Test 1. Econ.Anal 14.Oct. NEPTUN: Name:

1\. A. Compute $C = \begin{pmatrix} -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

B. Compute the square of the Eucledian length of $\{1, 1, -1, 0\}^T$!

C. Compute $\begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}^{-1}$.

A $\begin{pmatrix} -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = (-3) \cdot 1 + (-1) \cdot 3 = -6$ 2p

B $\left| \begin{pmatrix} 1 & 1 & -1 & 0 \end{pmatrix}^T \right| = \left\| \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\| = \sqrt{1^2 + 1^2 + (-1)^2 + 0^2} = \sqrt{3}$

$\left\| \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\|^2 = 3$ 2p

C

$A A^{-1} = E$

$\begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 1

$\begin{pmatrix} 4x + 3y & 4u + 3v \\ 0x + 2y & 0u + 2v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2

$4x + 3y = 1$
$2y = 0$
$y = 0, \; x = \frac{1}{4}$

$4u + 3v = 0$
$2v = 1$
$v = \frac{1}{2}, \; u = -\frac{3}{8}$ 2

$\begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1/4 & -3/8 \\ 0 & 1/2 \end{pmatrix}$ 1

2. A. Let $f(x) = 3x^2 + 1$. Compute $\frac{f(5+\Delta x_n) - f(5)}{\Delta x_n}$! What is the limit of this fraction if $\Delta x_n = 1/n$? What is $f'(5)$?

B. Study the monotonicity and the limit of the following sequences!

a) $\frac{4-1}{5n+6}$, b) $\frac{3n}{5}(-1)^n$.

A $\frac{\Delta f_n}{\Delta x_n} = \frac{[3\cdot(5+\Delta x_n)^2+1]-[3\cdot 5^2+1]}{\Delta x_n} = 3\cdot 2\cdot 5 + 3\Delta x_n$

$\lim_{n\to\infty} 3\cdot 2\cdot 5 + 3\cdot\frac{1}{n} = 3\cdot 2\cdot 5 + 3\cdot 0 = 30$

$f'(5) = 30$ 4

B a) $4-1=3$

$a_{n+1} - a_n = \frac{3}{5(n+1)+6} - \frac{3}{5n+6} = \frac{3\cdot(5n+6) - 3\cdot(5n+11)}{(5n+11)\cdot(5n+6)}$

$= \frac{-15}{(5n+11)(5n+6)} < 0$ if $n = 0, 1, 2 \ldots$ decreasing 2

$\lim_{n\to\infty} \frac{3}{5n+6} = \lim_{n\to\infty} \frac{3/n}{5+6/n} = \frac{0}{5+0} = 0$ 1

b) $\frac{3n}{5}(-1)^n$ not monotone, as its sign is alternating. 1

divergent sequence, as $\frac{3n}{5} \to \infty$, but the $(-1)^n$ factor turns it to an alternating sequence.

2

3.A. Compute the limit of the following sequence! $a_n = \left(1 - \frac{1}{3n}\right)^{-3n+7}\left(\frac{2n^2}{3n^2+1}\right)$.

B. Let $\phi(x) = 3x - 2$, $x_0 = 2$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\phi^n(1) = x_n$?

That should be $X_0 = 2$, not 1

1. Compute ϕ^{-1} !
2. Find the fixed point x_f of ϕ !
3. Compute x_n !

(A)

$$\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^{-3n+7} \cdot \left(\frac{2n^2}{3n^2+1}\right) =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-1/3}{n}\right)^n\right]^{-3} \cdot \left(1 - \frac{1}{3n}\right)^7 \cdot \left(\frac{2}{3 + 1/n^2}\right) =$$

$$= \left[e^{-1/3}\right]^{-3} \cdot 1^7 \cdot \frac{2}{3+0} = e \cdot \frac{2}{3}$$

(4)

(B) φ^{-1}: $y = 3x - 2$

$$x = \frac{y+2}{3}$$

$$\varphi^{-1}(y) = \frac{y+2}{3}$$

$$\varphi^{-1}(x) = \frac{x+2}{3}$$

(2)

Fixed point: $x_f = \varphi(x_f)$

$$x_f = 3x_f - 2 \longrightarrow x_f = 1$$

(1)

X_n: $X_n = \varphi^n(1) = \varphi^n(x_f) = x_f = 1$ (luck!)

$$X_n = 3^n \cdot (X_0 - X_f) + X_f$$

So if $X_0 = 2$, $X_n = 3^n(2-1) + 1$

if $X_0 = 1$, $X_n = 3^n(1-1) + 1 = 1$

(3)

Both interpredations are accepted

3

4. A. Let $\bar{v}_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 12 \\ 82 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$. Compute $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$!

B. Let T be a 2×2 matrix formed by the transition probabilities of a two state (labeled by 1 and 2) stochastic system, where
$T(1 \leftarrow 1) = T_{11} = 1$, $T(2 \leftarrow 1) = T_{21} = 0$, $T(1 \leftarrow 2) = T_{12} = 0.2$, $T(2 \leftarrow 2) = T_{22} = 0.8$.

1. Find an eigenvector $\bar{v}_1$ corresponding to the eigenvalue $\lambda_1 = 1$! (This is the equilibrium state.)
2. Find the eigenvalue λ_2 of T corresponding to the eigenvector $\bar{v}_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$!
3. Calculate α and β in $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$!
4. Calculate $T(\alpha\bar{v}_1 + \beta\bar{v}_2)$, $T^2(\alpha\bar{v}_1 + \beta\bar{v}_2)$, etc.

A $\begin{pmatrix} 12 \\ 82 \end{pmatrix} = \alpha\begin{pmatrix} 2 \\ -2 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\alpha + 2\beta \\ -2\alpha + 2\beta \end{pmatrix}$

$$\left.\begin{aligned} 2\alpha + 2\beta &= 12 \\ -2\alpha + 2\beta &= 82 \end{aligned}\right\} \longrightarrow \beta = \frac{12+82}{4} = 23.5$$

$$\alpha = \frac{12-82}{4} = -17.5$$

B

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0.2 \\ 0 & 0.8 \end{pmatrix}$$

1 $T\bar{V}_1 = 1 \cdot \bar{V}_1$ $\quad \begin{pmatrix} 1 & 0.2 \\ 0 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{aligned} x + 0.2y &= x \\ 0.8y &= 1y \end{aligned} \longrightarrow y = 0, \; x \text{ arbitrary}$$

general form of $\bar{V}_1$: $\begin{pmatrix} x \\ 0 \end{pmatrix}$, normalized one: $\bar{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
($x+y=1$, probabilities)

2 $\begin{pmatrix} 1 & 0.2 \\ 0 & 0.8 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\cdot(-2) + 0.2\cdot 2 \\ 0\cdot(-2) + 0.8\cdot 2 \end{pmatrix} = \begin{pmatrix} -1.6 \\ 1.6 \end{pmatrix} = 0.8\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

so $\lambda_2 = 0.8$

3 $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -2 \\ 2 \end{pmatrix} \longrightarrow \beta = \frac{1}{2}, \; \alpha = 1$

4 $T^n\left[1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right] = 1 \cdot 1^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \cdot 0.8^n \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

Test 1. Econ.Anal 14.Oct. NEPTUN: Name:

1. A. Compute $C = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & -3 \end{pmatrix}$.

B. Compute the square of the Eucledian length of $\{-2, -2, -1, -2\}^T$!

C. Compute $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}^{-1}$.

A $\begin{pmatrix} -1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & -3 \end{pmatrix} = \begin{pmatrix} -1\cdot 1 & -1\cdot(-3) \\ -2\cdot 1 & -2\cdot(-3) \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & 6 \end{pmatrix}$

B $\left| (-2, -2, -1, -2)^T \right| = \left\| \begin{pmatrix} -2 \\ -2 \\ -1 \\ -2 \end{pmatrix} \right\| = \sqrt{(-2)^2+(-2)^2+(-1)^2+(-2)^2}$

$= \sqrt{13}$

$\left\| \begin{pmatrix} -2 \\ -2 \\ -1 \\ -2 \end{pmatrix} \right\|^2 = 13$

C $A A^{-1} = E$

$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1x+0y & 1u+0v \\ 4x+1y & 4u+1v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$1x = 1$ $\quad\quad$ $1u = 0$

$4x+1y=0$ $\quad\quad$ $4u+v=1$

$x=1, y=-4$ $\quad\quad$ $u=0, v=1$

$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$

2. A. Let $f(x) = -8x + 7$. Compute $\frac{f(5+\Delta x_n) - f(5)}{\Delta x_n}$! What is the limit of this fraction if $\Delta x_n = 1/n(-1)^n$? What is $f'(5)$?

B. Study the monotonicity and the limit of the following sequences!

a) $\frac{-2n}{3n+6}$, b) $\frac{3}{n^2}(-1)^n$.

A $\frac{\Delta f_n}{\Delta x_n} = \frac{[-8(5+\Delta x_n)+7] - [-8\cdot 5+7]}{\Delta x_n} = -8$

$\lim_{n\to\infty} -8 = -8$, so $f'(5) = -8$ (3)

B a) $a_{n+1} - a_n = \frac{-2(n+1)}{3(n+1)+6} - \frac{-2n}{3n+6} = \frac{-2(n+1)(3n+6) - (-2n)\cdot(3n+9)}{(3n+9)(3n+6)}$

$= \frac{-12}{(3n+9)(3n+6)} < 0$ if $n = 0, 1, 2 \dots$ decreasing (2)

$\lim_{n\to\infty} \frac{-2n}{3n+6} = \lim_{n\to\infty} \frac{-2}{3+6/n} = \frac{-2}{3+0} = -\frac{2}{3}$ (2)

b) $\frac{3}{n^2}(-1)^n$ not monotone, as its sign is alternating. (1)

$\frac{3}{n^2} \longrightarrow 0$, so $\frac{3}{n^2}\cdot(-1)^n \longrightarrow 0$

(for odd n $\frac{3}{n^2}(-1)^n = -\frac{3}{n^2}$,

even n $= \frac{3}{n^2}$, and (2)

the sequences $\pm\frac{3}{n^2}$ have the common limit 0.)

3.A. Compute the limit of the following sequence! $a_n = \left(1+\frac{1}{-3n}\right)^{-n+7}\left(\frac{-2n}{-3n+1}\right)$.

B. Let $\phi(x) = 3x+2$, $x_0 = 3$, $x_{n+1} = \phi(x_n)$. What are ϕ^{-1} and $\phi^n(1) = x_n$?

It was intended to be $x_0 = 3$, not 1.

1. Compute ϕ^{-1} !
2. Find the fixed point x_f of ϕ !
3. Compute x_n !

(A) $$\lim_{n\to\infty}\left(1+\frac{1}{-3n}\right)^{-n+7}\left(\frac{-2n}{-3n+1}\right) =$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{-1/3}{n}\right)^{n}\right]^{-1}\cdot\left(1+\frac{1}{-3n}\right)^{7}\cdot\left(\frac{-2}{-3+1/n}\right) =$$

$$= \left[e^{-1/3}\right]^{-1}\cdot 1^{7}\cdot\frac{-2}{-3+0} = e^{1/3}\cdot\frac{2}{3} \qquad (4)$$

(B) φ^{-1}: $$y = 3x+2$$
$$x = \frac{y-2}{3}$$
$$\varphi^{-1}(y) = \frac{y-2}{3}$$
$$\varphi^{-1}(x) = \frac{x-2}{3} \qquad (2)$$

Fixed point: $x_f = \varphi(x_f)$
$$x_f = 3x_f + 2 \longrightarrow x_f = -1 \qquad (1)$$

x_n: $$x_n = 3^n(x_0 - x_f) + x_f$$

So if $x_0 = 3$, $x_n = 3^n(3-(-1)) + (-1) =$
$= 3^n \cdot 4 - 1$

if $x_0 = 1$, $x_n = 3^n(1-(-1)) + (-1) =$
$= 3^n \cdot 2 - 1 \qquad (3)$

$\rightarrow$ Both versions are accepted.

4. A. Let $\bar{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 12 \\ 82 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$. Compute $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$!

B. Let $T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

1. Find an eigenvector $\bar{v}_1$ corresponding to the eigenvalue $\lambda_1 = 3$!
2. Find the eigenvalue λ_2 of T corresponding to the eigenvector $\bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$!
3. Calculate α and β in $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\bar{v}_1 + \beta\bar{v}_2$!
4. Calculate $T(\alpha\bar{v}_1 + \beta\bar{v}_2)$, $T^2(\alpha\bar{v}_1 + \beta\bar{v}_2)$, etc.

A $\begin{pmatrix} 12 \\ 82 \end{pmatrix} = \alpha\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\alpha + 2\beta \\ 2\beta \end{pmatrix}$

$\left.\begin{matrix} 2\alpha + 2\beta = 12 \\ 2\beta = 82 \end{matrix}\right\} \longrightarrow \beta = 41, \quad \alpha = -35$

B 1 $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\cdot\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} 1x + 2y \\ 2x + 1y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$

$\left.\begin{matrix} 1x + 2y = 3x \\ 2x + 1y = 3y \end{matrix}\right\} \quad x = y$

general form of $\bar{v}_1 = \begin{pmatrix} x \\ x \end{pmatrix}$, pick one : $\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2 $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1\cdot 1 + 2\cdot(-1) \\ 2\cdot 1 + 1\cdot(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so $\lambda_2 = -1$

3 $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix} \longrightarrow \begin{matrix} \alpha = 1/2 \\ \beta = 1/2 \end{matrix}$

4 $T^n\left[\frac{1}{2}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{2}\cdot 3^n\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\cdot(-1)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$