n^2}{3n^2+1}\right)$.

B. Let $\phi(x) = 3x - 2$, $x_0 = 2$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\phi^n(1) = x_n$?

That should be $x_0 = 2$, not 1

1. Compute ϕ^{-1} !
2. Find the fixed point x_f of ϕ !
3. Compute x_n !

(A) $$\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^{-3n+7} \cdot \left(\frac{2n^2}{3n^2+1}\right) =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-1/3}{n}\right)^n\right]^{-3} \cdot \left(1 - \frac{1}{3n}\right)^7 \cdot \left(\frac{2}{3 + 1/n^2}\right) =$$

$$= \left[e^{-1/3}\right]^{-3} \cdot 1^7 \cdot \frac{2}{3+0} = e \cdot \frac{2}{3}$$ (4)

(B) φ^{-1}:
$$y = 3x - 2$$
$$x = \frac{y+2}{3}$$
$$\varphi^{-1}(y) = \frac{y+2}{3}$$
$$\varphi^{-1}(x) = \frac{x+2}{3}$$ (2)

Fixed point: $x_f = \varphi(x_f)$

$x_f = 3x_f - 2 \longrightarrow x_f = 1$ (1)

x_n: $x_n = \varphi^n(1) = \varphi^n(x_f) = x_f = 1$ (luck!)

$$x_n = 3^n \cdot (x_0 - x_f) + x_f$$

So if $x_0 = 2$, $x_n = 3^n(2-1) + 1$

if $x_0 = 1$, $x_n = 3^n(1-1) + 1 = 1$ (3)

Both interpredations are accepted

3.A. Compute the limit of the following sequence! $a_n = \left(1+\frac{1}{-3n}\right)^{-n+7}\left(\frac{-2n}{-3n+1}\right)$.

B. Let $\phi(x) = 3x+2$, $x_0 = 3$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\phi^n(1) = x_n$?

It was intended to be $x_0 = 3$, not 1.

1. Compute ϕ^{-1} !
2. Find the fixed point x_f of ϕ !
3. Compute x_n !

A $$\lim_{n\to\infty} \left(1+\frac{1}{-3n}\right)^{-n+7}\left(\frac{-2n}{-3n+1}\right) =$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{-1/3}{n}\right)^n\right]^{-1} \cdot \left(1+\frac{1}{-3n}\right)^7 \cdot \left(\frac{-2}{-3+1/n}\right) =$$

$$= \left[e^{-1/3}\right]^{-1} \cdot 1^7 \cdot \frac{-2}{-3+0} = e^{1/3} \cdot \frac{2}{3}$$ 4

B φ^{-1}: $y = 3x+2$

$$x = \frac{y-2}{3}$$

$$\varphi^{-1}(y) = \frac{y-2}{3}$$

$$\varphi^{-1}(x) = \frac{x-2}{3}$$ 2

Fixed point: $x_f = \varphi(x_f)$

$x_f = 3x_f + 2 \longrightarrow x_f = -1$ 1

x_n: $x_n = 3^n(x_0 - x_f) + x_f$

So if $x_0 = 3$, $x_n = 3^n(3-(-1)) + (-1) =$
$= 3^n \cdot 4 - 1$

if $x_0 = 1$, $x_n = 3^n(1-(-1)) + (-1) =$
$= 3^n \cdot 2 - 1$ 3

Both versions are accepted.

I.

1. Compute the derivatives of the following functions!

A • $\sqrt[6]{x^5} - \frac{4}{\sqrt[6]{x^5}} + \ln(3x)$

B • $e^x \cos(-x-1)$

C • $\frac{\cos(2x)}{x^4+1}$

5x2points

2. Compute the $\int f(x)\,dx$ indefinite integrals of the following $f(x)$ functions!

D • $e^{-x} + \sin(3x)$

E • $x^2 - 2x$

(A) $\left[\sqrt[6]{x^5} - \frac{4}{\sqrt[6]{x^5}} + \ln(3x)\right]' = \left[x^{5/6} - 4x^{-5/6} + \ln(3x)\right]'$

$= \frac{5}{6}x^{-1/6} - 4\cdot\left(-\frac{5}{6}\right)x^{-11/6} + \frac{1}{3x}\cdot 3$

(B) $\left[e^x\cdot\cos(-x-1)\right]' = [e^x]'\cdot\cos(-x-1) + e^x\cdot[\cos(-x-1)]'$

$= e^x\cdot\cos(-x-1) + e^x\cdot[-\sin(-x-1)\cdot(-1)]$

(C) $\left[\frac{\cos(2x)}{x^4+1}\right]' = \frac{[\cos(2x)]'\cdot(x^4+1) - \cos(2x)\cdot[x^4-1]'}{[x^4+1]^2}$

$= \frac{[-\sin(2x)\cdot 2]\cdot(x^4+1) - \cos(2x)\cdot[4\cdot x^3]}{[x^4+1]^2}$

(D) $\int e^{-x} + \sin(3x)\,dx = \frac{e^{-x}}{-1} + \frac{-\cos(3x)}{3} + C$

(E) $\int x^2 - 2x\,dx = \frac{x^3}{3} - 2\cdot\frac{x^2}{2} + C$