Végeselem módszer

① PDE: $\frac{d^2}{dx^2}u(x) + f(x)u(x) = 0, \quad u(0)=u(1)=0$

② Minimalizáld

$$S[u] = \int_0^1 \tfrac{1}{2}[u'(x)]^2 - f(x)u(x)\,dx \text{ -et,}$$

ha $u(0)=u(1)=0$

E-L egyenlet

$$\frac{d}{dx}\frac{\partial[\frac{1}{2}u'^2 - fu]}{\partial u'} - \frac{\partial[\frac{1}{2}u'^2 - fu]}{\partial u} = 0$$

$$\frac{d}{dx}u' - (-f) = 0 = u'' + f$$

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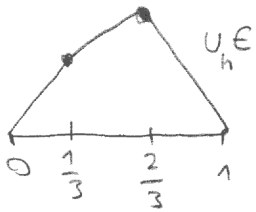
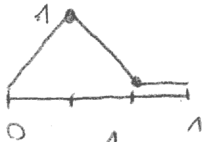
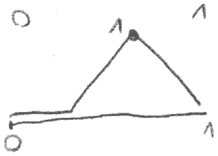
véges dim. tér $\rightarrow V \subset H \leftarrow \infty$ dim. tér



$u_h \in V$: a három intervallumon darabonként affín függvények, $u(0)=u(1)=0$

V bázisa: $\varphi_1 =$ 

$\varphi_2 =$ 

$u_h = c_1\varphi_1 + c_2\varphi_2$

$$S[u_h] \approx \frac{1}{2}\left[\left((3c_1)^2 + (-3c_1+3c_2)^2 + (-3c_2)^2\right)\cdot\frac{1}{3}\right]$$

u' dx

trapéz közelítés

$$-\left[\frac{1}{2}\left(f(0)\cdot 0 + f(\tfrac{1}{3})\cdot c_1\right) + \frac{1}{2}\left(f(\tfrac{1}{3})\cdot c_1 + f(\tfrac{2}{3})\cdot c_2\right) + \frac{1}{2}\left(f(\tfrac{2}{3})\cdot c_2 + f(1)\cdot 0\right)\right]\cdot\frac{1}{3} =$$

$$= \frac{1}{2}\sum_{i,j} c_i K_{ij} c_j - \sum_i F_i c_i = \frac{1}{2}\bar{c}^T K\bar{c} - \bar{F}\bar{c} = S[\bar{c}]$$

$$\frac{\partial S[\bar{c}]}{\partial c_i} = \sum_j K_{ij}c_j - F_i = 0 \qquad K\bar{c} = \bar{F}$$

$$\bar{c} = K^{-1}\bar{F}$$

Gyenge megoldás

$u''(x) + f(x) = 0$, $u(0)=u(1)=0$ $\iff \int_0^1 (u''(x)+f(x))\cdot v(x)\,dx = 0$ bármely? $v(x)$-re

$$\int_0^1 u''v + fv\,dx = [u'v]_0^1 - \int_0^1 u'v' - f\cdot v\,dx$$

ha $v(0)=v(1)=0$,

$u''+f=0 \iff \int_0^1 u'v' - f\cdot v\,dx = 0 \quad \forall v$-re.

Végeselem: $u_h = c_1\varphi_1 + c_2\varphi_2$, $\quad v = \varphi_1$ vagy φ_2

a) $v=\varphi_1$: u' v'

$$\frac{1}{3}\left((3c_1)\cdot 3 - \frac{1}{2}\left(f(0)\cdot 0 + f(\tfrac{1}{3})\cdot c_1\right)\right)$$

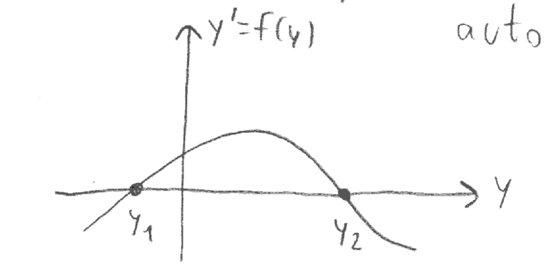
$$+\frac{1}{3}\left((-3c_1+3c_2)\cdot 3 - \frac{1}{2}\left(f(\tfrac{1}{3})\cdot c_1 + f(\tfrac{2}{3})\cdot 0\right)\right) + 0 = 0$$

b) $v=\varphi_2$: $\quad 0 + \ldots + \ldots = 0$

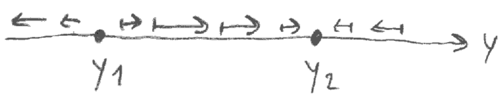
két ismeretlen: $c_1, c_2 \leftrightarrow$ két egyenlet

DE linearizációja

1dim: $y' = f(y)$ időfüggetlen, autonóm DE

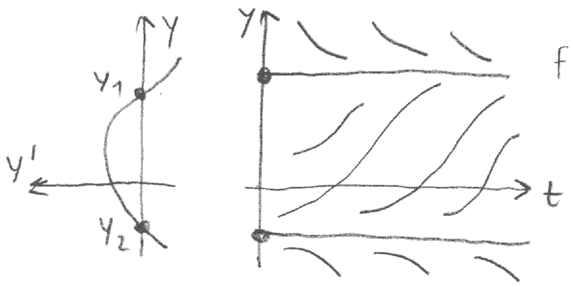


$y_{1,2}$ fixpont egyensúlyi állapot

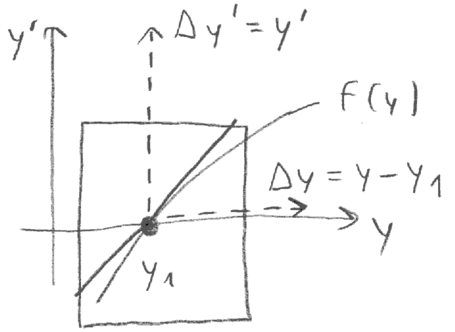


← fázistér (vonal)

instabil stabil fixpont



fixpont ⟺ konst. megold



$$\Delta y \approx 0 \longrightarrow f(y_1+\Delta y) \approx \underbrace{f(y_1)}_{=0\text{ (fixpont)}} + f'(y_1)\cdot\Delta y$$

$$\frac{d}{dt}\Delta y \approx \underbrace{f'(y_1)}_{\frac{df}{dy}}\cdot \Delta y$$

több dim: $\frac{d}{dt}\bar{y} = \bar{f}(\bar{y})$

y_1 fixpont $\longrightarrow \bar{f}(\bar{y}_1) = 0$, $\overline{\Delta y} = \bar{y} - y_1$, $\overline{\Delta y}' = \bar{y}'$

$$\frac{d}{dt}\overline{\Delta y} \approx \frac{\partial \bar{f}}{\partial \bar{y}}(y_1)\cdot\overline{\Delta y}$$

$$\frac{d}{dt}(\overline{\Delta y})_i = \sum_\gamma \frac{\partial f_i}{\partial y_\gamma}\cdot(\overline{\Delta y})_\gamma \qquad \left(\frac{\partial \bar{f}}{\partial \bar{y}}\right)_{i\gamma} = \frac{\partial f_i}{\partial y_\gamma} \quad \text{Jacobi mátrix}$$

Pl.: 1dim: $y' = y(1-y) = y - y^2$ $\qquad \frac{d(y-y^2)}{dy} = 1-2y$

fixpontok	$y_1 = 0$	$y_2 = 1$
	$1-2\cdot 0 = 1$	$1-2\cdot 1 = -1$
lin. egy.	$\Delta y = y - 0$	$\Delta y = y - 1$
	$\frac{d}{dt}\Delta y = 1\cdot\Delta y$	$\frac{d}{dt}\Delta y = -1\cdot\Delta y$

2dim, Lotka-Volterra

$$\frac{d}{dt}\bar{r}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t)\end{pmatrix} = \begin{pmatrix} 2x - 4xy \\ -y + xy\end{pmatrix} = \begin{pmatrix} 2x(1-2y) \\ y(-1+x)\end{pmatrix} \quad \text{fixpont} \quad P_1: \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad P_2: \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

Jacobi mátrix

$$J = \begin{pmatrix} \frac{\partial}{\partial x}(2x-4xy) & \frac{\partial}{\partial y}(2x-4xy) \\ \frac{\partial}{\partial x}(-y+xy) & \frac{\partial}{\partial y}(-y+xy)\end{pmatrix} = \begin{pmatrix} 2-4y & -4x \\ y & -1+x \end{pmatrix}$$

$$J(P_1) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad J(P_2) = \begin{pmatrix} 0 & -4 \\ 1/2 & 0 \end{pmatrix}$$

$$\overline{\Delta r} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \overline{\Delta r} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$\frac{d}{dt}\overline{\Delta r} \approx \frac{d}{dt}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad \frac{d}{dt}\overline{\Delta r} = \begin{pmatrix} 0 & -4 \\ 1/2 & 0 \end{pmatrix}\overline{\Delta r}$$