Math.Econ.Anal.Test.2. 15.nov.28.

1. (2+3+2+3 points)
Compute the following integrals:

(a) $\int \sqrt[4]{3x^5} + \sqrt[2]{(5x)^6} + \frac{7}{3x}\, dx$

(b) $\int x \cdot e^{5x}\, dx$

(c) $\int_2^5 3 + 4x\, dx$

Solve the following DE: $y'(x) = -10x + 3, y(1) = 2$.

$\int \frac{1}{3x} dx = \frac{1}{3} \int \frac{1}{x} dx = \frac{1}{3} \ln|x| + C$

or $= \int \frac{1}{(3x)} dx = \frac{\ln|3x|}{3} + \tilde{C}$

a) $\int \sqrt[4]{3x^5} + \sqrt[2]{(5x)^6} + \frac{7}{3x} dx = \int \sqrt[4]{3} \cdot x^{5/4} + (5x)^{6/2} + \frac{7}{3} \cdot \frac{1}{x} dx$

$= \sqrt[4]{3}\, \frac{x^{9/4}}{9/4} + \frac{\frac{(5x)^4}{4}}{5} + \frac{7}{3} \ln|x| + C$

b) $\int x\, e^{5x} dx = \left| \begin{matrix} f' = e^{5x} & g = x \\ f = \frac{e^{5x}}{5} & g' = 1 \end{matrix} \right| = \frac{e^{5x}}{5} \cdot x - \int \frac{e^{5x}}{5} \cdot 1\, dx = \frac{e^{5x}}{5} \cdot x - \frac{e^{5x}}{5 \cdot 5} + C$

Part.int:
$\int f'g = fg - \int fg'$

c) $\int_2^5 3 + 4x dx = \left[3x + 4\frac{x^2}{2} \right]_2^5 = \left(3 \cdot 5 + 4 \cdot \frac{5^2}{2}\right) - \left(3 \cdot 2 + 4 \cdot \frac{2^2}{2}\right) = 65 - 14$
$= 51$

$y'(x) = -10x + 3$

general solution: $y_{gen}(x) = \int -10x + 3\, dx = -10 \cdot \frac{x^2}{2} + 3x + C = -5x^2 + 3x + C$

initial cond.: $y(1) = 2 \longrightarrow -5 \cdot 1^2 + 3 \cdot 1 + C = 2 \longrightarrow C = 4$

particular solution: $y_{part}(x) = -5x^2 + 3x + 4$

2. (2+4+4 points)
A) Solve the following DE: $y'(x) = -4y(x)$, $y(0) = 3$.
B) Solve the following DE: $y'(x) = -4y(x) + 8$.
C) Let $f(x) = (x+2)^5 y^6$. Compute $f'_x, f'_y, f''_{xy}, f''_{yx}$!

A) $y'(x) = -4y(x)$
general solution $y(x) = C \cdot e^{-4x}$
initial condition: $y(0)=3 \longrightarrow C \cdot e^{-4 \cdot 0} = 3 \longrightarrow C = 3$
particular solution: $y(x) = 3 \cdot e^{-4x}$

B) $y'(x) = -4y(x) + 8$
Fixed point: $0 = -4y_{fix} + 8 \longrightarrow y_{fix} = 2$
$\Delta y(x) = y(x) - y_{fix}$, $\Delta y' = y'$, so $\Delta y' = -4\Delta y$.
Consequently $\Delta y = C \cdot e^{-4x}$.
$y(x) = \Delta y(x) + y_{fix} = \Delta y(x) + 2$,
so the general solution is $y_{gen}(x) = C \cdot e^{-4x} + 2$

c) $f(x) = (x+2)^5 y^6$

$f'_x = 5(x+2)^4 y^6$

$f'_y = (x+2)^5 \cdot 6 \cdot y^5$

$f''_{xy} = (f'_x)'_y = [5 \cdot (x+2)^4 y^6]'_y = 5 \cdot (x+2)^4 \cdot 6 \cdot y^5$

$f''_{yx} = f''_{xy}$

3. (6+2+2 points)
A) Find the critical point of $f(x,y) = x^2 - 2y^2 - 2xy + 6x + 7$ and determine the critical point's type!
Ba) Roll a fair dice. Are the events $odd =$ "*the outcome is odd*" and $large =$ "*the outcome is larger then three*". are independent?
Bb) Compute the conditional probability $p(odd\,|large)$!

A) $f = x^2 - 2y^2 - 2xy + 6x + 7$ $\boxed{\text{3points}}$

$f'_x = 2x - 0 - 2y + 6 + 0$

$f'_y = 0 - 4y - 2x + 0 + 0$

$f''_{xx} = (f'_x)'_x = (2x - 2y + 6)'_x = 2$

$f''_{yx} = f''_{xy} = (f'_x)'_y = (2x - 2y + 6)'_y = -2$

$f''_{yy} = (f'_y)'_y = (-4y - 2x)'_y = -4$

location of crit. point: $\boxed{\text{2points}}$

$\left.\begin{array}{l} f'_x = 0 = 2x - 2y + 6 \\ f'_y = 0 = -4y - 2x \end{array}\right\} \rightarrow y = 1, \; x = -2$

$P_{crit} = (-2, 1)$

Type of P_{crit}: $\text{Hessian}(f) = \begin{pmatrix} 2 & -2 \\ -2 & -4 \end{pmatrix}$

$(\text{Hessian}(f))(P_{crit}) = \begin{pmatrix} 2 & -2 \\ -2 & -4 \end{pmatrix}$

$\boxed{\text{1point}}$

1 $2 \cdot (-4) - (-2) \cdot (-2) = -12$

$-12 < 0 \rightarrow$ saddle point

Ba) $odd = \{1, 3, 5\}$ $large = \{4, 5, 6\}$ $odd \cap large = \{5\}$

"odd" and "large" are independent $\Longleftrightarrow p(odd \cap large) = p(odd) \cdot p(large)$

but $\frac{1}{6} \neq \frac{3}{6} \cdot \frac{3}{6}$, False

Bb)

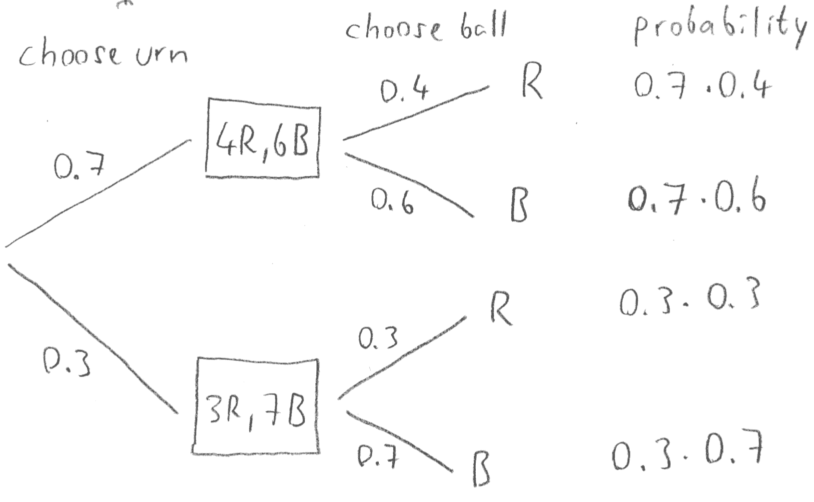
$$p(odd \mid large) = \frac{p(odd \cap large)}{p(large)} = \frac{1/6}{3/6} = \frac{1}{3}$$

4. (6+(2+2) points)

A) There are two urns containing colored balls. The first urn contains 4 red balls and 6 blue balls. The second urn contains 3 red balls and 7 blue balls. One of the two urns is randomly chosen in such a way that the first urn is chosen with 0.7 probability. Then a ball is drawn at random from one of the ~~two~~ chosen urn~~s~~. If a red ball is drawn, what is the probability that it comes from the first urn?

B) Toss ~~a~~ fair coin two times, and let h denote the number of heads. Compute $E[h]$ and $Var[h]$!

A)



$$p(1^{st}\text{ urn} \mid \text{Red drawn}) = \frac{0.7\cdot 0.4}{0.7\cdot 0.4 + 0.3\cdot 0.3}$$

Or by Bayes th.:

$$P(1^{st}\text{ urn} \mid \text{Red drawn}) = \frac{P(\text{Red drawn} \mid 1^{st}\text{ urn})\cdot p(1^{st}\text{ urn})}{p(\text{Red drawn})}$$

$$= \frac{0.4\cdot 0.7}{0.7\cdot 0.4 + 0.3\cdot 0.3}$$

B) $\Omega = \{HH, HT, TH, TT\}$, $p(HH) = \ldots = p(TT) = \frac{1}{4}$

$h(HH) = 2$, $h(HT) = h(TH) = 1$, $h(TT) = 0$

$$E[h] = \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 0 = 1$$

$$Var[h] = \frac{1}{4}\cdot(2-1)^2 + \frac{1}{4}(1-1)^2 + \frac{1}{4}(1-1)^2 + \frac{1}{4}\cdot(1-0)^2 = \frac{1}{2}$$

*Remark: $E[h_1+h_2] = E[h_1] + E[h_2]$ is alway true, but $Var[h_1+h_2] = Var[h_1] + Var[h_2]$ is not necessarily true if h_1 and h_2 are NOT independent rand. vars.

Or: Firstly solve ex. for $\Omega = \{H, T\}$, $p(H) = p(T) = \frac{1}{2}$, $h_0(H) = 1$, $h_0(T) = 0$.

$E[h_0] = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{2}$, $Var[h_0] = \frac{1}{2}(1-\frac{1}{2})^2 + \frac{1}{2}(0-\frac{1}{2})^2 = \frac{1}{4}$

Then $h = h_1 + h_2$, where h_1 counts the heads for the first toss,
while h_2 ——"—— second —"—.

h_1 and h_2 are independent of each other, they have the same expected value and variance as h_0 has.

So $E[h] = E[h_1+h_2] = E[h_1] + E[h_2] = 2E[h_0] = 2\cdot\frac{1}{2} = 1$

$Var[h] = Var[h_1+h_2] = Var[h_1] + Var[h_2] = 2\cdot Var[h_0] = 2\cdot\frac{1}{4} = \frac{1}{2}$ *