First and second weeks

1. (a) A melon weights 4 kg and costs 2 EUR, while a pumpkin's weight is 3 kg and its price is 1 EUR. Jim buys altogether 10 kg of pumpkins and melons and pays 4 EUR. How many pumpkins and melons are bought?
   (b) $\overline{m} = (4, 2)^T$, $\overline{p} = (3, 1)^T$. Solve $\alpha\overline{m} + \beta\overline{p} = (10, 4)^T$ for α and β !

2. (a) The price of apple is 2 EUR/kg, while orange costs 5 EUR/kg. How much do 3 kg of apple and 4 kg of orange cost?
   (b) $\overline{p} = (2, 5)^T$, $\overline{q} = (3, 4)^T$. Compute the $\overline{p}\,\overline{q}$ scalar product!

3. (a) You got now 7 HUF and you spend 2 HUF a day. When will you run out of money? What is the change of your wealth in a day? At what time will you have 3 HUF?
   (b) $y = f(x) = 7 - 2x$.
      i. Where are the intersections of this line with the x and y axes?
      ii. What is the slope of this function?
      iii. Express x with y !
      iv. What is the f^{-1} inverse of the function f ?
      v. Plot f and f^{-1} together!
   (c) $y = f(x) = \sqrt{9 - x}$.
      i. Where are the intersections of this curve with the x and y axes?
      ii. Express x with y !
      iii. What is the f^{-1} inverse of the function f ?
      iv. Plot f and f^{-1} together!
   (d) Repeat exercise 3c for the following functions:

$$y = f(x) = 2^{3-x}, \quad y = f(x) = \log_3(9 + x).$$

   (e) Repeat exercise 3c for the following functions:

$$y = f(x) = 5 \cdot 2^{3-4x} + 2, \quad y = f(x) = 3\log(9 + 4x) - 1.$$

4. (a) Review the meaning of the phrases: "arithmetic mean", "geometric mean", "median" !
   (b) Let $(x_1, x_2, x_3) = (1, 2, 4)$.
      - Plot the graph of the function

$$f(x) = (x - x_1)^2 + (x - x_2)^2 + (x - x_3)^2 = \sum_{i=1}^{3}(x - x_i)^2.$$

      - Where is the location x_{min} of the minimum of f ? (Recall the solution method of the quadratic equation!)
      - What is the name of x_{min} (in relation to (x_1, x_2, x_3))?
   (c) Repeat this exercise for $f(x) = \sum_{i=1}^{3} |x - x_i|$.
      - Plot the function $f(x) = |x - x_1| + |x - x_2| + |x - x_3|$!
         i. How much is $f(1)$?
         ii. What is the slope of f on the intervals $(-\infty, 1)$, $(1, 2)$, $(2, 4)$, $(4, \infty)$?
      - What is the name of x_{min} ?
      - What would happen for $(x_1, x_2) = (1, 2)$ and $(x_1, x_2, x_3, x_4) = (1, 2, 4, 7)$?

5. (a) The lengths of the legs (catheti) of a right triangle are 3 and 4. What is the length of its longest side (hypotenuse)?
   How much is is $|(3,4)^T|$?
   (b) A 3d rectangle has sides 3, 4, 5. What is the distance between two opposite vertices?
   How much is $|(3,4,5)^T|$?
   (c) $\overline{a} = (3,4,5,6)^T$ How much is $|\overline{a}|$?
   How does $|\overline{a}|$ relate to $\overline{a}\,\overline{a}$?

6. (Computer ex.) Plot the following set of (x_i, y_i) points: $\{(0,0),(1,1),(2,1)\}$! Find the line $y(x) = ax + b$ that fits "best" to these points! Solve this exercise when the word "best" means the minimization of:
   (a) $\sum_{i=1}^{3}(y_i - (ax_i + b))^2$,
   (b) $\sum_{i=1}^{3}|y_i - (ax_i + b)|$,
   (c) $\sum_{i=1}^{3}(y_i - (ax_i + b))^4$,

7. (a) At the market you can buy apples, oranges, lemons, peaches for 3, 5, 15, 1 HUF/kg.
   i. Jim buys (x) kg of apples.
   ii. Jill buys (x, y) kg of apples and oranges.
   iii. Jacob buys (x, y, z) kg of apples, oranges and lemons.
   iv. Joseph buys (x, y, z, u) kg of apples, oranges, lemons and peaches.

   All of them pay 15 HUF. Draw pictures of the following sets:
   i. $\{(x) \in \mathbb{R}^1 \,|\, 3x = 15\}$,
   ii. $\{\{(x,y) \in \mathbb{R}^2 \,|\, 3x + 5y = 15\}$, (budget line)
   iii. $\{(x,y,z) \in \mathbb{R}^3 \,|\, 3x + 5y + 1z = 15\}$,
   iv. $\{(x,y,z,u) \in \mathbb{R}^4 \,|\, 3x + 5y + 1z + 15u = 15\}$.

   Keep in mind that $x, y, z, u \geq 0$.
   Explain the your last drawing!
   Can you continue this exercise in higher dimensions?
   Express $y(x)$ from $3x + 5y = 15$ and $z(x,y)$ from $3x + 5y + 1z = 15$.
   What are the slopes of the line $y(x)$ and the plane $z(x,y)$?
   What can you say about $u(x,y,z)$?

   (b) i. Let $\overline{r}_0 = (0,0,15)^T$ and $\overline{n} = (3,5,1)^T$.
   ii. Find an equation of the plane that contains $\overline{r}_0$ and has normal vector $\overline{n}$!
   iii. Express the third coordinate $z(x,y)$ of a point (x,y,z) of the plane with x and y !
   iv. How much does $z(x,y)$ change when x is increased by 1 while y is left unchanged?
   v. How much does $z(x,y)$ change when y is increased by 1 while x is left unchanged?
   vi. Find the intersection of the plane with the x, y and z axes!

8. Logarithm, exponential function.
   (a) Compute the following expressions:
   $2^3 \cdot 2^4$, $2^3/2^4$, $\lg(1)$, $\lg(10)$, $\lg(100)$, $\lg(10000)$, $\lg(0.1)$, $\lg(0.0001)$, $\log_2(8)$, $\log_2(1024)$, $\log_2(1/2)$, $\log_2(1/32)$, $\log_{1/2}(32)$.
   (b) Rewrite these expressions:
   $\lg(xy)$, $\lg(x/y)$, $\lg(x^n)$, $\lg(1/x^n)$.
   (c) What is the relation between $\log_a(x)$ and $\log_b(x)$?
   (d) Plot the functions 2^x and $\log_2(x)$!
   (e) How much are $\lg(10^x)$ and $10^{\lg(x)}$?

9. The initial balance of your bank account is 3000 HUF, the interest rate for a year is 2%.

   (a) i. Give a rough estimate of your balance after tree years! (Do not use calculator!)

   ii. Compute the balance after three years! What was the error of the previous estimate? (Study the error for 0.01, 0.1, 1, 10, 100 percent interest rates, too!)

   iii. What is the balance after n years?
   When will you have 6000 HUF on your account?
   What is the balance after $n = 1/0.02 = 50$ years?

   iv. If the interest rate is 25% and the initial balance is 1 HUF, what is the balance after $n = 1/0.25 = 4$ years?
   If the interest rate is 1% and the initial balance is 1 HUF, what is the balance after $n = 1/0.01 = 100$ years?
   If the interest rate is 0.1% and the initial balance is 1 HUF, what is the balance after $n = 1/0.001 = 1000$ years?

   (b) i. Solve the equation $3000 \cdot 1.02^x = 6000$ for x !

   ii. Solve the equation $aq^x = b$ for x !

   iii. If $1.02^{50} \approx 2.7$, then how much is 1.02^{100} approximately?

10. (a) Arithmetic sequence. Let $f(x) = x + d$. If a_0 is given and $a_{n+1} = f(a_n) = a_n + d$, then how much is a_n ?

   (b) Compute $S_{100} = 1 + 2 + \cdots + 99 + 100$! (How much are $1 + 100$, $2 + 99$, etc. ?)

   (c) Compute $\sum_{k=0}^{n}(a_0 + nd)$!

11. (a) i. Geometric sequence. Let $f(x) = qx$. If a_0 is given and $a_{n+1} = f(a_n) = qa_n$, then how much is a_n ?

   ii. Compute
   $$S_{100} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{10}}.$$
   (Concatenate intervals of lengths $1, \frac{1}{2}, \frac{1}{2^2}, \ldots$. After n steps, how much is missing from the interval of length 2 ?)

   iii. A. Let $S_n = 1 + q + q^2 + \cdots + q^n$. What is $S_n - qS_n$? What is S_n ?
   B. Compute $\sum_{k=0}^{n} a_0 q^n$!

   (b) Net present value. Let us suppose that you buy a house for 10000 HUF, but you need to pay only 1000 HUF yearly for ten years. You can invest your money in government bonds with 2% yield. How much money is necessary right now to cover the yearly payments?

12. (a) Let $f(x) = 2x + 1$. If $a_0 = 9$ and $a_{n+1} = f(a_n) = 2a_n + 1$, then how much is a_n ?

   i. Find the fixed point x_{fix} of f, i.e. solve the $f(x_{fix}) = x_{fix}$ equation!

   ii. Then
   $$a_n = 2^n(a_0 - x_{fix}) + x_{fix} = 2^n(9 - (-1)) + (-1).$$

   iii. Give a "graphical" explanation of this result!

   (b) The initial balance of your bank account is $a_0 = 1500$ HUF. The bank pays 10% interest, but also deducts a 50 HUF fee at the end of each year. In other words, $a_{n+1} = f(a_n) = 1.1 \cdot a_n - 50$. What will be the balance after n years?

   i. If Jill's balance does not change in a year (i.e. the interest yield and the fee cancel each other), then how much money does Jill has in her account? Denote this by x_{fix}.

   ii. Compute $\Delta a_0 = 1500 - x_{fix}$ and $\Delta a_1 = f(1500) - x_{fix}$! How are Δa_0 and Δa_1 related?

   iii. Compute Δa_n and a_n!

   (c) The initial balance of John's account is 77 HUF. He deposits 33 HUF each year and the bank pays 5 percent interest. What will be the balance after 20 years?

(d) The initial balance of John's account is 77 HUF. He deposits 33 HUF each year and the bank pays zero percent interest. What will be the balance after 20 years?

13. (a) $p(x) = 2x^3 + 13x^2 + 8x$.

    i. If $x \approx 0$, then which term is the dominant one in $p(x)$? (Take for example $x = 0.001$ or $x = 10^{-6}$.)

    ii. If $|x|$ is large, then which term is the dominant one in $p(x)$? (Take for example $x = 1000$ or $x = 10^6$.)

    (b) $q(x) = 2x^{-3} + 13x^{-2} + 8x$.

    i. If $x \approx 0$, then which term is the dominant one in $q(x)$? (Take for example $x = 0.001$ or $x = 10^{-6}$.)

    ii. If $|x|$ is large, then which term is the dominant one in $q(x)$? (Take for example $x = 1000$ or $x = 10^6$.)

14. In these exercises compute the result up to 0.1% percent accuracy and do not use calculators!

    (a) i. The growth rate of the GDP was 1% last year, now it is 3%. What is the compound growth in two years?

    ii. The interest yield of your account is 3%. If you want a 1000 HUF balance at the end of a two year period, what should be your initial deposit?

    iii. Determine the approximate values of

$$(1.02)^2, \quad (1.02)^, \quad \frac{1}{1.02^3}, \quad \frac{1}{1.02^3}, \quad \frac{1.05}{1.02}.$$

    iv. June buys a house for 3000 USD. She needs to pay 1000 USD yearly for three consecutive years. She can invest her money in government bonds with 1% yield. What is the (approximate) difference between 3000 USD and the net present value of the transaction?

    (b) Discuss the validity of the following statements:

    i. Assume that $x \approx 0$. Then

$$(1+x)^2 \approx 1 + 2x, \quad \frac{1}{1+x} \approx 1 - x, \quad \frac{1}{1-x} \approx 1 + x.$$

    How much is $(1+x)^3$ in this approximation?

    ii. For what values of n can you say that $(1+x)^n \approx 1 + nx$?

15. Consider the deterministic nonlinear dynamical systems generated by the functions

$$f_1(x) = \frac{1}{3}x(1-x), \qquad f_2(x) = 3x(1-x),$$

with initial condition $a_0 = 0.01$. Note that $x = 0$ is a fixed point of these systems and $0 \approx a_0$. If $x \approx 0$, then $x(1-x) = x - x^2 \approx x$, so it can be said that:
The linear dynamical systems

$$\overline{f}_1(x) = \frac{1}{3}x, \qquad \overline{f}_2(x) = 3x, \qquad a_0 = 0.01$$

are fairly good approximations of the nonlinear ones.

(a) Discuss the validity of this statement!

(b) How does your answer depend on the stability of the fixed point $x = 0$?

(c) Plot the following functions of x:

$$1-x, \quad x^2-1, \quad 1-x^2, \quad x(x-1), \quad x(1-x), \quad x^2(x-1), \quad x-x^3.$$

(d) Try to do find the first few members of the sequences

$$a_0,\ a_1 = f_i(a_0),\ a_2 = f_i(a_1) = f_i^2(a_0),\ \ldots\ldots\ a_n = f_i(a_{n-1}) = f_i^n(a_0),\ \ldots \qquad i = 1, 2.$$

by graphical iteration!

16. Denote by a_n the worth of Alice's wealth in the n^{th} year. Initially $a_0 = 100000$ EUR. Suppose that Alice loses 10% of her wealth every year.

(a) Compute a_n !

(b) When will a_n be smaller than 1000 EUR? And 10, 1, ϵ EUR?

(c) What is $\lim_{n\to\infty} a_n$? Prove your claim!

(d) What happens if Alice not loses, but gains 10% each year? At what time will her wealth surpass 200000, 1 million, 1 billion, 1 trillion EUR?

17. Check the following properties:

convergent, divergent, increasing, decreasing, bounded

for the following sequences:

$$13,\quad (-1)^n,\quad n+1,\quad n^3-n,\quad n^3-13n,\quad \frac{2}{3n+4},\quad \frac{(-2)^n}{3n+4},\quad \frac{n+6}{2n+5},\quad (-1)^n\frac{n+6}{2n+5},$$

$$2^n,\quad (-2)^n,\quad \left(\frac{1}{2}\right)^n,\quad \left(-\frac{1}{2}\right)^n,\quad 2^{-n},\quad \log_2(n),\quad \log_2(n^3),\quad \log_2(\sqrt{n}),\quad \log_{0.5}(n),\quad \frac{2^n3^{2n+1}}{5^n4^{n+9}}.$$

18. Find the increasing and the decreasing sequences in the following list:

$$\frac{n+6}{2n+5},\qquad \frac{7n+6}{2n+5},\qquad \frac{n+6}{2n-5},\qquad \frac{7n+6}{2n-5}.$$

Prove your claims!

19. **Quiz 1.** Exercises: 1b, 2b, 3b, 5, 7b, 8, 9b, 12a 17, 18. The quiz will consist of three or four of these types of problems.

20. **Quiz 1.**

(a) $\overline{m} = (0,2)^T$, $\overline{p} = (3,1)^T$. Solve $\alpha\overline{m} + \beta\overline{p} = (10,4)^T$ for α and β !

(b) How much is $|(3,4,5)^T|$? How much is $(3,1,4,5)^T(1,3,4,5)^T$?

(c) Let $\overline{r}_0 = (1,1,1)^T$ and $\overline{n} = (2,2,2)^T$. Find an equation of the plane that contains $\overline{r}_0$ and has normal vector $\overline{n}$!
Express the third coordinate $z(x,y)$ of the point (x,y,z) of the plane with x and y !

(d) Let $f(x) = 3x+4$. If $a_0 = 13$ and $a_{n+1} = f(a_n) = 3a_n + 4$, then how much is a_n ?

Third week

21. (a) Your initial deposit is 1 HUF, the bank pays (100/12)% percent interest for (1 month)=(1/12 year).

i. What is your balance after 7 months (where 7 months equals to 7/12 year) ?

ii. What is the rate of change of your balance between the begining of the seventh and the eighth months of the year?

iii. What is the relation between the answers for the previous two questions?

(b) Consider the sequence $a_n = \left(1+\frac{1}{N}\right)^n$, where N is a positive integer. Plot a_n as the set of points $\{(n/N, a_n) \mid n \in \mathbb{N}\}$ and join the consecutive points by straight line segments. In this way you get a function $\exp_N(x)$, defined on $[0,\infty)$.

i. How much is $\exp_N(1)$? How much is $\lim_{N\to\infty}\exp_N(1)$?

ii. If x is a multiple of $1/N$, then how much is $\exp_N(x)$? How much is $\lim_{N\to\infty}\exp_N(x)$?

iii. What is the slope of $\exp_N(x)$ in the interval $[x, x+1/N]$?

iv. What is the relation between the answers for the previous two questions?

22. Given a sequence a_n, define the a new sequence $\Delta a_n = a_{n+1} - a_n$. Plot a_n as the set of points $\{(n, a_n) \mid n \in \mathbb{N}\}$ and join the consecutive points by straight line segments. Call the obtained function $a(x)$.

(a) Compute Δa_n for the following a_n sequences:

$$n, \qquad n^2, \qquad n^3, \qquad 2^n, \qquad 3^n, \qquad \frac{1}{2^n}.$$

What is the slope of a ?

(b) Find some (maybe approximative) relation between a_n and Δa_n !

(c) Find the sequence a_n if $\Delta a_n = n^k$, $k = 0,1,2$ and $a_0 = 0$!

23. 19. Study the monotonicity, boundedness and convergence of the following sequences!

2^n, $(-2)^n$, 0.99^n, $(-0.99)^n$, $\frac{2n+2}{3n+6}$, $(-1)^n\frac{2n+2}{3n+6}$, $(-1)^n\frac{2}{3n+6}$,

24. 20. Compute the limits of the following sequences!

$\frac{2n^3+2n}{3n^3+6}$, $\frac{2n^3+2n}{3n^4+6}$, $\frac{2n^4+2n}{3n^3+6}$, $\left(1-\frac{3}{2n}\right)^{4n+3}$, $\left(2-\frac{3}{2n}\right)^{4n+3}$, $\left(0.9-\frac{3}{2n}\right)^{4n+3}$, $\frac{2^{3n}}{3^n5^{n+6}}$.

25. 21. One receives 1%/year interest for a 1 EUR deposit. What will be the balance of the account after 100 years? What is the relation of the result with the number e? Repeat the exercise for 2%/year interest rate!

26. **Quiz 2.** Exercises:

(a) Compute $\lim_{n\to\infty}\left(1+\frac{4}{3n}\right)^{2n+2}$!

(b) Compute $\lim_{n\to\infty}\left(2-\frac{6}{5n}\right)^{n-2}$!

(c) Compute $\lim_{n\to\infty}\left(0.4-\frac{6}{5n}\right)^{5n-2}$!

(d) Compute $\lim_{n\to\infty}\left(1+\frac{1}{n/2}\right)^{3n-2}$!

(e) Let $f(x) = x^3 - 5x$, $x_0 = 1$. Compute $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$!

(f) Let $f(x) = 1 - 5x$, $x_0 = 1$. Compute $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$!

(g) Let $f(x) = 2^x$. Compute $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$!

(h) Let $f(x) = -x^2 + 5x$, $x_0 = 3$. What is the prediction of the linear approximation of f around x_0 for the value of $f(3+\Delta x)$?

(i) Let $f(x) = e^{3x}$, $x_0 = 4$. What is the prediction of the linear approximation of f around x_0 for the value of $f(3+\Delta x)$?

(j) Compute $(x^3\sin(4x))'$!

(k) Compute $(x^3/\sin(4x))'$!

(l) Compute $(\cos(\sin(4x)))'$!

(m) Compute $((\sin(4x))^3)'$!

(n) Compute $((5x)^3 + \sin(4x) - \sqrt[3]{x^5})'$!

The quiz will consist of three or four of these types of problems.

27. **Quiz 2.**

(a) Compute $\lim_{n\to\infty} \left(1 - \frac{6}{5n}\right)^{3n-2}$!

(b) Let $f(x) = 3x^2 - 5x,\ x_0 = 7$. Compute $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$!

(c) Let $f(x) = 3x^2 - 5x,\ x_0 = 7$. What is the prediction of the linear approximation of f around x_0 for the value of $f(7 + \Delta x)$?

(d) Compute $((x + 3)\cos(4x))'$!

Fourth week

28. 23. What is the prediction of the linear approximation of the function $f(x)$ at $x = x_0$ for the value of $f(x_0 + \Delta x)$?

- $f(x) = 1/x,\ x_0 = 3,\ \Delta x = 0.1,\ 0.01,\ 0.001.$
- $f(x) = \sqrt{x},\ x_0 = 9,\ \Delta x = 0.1,\ 0.01,\ 0.001.$

29. 24. Compute the derivatives of the following functions!

- $\sqrt[3]{x^4} + \frac{1}{x^6} + \cos(3x) + \ln(2x)$
- $\sin\left(\sqrt[3]{x}\right)$
- $x^7 \cos(2x - 1)$
- $\frac{\sin(2x)}{x^2+1}$
- $\frac{\ln(2x^2+8)}{\ln(x)}$

30.
- 25.* What is the price elasticity of the demand function $q(p) = 1/p^2$.
  ("price elasticity of the demand function $q(p)$": $\frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q} q'(p)$.
- Plot the following $q(p)$ functions together with their price elasticity! Mark the "economically relevant" regions of the plots!

$$1 - p, \quad 3 - 4p, \quad 1/p, \quad 3/(2p + 5), \quad 3/(2p + 5)^2, \quad 2^{-p}, \quad \sqrt[2]{-x + 5}.$$

31. Draw the graphs of the functions $y(x) =$

$$\sqrt{x}, \quad 1/x^2, \quad 5 \cdot 10^{2x}$$

in the $x - y,\ x - \log y,\ \log x - \log y$ coordinate systems!

32. 26. Study the monotonicity, convexity and local extremal values of the following functions!

$$x^2 - 2x,\ x^3 - x,\ x^2 - x^4,\ x\ln(x),\ xe^{-3x}.$$

Draw their graphs!

33. 27. Calculate the following derivatives!

$$\frac{d(x^4a^3b^2)}{dx}, \frac{d(x^4a^3b^2)}{da}, \frac{d(x^4a^3b^2)}{db},$$

34. Linear approximation: $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$.

- Let $f(x) = x^2$, $x_0 = 3$. What is the prediction of the linear approximation for $f(x_0 + \Delta x)$? Plot f and its tangent line at x_0!
- Let $f(x) = 3x + 4$, $x_0 = 3$. What is the prediction of the linear approximation for $f(x_0 + \Delta x)$?
- Let $f(x) = e^x$, $x_0 = 1$. What is the prediction of the linear approximation for $f(x_0 + \Delta x)$?

35. (Computer ex.) Error of the linear approximation. I.

Let $x_0 = 0$. Compute $|f(x_0 + \Delta x) - (f(x_0) + f'(x_0)\Delta x)| = error(\Delta x)$ for the following functions and for $\Delta x = 0.1, 0.01, 0.001, 0.0001$,

$$f(x) = (x-1)^2, \quad 3x - 5, \quad \sin(x), \quad \cos(3x).$$

- Plot $error(\Delta x)$ versus Δx. Assume that for small Δx the error behaves as $error(\Delta x) \approx C\Delta x^\alpha$. Try to read off the value of α from the plot!
- Plot $\ln(error(\Delta x))$ versus $\ln(\Delta x)$. Assume that for small Δx the error behaves as $error(\Delta x) \approx C\Delta x^\alpha$. Read off the value of α from the plot!

36. Error of the linear approximation. II.
Four cars take part of a car race: F, Lin, Fast, Slow. At $x = 0$ (i.e. at time zero) their positions and their velocities are the same: $pos = 3$, $velocity = 5$. Assume that the absolute value of their accelerations can not be more than 7 (in SI units). Suppose that they behave as follows:

- there is no additional constraint on F,
- Lin travels with constant speed,
- the acceleration of Fast is constantly 7,
- the acceleration of Slow is constantly -7.

Verify that the positions of the cars are described by the followig functions:

- $F(x)$,
- $Lin(x) = 3 + 5x$,
- $Fast(x) = 3 + 5x + \frac{1}{2} \cdot 7x^2$,
- $Slow(x) = 3 + 5x - \frac{1}{2} \cdot 7x^2$.

Plot $Lin(x)$, $Fast(x)$, $Slow(x)$ for $x = 0 \ldots 0.3$!
What are the possible orders of the cars?
Prove that

$$|F(x) - Lin(x)| \leq \frac{1}{2} 7x^2.$$

Prove (or at least argue in favour of) the following statements:

- If $\max_{z\in[0,x]} |f''(z)| \leq A$, then
$$|f(x) - (f(0) + f'(0)x)| \leq \frac{A}{2}x^2$$
- If $\max_{z\in[x,x+\Delta x]} |f''(z)| \leq A$, then
$$|f(x + \Delta x) - (f(x) + f'(x)\Delta x)| \leq \frac{A}{2}\Delta x^2$$

37. • Find the zeroth order Taylor approximation of the following functions around $x = 0$:

$$5, \quad (x-1)^2, \quad e^x, \quad e^{2x}, \quad \sin(3x), \quad 1/(x-2), \quad \ln(1-x).$$

• Find the first order Taylor approximation of the following functions around $x = 0$:

$$5, \quad (x-1)^2, \quad e^x, \quad e^{2x}, \quad \sin(3x), \quad 1/(x-2), \quad \ln(1-x).$$

What is the other name of the first order approximation?

• Find the third order Taylor approximation of the following functions around $x = 0$:

$$5, \quad (x-1)^2, \quad e^x, \quad e^{2x}, \quad \sin(3x), \quad 1/(x-2), \quad \ln(1-x).$$

38. • Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!

$$5,\ x,\ x^2-2,\ x^2-x,\ \sqrt[3]{2x^7}+\sqrt[3]{(2x)^7}+\frac{7}{x^7},$$
$$e^x+\sin(x),\ e^{3x}+\sin(3x).$$

• Find the general solutions of the following differential equations!

$$y'(x)=5,\ y'(x)=x,\ y'(x)=x^2-2,\ y'(x)=x^2-x,\ y'(x)=\sqrt[3]{2x^7}+\sqrt[3]{(2x)^7}+\frac{7}{x^7},$$
$$y'(x)=e^x+\sin(x),\ y'(x)=e^{3x}+\sin(3x).$$

• Find the particular solutions of the following differential equations!

$$y'(x)=5,\ y(1)=4, \quad y'(x)=x,\ y(-1)=7, \quad y'(x)=x^2-2,\ y(0)=0.$$

39. Compute the $\int f(x)\,dx$ indefinite integrals of the following functions!

$$x\cdot e^x, \quad x\cdot\sin(x), \quad x\cdot\cos(x), \quad x^3\ln(x), \quad 1\cdot\ln(x),$$
$$x\cdot e^{-x}, \quad x\cdot\sin(2x), \quad x\cdot\cos(3x), \quad x^3\ln(4x), \quad 1\cdot\ln(5x).$$

40. • A car moves in one dimension, its position and velocity are described by $x(t)$ and $v(t)$, where t is the time variable. Let $x(1) = 2\,m$, $v(1) = 3\,m/sec$. Assume that the car has constant $4\,m/sec^2$ acceleration.

  – Compute $v(t)$!
  – Compute $x(t)$!

Solve the following differential equations! Find first the general solutions, then the particular ones!

  – DE: $v'(t) = 4$, $v(1) = 3$. Compute $v(t)$!
  – DE: $x''(t) = 4$, $x'(1) = 3$, $x(1) = 2$. Compute $x(t)$!

41. **Test 1.** Exercises: 1b, 2b, 3b,3c,3d, 5c,7b,8,9b,11(a)iii, 12a,17,18,19, 20,23,24,26,27, 29,32,33,34,37,38, 39

42. **Sample Test 1.**

• (a) $\overline{m} = (4,2)^T$, $\overline{p} = (0,1)^T$. Solve $\alpha\overline{m} + \beta\overline{p} = (10,4)^T$ for α and β !

(b) $y = f(x) = 7 - 2x$.

i. Where are the intersections of this line with the x and y axes?
ii. What is the slope of this function?
iii. Express x with y !
iv. What is the f^{-1} inverse of the function f ?
v. Plot f and f^{-1} together!

(c) Compute $\log_2(1/2)$, $\log_2(1/32)$, $\log_{1/2}(32)$!
Solve the equation $5e^x = 6$ for x !

(d) Compute

$$S_{10} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{10}}.$$

(e) Check the following properties:

convergent, divergent, increasing, decreasing, bounded

for the sequences in the table! Fill in the table with Y and N (Yes or No)!

	conv.	div.	bound.	inc.	dec.
$(-2)^n$					
$2 - 1/n$					
$(-1)^n(2 - 1/n)$					
$\ln(3n + 1)$					

(f) Compute $\lim_{n\to\infty} \left(1 + \frac{5}{n/7}\right)^{2n-2}$!

(g) Let $f(x) = -x^2 + 5x$, $x_0 = 3$. What is the prediction of the linear approximation of f around x_0 for the value of $f(3 + \Delta x)$?

(h) – Compute $((\sin(4x))^3)'$!
– Compute $((5x)^3 + \sin(4x) - \sqrt[3]{x^5})'$!

(i) Find the particular solutions of the following differential equations!

$$y'(x) = 5,\ y(1) = 4, \quad y'(x) = x,\ y(-1) = 7, \quad y'(x) = x^2 - 2,\ y(0) = 0.$$

- (a) i. Let $\overline{r}_0 = (0, 0, 15)^T$ and $\overline{n} = (3, 5, 1)^T$.
ii. Find an equation of the plane that contains $\overline{r}_0$ and has normal vector $\overline{n}$!
iii. Express the third coordinate $z(x, y)$ of a point (x, y, z) of the plane with x and y !
iv. How much does $z(x, y)$ change when x is increased by 1 while y is left unchanged?
v. How much does $z(x, y)$ change when y is increased by 1 while x is left unchanged?
vi. Find the intersection of the plane with the x, y and z axes!

(b) Solve the following equations for x:
i. $2^{5x-6} = 12$,
ii. $\ln(x^2) - 4\ln(x) = 5$.

(c) Let $f(x) = 2 - 3x$. If $a_0 = 9$ and $a_{n+1} = f(a_n) = 2 - 3a_n$, then how much is a_n ?

(d) What is the limit of the sequence $a_n = \frac{4n+6}{6n-5}$?
Is a_n increasing or decreasing? Prove your answer!

(e) i. Compute $\lim_{n\to\infty} \left(1/2 + \frac{4}{3n}\right)^{2n+2}$!
ii. Compute $\lim_{n\to\infty} \frac{6+n^2-n^3}{5n^3-3n+1}$!

(f) Let $f(x) = x^3$, $x_0 = 1$.
i. Compute $\frac{\Delta f}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$!
ii. Compute $\lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$!

(g) Compute
i. $(\cos(\sin(4x)))'$,

ii. $(x^3/\sin(4x))'$.

(h) Study the monotonicity, convexity and local extremal values of the $f(x) = x^4 - x^2$ function! Draw its graphs!

(i) Find the third order Taylor approximation of the $f(x) = e^{-3x}$ function around $x = 0$!