# Math.Econ.Anal.MakeUp.Test.1. 15.dec.2.

1. (3+4+3 points)
   A) $\overline{m} = (-2,2)^T$, $\overline{p} = (2,2)^T$. Solve $\alpha\overline{m} + \beta\overline{p} = (10,4)^T$ for $\alpha$ and $\beta$ !
   B)Let $a_n = \frac{3n+2}{5n+4}$. Is $a_n$ increasing or decreasing? (Prove it!)
   Is $a_n$ convergent as $n \to \infty$?
   If the answer is yes, what is the limit of $a_n$ ?
   C) Compute $\lim_{n\to\infty} \left(1 - \frac{6}{5n}\right)^{5n-2}$ !

2. (2+4+4 points)
   A) Let $\overline{r}_0 = (1,1,1)^T$ and $\overline{n} = (2,2,2)^T$. Find an equation of the plane that contains $\overline{r}_0$ and has normal vector $\overline{n}$ !
   Express the third coordinate $z(x,y)$ of the point $(x,y,z)$ of the plane with $x$ and $y$ !
   B) Let $f(x) = 3x+4$. If $a_0 = 13$ and $a_{n+1} = f(a_n) = 3a_n + 4$, then how much is $a_n$ ?
   B) How much is $|(3,4,5)^T|$ ? How much is $(3,1,4,5)^T(1,3,4,5)^T$ ?

3. (5 × 2 points)
   (a) Compute $(x^3\cos(4x))'$ !
   (b) Compute $(\cos(\sin(4x)))'$ !
   (c) Compute $((\sin(-2x))^2)'$ !
   (d) Compute $((-x)^3 + \sin(4x) - \sqrt[3]{(3x)^5})'$ !

   B) Let $f(x) = e^{-x}$, $x_0 = 1$. What is the prediction of the linear approximation of $f$ around $x_0$ for the value of $f(x_0 + \Delta x)$ ?

4. (3+2+5 points)
   A) Let $f(x) = 2^x$, $x_0 = 1$. Compute $\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ !
   B) Compute $\lim_{n\to\infty} \left(0.4 - \frac{6}{5n}\right)^{5n-2}$ !
   C) Study the monotonicity, convexity and the local extremal values of the function $f(x) = x - x^2$ ! Draw the graphs of $f$ and $f'$ in the same coordinate system!

A) $\alpha\begin{pmatrix}-2\\2\end{pmatrix} + \beta\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}10\\4\end{pmatrix} \longrightarrow \begin{pmatrix}-2\alpha+2\beta\\2\alpha+2\beta\end{pmatrix} = \begin{pmatrix}10\\4\end{pmatrix} \longrightarrow \begin{matrix}-2\alpha+2\beta=10\\2\alpha+2\beta=4\end{matrix} \rightarrow \begin{matrix}4\beta=14\\4\alpha=-6\end{matrix}$

$\longrightarrow \alpha = -\frac{3}{2}, \; \beta = \frac{7}{2}$

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B) $a_{n+1} - a_n = \frac{3(n+1)+2}{5(n+1)+4} - \frac{3n+2}{5n+4} = \frac{(3n+5)(5n+4)-(5n+9)(3n+2)}{(5n+9)(5n+4)} = \frac{2}{(5n+9)(5n+4)} > 0$

for $n = 1,2,3\ldots$

so $a_n$ is increasing.

$\lim_{n\to\infty} \frac{3n+2}{5n+4} = \lim_{n\to\infty} \frac{3+2/n}{5+4/n} = \frac{3}{5}$

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C) $\lim_{n\to\infty} \left(1-\frac{6}{5n}\right)^{5n-2} = \lim_{n\to\infty} \left[\left(1+\frac{-6/5}{n}\right)^n\right]^5 \cdot \left(1-\frac{6}{5n}\right)^{-2} =$

$= \left[e^{-6/5}\right]^5 \cdot 1^{-2} = e^{-6}$

2. (2+4+4 points)
A) Let $\overline{r}_0 = (1,1,1)^T$ and $\overline{n} = (2,2,2)^T$. Find an equation of the plane that contains $\overline{r}_0$ and has normal vector $\overline{n}$ !
Express the third coordinate $z(x,y)$ of the point $(x,y,z)$ of the plane with $x$ and $y$ !
B) Let $f(x) = 3x+4$. If $a_0 = 13$ and $a_{n+1} = f(a_n) = 3a_n + 4$, then how much is $a_n$ ?
C ~~B~~) How much is $|(3,4,5)^T|$ ? How much is $(3,1,4,5)^T(1,3,4,5)^T$ ?

(A) $\overline{n}\cdot(\overline{r}-\overline{r}_0)=0 = \begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\left[\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right] = \begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}x-1\\y-1\\z-1\end{pmatrix} = 2(x-1)+2(y-1)+2(z-1)$

$= 2x+2y+2z-6 = 0$

$z(x,y) = \dfrac{6-2x-2y}{2} = 3-x-y$

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(B) Fixed point $x_f$: $\quad x_f = 3x_f + 4 \longrightarrow x_f = -2$

Linerized dynamics around $x_f$:

$\Delta a_n = a_n - x_f = a_n - (-2) \longleftrightarrow a_n = \Delta a_n + (-2)$

$\Delta a_{n+1} = 3\,\Delta a_n \quad , \Delta a_0 = 13-(-2) = 15$

$\Delta a_n = 3^n \cdot [13-(-2)]$

$a_n = \Delta a_n + (-2) = 3^n \cdot [13-(-2)] + (-2)$

$= 3^n \cdot 15 - 2$

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(C) $\left|\begin{pmatrix}3\\4\\5\end{pmatrix}\right| = \sqrt{3^2+4^2+5^2} = \sqrt{50}$

$\begin{pmatrix}3\\1\\4\\5\end{pmatrix}\cdot\begin{pmatrix}1\\3\\4\\5\end{pmatrix} = 3\cdot1+1\cdot3+4\cdot4+5\cdot5 = 47$

3. ($5 \times 2$ points)

(a) Compute $(x^3 \cos(4x))'$ !

(b) Compute $(\cos(\sin(4x)))'$ !

(c) Compute $((\sin(-2x))^2)'$ !

(d) Compute $((-x)^3 + \sin(4x) - \sqrt[3]{(3x)^5})'$ !

B) Let $f(x) = e^{-x}$, $x_0 = 1$. What is the prediction of the linear approximation of $f$ around $x_0$ for the value of $f(x_0 + \Delta x)$ ?

a) $\left[x^3 \cos(4x)\right]' = (x^3)' \cdot \cos(4x) + x^3 \cdot \left[\cos(4x)\right]'$

$= 3x^2 \cdot \cos(4x) + x^3 \cdot \left[-\sin(4x) \cdot 4\right]$

b) $\left[\cos(\sin(4x))\right]' = \cos'(\sin(4x)) \cdot \left[\sin(4x)\right]'$

$= -\sin(\sin(4x)) \cdot \cos(4x) \cdot 4$

c) $\left[\left(\sin(-2x)\right)^2\right]' = \underset{\substack{\uparrow \\ \text{since } (x^2)' = 2x}}{\boxed{2(\sin(-2x))}} \cdot \underset{\substack{\uparrow \\ \sin'}}{\left[\cos(-2x) \cdot (-2)\right]}$

d) $\left[(-x)^3 + \sin(4x) - (3x)^{5/3}\right]' = 3(-x)^2 \cdot (-1) + \cos(4x) \cdot 4 - \frac{5}{3}(3x)^{\frac{2}{3}} \cdot 3$

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B) $f(x_0) = e^{-1} = \frac{1}{e}$

$f'(x) = \left[e^{-x}\right]' = -e^{-x}$

$f'(x_0) = -e^{-1} = -\frac{1}{e}$

Lin. approx.: $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$

$e^{-(1+\Delta x)} \approx \frac{1}{e} + \left(-\frac{1}{e}\right) \cdot \Delta x$

4. (3+2+5 points)

A) Let $f(x) = 2^x$, $x_0 = 1$. Compute $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ !

B) Compute $\lim_{n\to\infty} \left(0.4 - \frac{6}{5n}\right)^{5n-2}$ !

C) Study the monotonicity, convexity and the local extremal values of the function $f(x) = x - x^2$ ! Draw the graphs of $f$ and $f'$ in the same coordinate system!

(A) $$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \frac{2^{1+\Delta x}-2^1}{\Delta x} = \frac{2^1\cdot(2^{\Delta x}-1)}{\Delta x} = 2^1\cdot\frac{(2^{\Delta x}-1)}{2}$$

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(B) $$\lim_{n\to\infty}\left(0.4-\frac{6}{5n}\right)^{5n-2} = \lim_{n\to\infty}(0.4)^{5n-2} = 0$$

Since $\lim_{n\to\infty}\left(0.4-\frac{6}{5n}\right) = 0.4$

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(C) $f(x) = x - x^2$

$f'(x) = 1 - 2x$

$f''(x) = -2$

Location of critical point: $f'(x) = 0 = 1 - 2x \longrightarrow x_{crit} = \frac{1}{2}$

type of $x_{crit}$: $f''(\frac{1}{2}) = -2 < 0 \longrightarrow$ MAXIMUM